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On the relative extrema of the Laguerre orthogonal functions.

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Summary. - *It is shown that the relative extrema, of fraed rank of the* LAGUERRE *orthogonal functions form monotone sequences.*

1. We define the LAGUERRE polynomials $L_n(x)$ by the equations

$$n\,!\,L_n(x) = e^x \left(\frac{d}{dx}\right)^n (x^n e^{-x}), \qquad n = 0,\ 1,\ 2, \ldots$$

and the corresponding orthogonal functions by

$$\Phi_n(x) = e^{-\frac{1}{2}x} L_n(x), \qquad n = 0,\ 1,\ 2, \ldots .$$

We denote by $x_{1,n}, x_{2,n}, \ldots, x_{n,n}$ the zeros of $\Phi'_n(x)$, arranged in ascending order of magnitude and we write

$$\Phi_n(x_{r,n}) = \mu_{r,n}$$

so that $\mu_{r,n}$ is the r^{th} relative extremum of $\Phi_n(x)$. We shall establish the following result.

THEOREM. - *The sequence* $\{\mu_{r,n}\}$ *is monotone for fixed* r; *more precisely, it is increasing for* r *odd and decreasing for* r *even.*

This theorem, and the corresponding result for the LEGENDRE polynomials, were suggested by a study of the tables and graphs of these functions. (In particular see JAHNKE-EMDE [2] 32-33, 115-120, or F. TRICOMI [6]). A proof of the *ultimate* monotony in the LEGENDRE case was first given by R. COOPER [1]. G. SZEGÖ [5] established the full result in the LEGENDRE case; O. SZÁSZ [3] obtained a corresponding result in the ultra-spherical case. The following proof runs parallel to that of SZEGÖ, but there are certain complications.

It can be shown that

$$\lim_{n \to \infty} \mu_{r,n} = J_0(j_r)$$

where j_r is the r^{th} positive zero of $J_1(x)$.

The following set of approximate values show the numerical behavior of $\mu_{1,n}$:

$\mu_{1,1} = -.4463$, $\mu_{1,2} = -.4141$, $\mu_{1,3} = -.4081$, $\mu_{1,4} = -.4059$,
$\mu_{1,5} = -.4049$, $\mu_{1,6} = .4042$, $\mu_{1,7} = -.4039$, $\mu_{1,8} = -.4036$.

The limit of the sequence is $-.4028 \ldots$.

2. Lemma. - *For each* n *we have*

$$x_{1,n+1} < x_{1,n} < x_{2,n+1} < x_{2,n} < \ldots < x_{n,n+1} < x_{n,n} < x_{n+1,n+1} .$$

Proof. - We observe that the points $x_{r,n}$ are the zeros of the polynomial

$$\mathfrak{L}_n(x) = L'_n(x) - \frac{1}{2} L_n(x).$$

Denote by $y_{r,n}$ the zeros of $L_n(x)$ and by $\bar{y}_{r,n}$ the zeros of $L'_n(x)$ (which of course separate the $y_{r,n}$). Consider the behavior of the polynomials L_n, L'_n and $\mathfrak{L}_n$ in the interval $y_{r,n} \leq x \leq \bar{y}_{r,n}$. At $y_{r,n}$ we have $L_n = 0$, $(-)^r L'_n > 0$ and therefore $(-)^r \mathfrak{L}_n > 0$; at $\bar{y}_{r,n}$ we have $L'_n = 0$, $(-)^r L_n > 0$ and therefore $(-)^r \mathfrak{L}_n < 0$. We conclude that $\mathfrak{L}_n$ vanishes at some point between $y_{r,n}$ and $\bar{y}_{r,n}$ and that the sign of L'_n at this point is $(-)^r$. This is true also for $r = n$, $\bar{y}_{n,n} = +\infty$.

We now recall the well-known fact (see, e. g., Szegö [4, p. 44]) that $E_n(x) = L'_{n+1}(x)L_n(x) - L'_n(x)L_{n+1}(x) < 0$.
From this it follows that at $x_{r,n}$, where $L'_n = \frac{1}{2} L_n$, we have

$$E_n = 2L'_{n+1}L'_n - L'_n L_{n+1} = 2L'_n \mathfrak{L}_{n+1} < 0.$$

In other words, $\mathfrak{L}_{n+1}$ and L_n' have opposite signs at each point $x_{r,n}$. In the preceding paragraph we have shown that the sign of L'_n at $x_{r,n}$ is $(-)^r$ and therefore is opposite at consecutive zeros of $\mathfrak{L}_n$. It follows that $\mathfrak{L}_{n+1}$ vanishes at least once between any two zeros of $\mathfrak{L}_n$ (also in 0, $x_{1,n}$ and $x_{n,n}$, $+\infty$). From this the lemma follows.

3. Some formulae. — The basic relations for the Laguerre functions are the following

$$x\Phi''_n + \Phi'_n + \left(n + \frac{1}{2} - \frac{1}{4}\right)\Phi_n = 0, \tag{3.1}$$

$$(n + 1)\Phi_{n+1} = (2n + 1 - x)\Phi_n - n\Phi_{n-1}, \tag{3.2}$$

$$x\Phi'_n = \left(n - \frac{1}{2} x\right)\Phi_n - n\Phi_{n-1}. \tag{3.3}$$

(Compare, e. g., Szegö ([4], 5.1.2, 5.1.10, 5.1.14)).

In (3.3) we replace n by $n+1$ and add and subtract (3.3) k obtain

$$x\Phi'_{n+1}+x\Phi'_n=\left(n+1-\frac{1}{2}x\right)\Phi_{n+1}+\left(n-\frac{1}{2}x\right)\Phi_n-(n+1)\Phi_n-n\Phi_{n-1},$$

$$x\Phi'_{n+1}-x\Phi'_n=\left(n+1-\frac{1}{2}x\right)\Phi_{n+1}-\left(n-\frac{1}{2}x\right)\Phi_n-(n+1)\Phi_n+n\Phi_{n-1}.$$

Inserting the expression obtained from (3.2) for Φ_{n-1}, the following relations arise:

$$\Phi'_{n+1}+\Phi'_n=\frac{2n+2-\frac{1}{2}x}{x}(\Phi_{n+1}-\Phi_n), \tag{3.4}$$

$$\Phi'_{n+1}-\Phi'_n=-\frac{1}{2}(\Phi_{n+1}+\Phi_n). \tag{3.5}$$

Eliminating Φ_{n+1} we have another identity which will be useful later:

$$(n+1)\Phi'_{n+1}-\left(n+1-\frac{1}{2}x\right)\Phi'_n=-\left(n+1-\frac{1}{4}x\right)\Phi_n. \tag{3.6}$$

Finally, multiplying (3.4) by (3,5) we find

$$\Phi^2_{n+1}-\Phi^2_n=\frac{x}{n+1-\frac{1}{4}x}(\Phi'^2_n-\Phi'^2_{n+1}). \tag{3.7}$$

4. An auxiliary function. — We write

$$f_n(x)=\Phi^2_n(x)+\frac{x}{n+1-\frac{1}{4}x}\Phi'^2_n(x). \tag{4.1}$$

Differentiating we find

$$f'_n(x)=2\Phi_n\Phi'_n+\frac{x}{n+1-\frac{1}{4}x}2\Phi'_n\Phi''_n+\Phi'^2_n\left[\frac{1}{n+1-\frac{1}{4}x}+\frac{\frac{1}{4}x}{\left(n+1-\frac{1}{4}x\right)^2}\right].$$

If we substitute for Φ''_n from (3.1) this gives

$$f'_n(x)=2\Phi'_n\left[\left\{\frac{\left(n+1-\frac{1}{4}x\right)\Phi_n-\Phi'_n-\left(n+\frac{1}{2}-\frac{1}{4}x\right)\Phi_n}{n+1-\frac{1}{4}x}\right\}+\frac{(n+1)\Phi'_n}{2\left(n+1-\frac{1}{4}x\right)^2}\right]$$

$$=\frac{\Phi'_n}{\left(n+1-\frac{1}{4}x\right)^2}\left\{\left(n+1-\frac{1}{4}x\right)\Phi_n-\left(n+1-\frac{1}{2}x\right)\Phi'_n\right\}.$$

If we now use (3.6) we obtain

$$f'_n(x) = \frac{-(n+1)\Phi'_n\Phi'_{n+1}}{\left(n+1-\frac{1}{4}x\right)^2}. \tag{4.2}$$

5. Proof of Theorem. – At $x_{r,n+1}$ we have $\Phi'_{n+1} = 0$ and so, from (3.7),

$$\Phi^2_{n+1} = \Phi^2_n + \frac{x}{n+1-\frac{1}{4}x}\Phi'^2_n,$$

that is: $\Phi^2_{n+1} = f_n$. On the other hand, at $x_{r,n}$ we have $\Phi'_n = 0$ and so, by definition of f_n, we have

$$f_u = \Phi^2_n.$$

From our Lemma and the fact that $\Phi'_n(0) = -n - \frac{1}{2}$ it is clear that Φ'_n and Φ'_{n+1} have opposite signs in the interval $x_{r,n+1} < x < x_{r,n}$ and this, with (4.2), implies that f_n is increasing in this interval, that is to say

$$\mu^2_{r,n+1} < \mu^2_{r,n}.$$

From this the required result follows immediately.

REFERENCES

[1] R. Cooper. « Proc. Cambridge Phil. Soc. », (1950).
[2] E. Jahnke and F. Emde, *Tables of higher functions* } 4 {, (1948).
[3] O. Szász. this journal.
[4] G. Szegö, *Orthogonal polynomials*, (New York, 1939).
[5] G. Szegö, this journal.
[6] F. Tricomi, « Atti R. Sc. Torino », **76**, 281 to 316, (1941).