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On the relative extrema of Bessel functions (*).

Nota di OTTO SZÁSZ (a Los Angeles, Calif.).

Summary. - *Denote the relative maxima of* $|\Lambda_\alpha(t)|$ *for* $t>0$ *by* $\mu_{r,\alpha}$, $r=1,2,\ldots$; *our main result is that* $\mu_{r,\alpha}>\mu_{r,\alpha+1}$ *for* $r\geq 1$, $\alpha>-1$.

1. The BESSELfunction of order α is defined for $\alpha>-1$ by the power series

$$J_\alpha(t)=\left(\frac{t}{2}\right)^\alpha \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{4^n n!\,\Gamma(\alpha+n+1)};\qquad \alpha>-1. \tag{1.1}$$

We write

$$\Lambda_\alpha(t)=\left(\frac{2}{t}\right)^\alpha \Gamma(\alpha+1)J_\alpha(t),$$

so that

$$\Lambda_\alpha(0)=1,\quad \Lambda_\alpha(-t)=\Lambda_\alpha(t).$$

It is known that for $\alpha>-1$ the zeros of $\Lambda_\alpha(t)$ are all real [see G. N. WATSON, *Bessel functions*, 1948, p. 483], and it is seen from the differential equation

$$\Lambda_\alpha''(t)+\frac{1+2\alpha}{t}\Lambda_\alpha'(t)+\Lambda_\alpha(t)=0, \tag{1.2}$$

that all zeros of Λ_α are simple. Λ_α has infinitely many positive zeros which we denote in increasing magnitude by

$$0<t_{1,\alpha}<t_{2,\alpha}<\ldots.$$

The formula

$$\frac{d}{dt}(t^{-\alpha}J_\alpha(t))=-t^{-\alpha}J_{\alpha+1}(t)$$

yields

$$\Lambda_\alpha'(t)=-\frac{t}{2(\alpha+1)}\Lambda_{\alpha+1}(t), \tag{1.3}$$

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so that the extrema of Λ_α correspond to the points $t_{r,\alpha+1} (r = 1, 2, 3, \ldots)$, and we have

$$0 < t_{1,\alpha} < t_{1,\alpha+1} < t_{2,\alpha} < t_{2,\alpha+1} < \ldots . \tag{1.4}$$

Denote the relative extrema of $|\Lambda_\alpha|$ by $\mu_{r,\alpha} (r \geqq 1)$; we shall prove the following monotonicity properties:

(I) $\mu_{1,\alpha} > \mu_{2,\alpha} > \mu_{3,\alpha} > \ldots$, for $\alpha > -\frac{1}{2}$,

(II) $\mu_{1,\alpha} < \mu_{2,\alpha} < \mu_{3,\alpha} < \ldots$, for $-1 < \alpha < -\frac{1}{2}$,

(III) $\mu_{r,\alpha} > \mu_{r,\alpha+1} > \mu_{r,\alpha+2} > \ldots$, for all $\alpha > -1$; $r \geqq 1$.

2. The differential equation (1.2) yields

$$\frac{d}{dt}(t^{1+2\alpha} \Lambda_\alpha') + t^{1+2\alpha} \Lambda_\alpha = 0; \tag{2.1}$$

we now quote the following theorem, due to Pólya [see G. Szegö, *Orthogonal polynomials*, 1939, pag. 161].

Let $y(t)$ satisfy the differential equation:

$$\frac{d}{dt}(k(t)y') + \Phi(t)y = 0, \tag{2.2}$$

where $k(t) > 0$, $\Phi(t) > 0$, and both functions have a continuous derivative. Then the relative maxima of $|y|$ form an increasing or decreasing sequence according as $k(t)\ \Phi(t)$ is decreasing or increasing.

Applied to the equation (2.1), we have

$$k(t) = \Phi(t) = t^{1+2\alpha} > 0 \quad \text{for} \quad t > 0;$$
$$\frac{d}{dt}(k\Phi) = 2(1 + 2\alpha)t^{1+4\alpha}.$$

This yields immediately the properties (I) and (II).

Note that in the limiting case $\alpha = -\frac{1}{2}$ we have $\Lambda_{-\frac{1}{2}}(t) = \cos t$, and $\mu_{r,-\frac{1}{2}} = 1$ for all r.

For the proof of Pólya's theorem consider the function

$$f(t) = y(t)^2 + \frac{k(t)}{\Phi(t)}(y'(t))^2, \tag{2.3}$$

then

$$f(t) = y(t)^2 \quad \text{if} \quad y'(t) = 0.$$

Furthermore, using (2.2) we find

$$(2.4)\qquad f'(t) = -\left(\frac{y'(t)}{\varphi(t)}\right)^2 \frac{d}{dt}(k(t)\varphi(t)),$$

and the theorem follows directly. For related results see WATSON, loc. cit., pag. 488. On similar lines (I) was proved recently by MIN-TEH CHENG, [« Duke Math. Journal », 17 (1950), Lemma 3].

It follows from (I) that

$$|\Lambda_\alpha(t)| < \Lambda_\alpha(0) = 1 \qquad \text{for} \qquad \alpha > -\frac{1}{2}, \quad t > 0.$$

This also follows from the integral representation

$$\Lambda_\alpha(t) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi e^{it\cos x} \sin^{2\alpha} x\, dx.$$

3. The recurrence formula

$$J_{\alpha+1} = \frac{2\alpha}{t} J_\alpha - J_{\alpha-1}, \qquad \alpha > 0,$$

yields

$$(3.1)\qquad \frac{t^2 \Lambda_{\alpha+1}}{4\alpha(\alpha+1)} = \Lambda_\alpha - \Lambda_{\alpha-1}.$$

From (1.3) and (3.1)

$$(3.2)\qquad \Lambda_\alpha - \Lambda_{\alpha-1} = -\frac{t}{2\alpha}\Lambda_\alpha';$$

again from (1.3)

$$(3.3)\qquad \Lambda_\alpha = -\frac{2\alpha}{t}\Lambda'_{\alpha-1}.$$

From (3.2) for $t = t_{r,\alpha+1}$ (as $\Lambda_\alpha' = 0$)

$$(3.4)\qquad \Lambda_\alpha(t_{r,\alpha+1}) = \Lambda_{\alpha-1}(t_{r,\alpha+1}).$$

We obviously have $\Lambda_\alpha(t) > 0$ for $0 < t < t_{1,\alpha}$ hence

$$(3.5)\qquad \operatorname{sgn} \Lambda_\alpha(t) = (-1)^r \text{ for } t_{r,\alpha} < t < t_{r+1,\alpha};$$

it now follows from (3.3) that

$$\operatorname{sgn} \Lambda'_{\alpha-1}(t) = (-1)^{r+1} \text{ for } t_{r,\alpha} < t < t_{r+1,\alpha},$$

so that $\Lambda_{\alpha-1}(t)$ is monotone increasing if r is odd, decreasing if r is even, in the interval $t_{r,\alpha} < t < t_{r+1,\alpha}$. From (1.4)

$$(3.6)\qquad t_{r,\alpha} < t_{r,\alpha+1} < t_{r+1,\alpha},$$

hence, using (3.4), (3.5) and (3.6)

$$(3.7) \qquad \Lambda_{\alpha-1}(t_{r,\alpha+1}) > \Lambda_\alpha(t_{r,\alpha+1}) = (-1)^r \mu_{r,\alpha}.$$

Replacing here r by $r+1$ and α by $\alpha-1$ we get

$$(3.8) \qquad \Lambda_{\alpha-1}(t_{r+1,\alpha}) = (-1)^{r+1}\mu_{r+1,\alpha-1}.$$

Thus $\Lambda_{\alpha-1}$ has opposite signs at the points $t_{r,\alpha+1}$ and $t_{r+1,\alpha}$; furthermore being monotone in the larger interval $(t_{r,\alpha}, t_{r+1,\alpha})$, $\Lambda_{\alpha-1}$ has exactly one zero in the interval $t_{r,\alpha+1} < t < t_{r+1,\alpha}$. It now follows from (1.4) and from

$$0 < t_{1,\alpha-1} < t_{1,\alpha} < t_{2,\alpha-1} < t_{2,\alpha} < \ldots$$

that for $\alpha > 0$

$$(3.9) \quad t_{r,\alpha-1} < t_{r,\alpha} < t_{r,\alpha+1} < t_{r+1,\alpha-1} < t_{r+1,\alpha} < t_{r+1,\alpha+1} < \ldots .$$

4. To prove now (III), consider

$$f(t) = \Lambda^2_{\alpha-1}(t).$$

From (3.7)

$$\mu^2_{r,\alpha} = f(t_{r,\alpha+1});$$

furthermore

$$f(t_{r,\alpha}) = \mu^2_{r,\alpha-1}.$$

Now, using (3.3)

$$f'(t) = 2\,\Lambda_{\alpha-1}(t)\,\Lambda'_{\alpha-1}(t) = -\frac{t}{\alpha}\,\Lambda_{\alpha-1}\,\Lambda_\alpha;$$

in the interval $(t_{r,\alpha}, t_{r+1,\alpha})$ $\operatorname{sgn}\Lambda_\alpha = (-1)^r$; in the interval

$$(t_{r,\alpha-1},\ t_{r+1,\alpha-1}) \operatorname{sgn} \Lambda_{\alpha-1} = (-1)^r:$$

Hence, in view of (3.9) $f'(t) < 0$ in the interval $(t_{r,\alpha},\ t_{r,\alpha+1})$, so that $f(t)$ is decreasing. Hence $f(t_{r,\alpha}) > f t_{r,\alpha+1})$, or

$$\mu^2_{r,\alpha-1} > \mu^2_{r,\alpha}, \qquad \alpha > 0, \qquad r = 1, 2, 3, \ldots .$$

Thus (III) is proved. It would be interesting to prove monotony for continuously increasing α.

Similar properties were established for the LEGENDRE *polynomials, ultraspherical polynomials,* LAGUERRE *functions and Hermite functions;* see the notes of G. SZEGÖ, JOHN TODD and the author, in « Bollettino della Unione Matematica Italiana », (3), 5 (1950), pp. 120–127.

Observe that for $y(t) = \Lambda_\alpha(t)$ the formulas (2.3) and (2.4) reduce to

$$f(t) = \Lambda_\alpha(t)^2 + \frac{t^2}{4(\alpha+1)}\Lambda_{\alpha+1}(t)^2,$$

$$f'(t) = -\frac{2(1+2\alpha)}{t}(\Lambda'_\alpha(t))^2.$$

Hence $f(t)$ is monotone increasing if $1 + 2\alpha < 0$, and monotone decreasing if $1 + 2\alpha > 0$. It follows that for $\alpha > -\frac{1}{2}$:

$$f(t_{r,\alpha}) = \Lambda'_{\alpha}(t_{r,\alpha})^2 > f(t_{r,\alpha+1}) = \mu^2_{r,\alpha}.$$

From (3.2) for $\alpha > 0$

$$\Lambda'_{\alpha}(t_{r,\alpha})^2 = \left(\frac{2\alpha}{t_{r,\alpha}}\right)^2 \Lambda_{\alpha-1}(t_{r,\alpha})^2 = \left(\frac{2\alpha}{t_{r,\alpha}}\right)^2 \mu^2_{r,\alpha-1},$$

so that

(III′) $$4\alpha^2\mu^2_{r,\alpha-1} > t^2_{r,\alpha}\,\mu^2_{r,\alpha}, \qquad \alpha > 0.$$

This is sharper than (III) if $t_{r\alpha} > 2\alpha > 0$. It is known [Szegö, « Trans. Am. Math. Soc. », 39 (1936), pp. 8–9] that for $-\frac{1}{2} < \alpha < \frac{1}{2}$.

$$\left(r + \frac{\alpha}{2} - \frac{1}{4}\right)\pi < t_{r,\alpha} < r\pi$$

hence $t_{1,\alpha} > 2\alpha$ for $0 < \alpha < \frac{1}{2}$. So that in this range (III′) is sharper than (III).