GEORGE PIRANIAN

The orders of lacunarity of a power series.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **11** (1956), n.2, p. 198–199.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1956_3_11_2_198_0>

The orders of lacunarity of a power series (*)

Nota di GEORGE PIRANIAN (a Ann Arbor U. S. A.)

Sunto. - *The proof of a theorem of* G. RICCI *[(1), pp. 610, 615-622] on the* HADAMARD-OSTROWSKI *and the* FABRY-PÓLYA *lacunarity of power series is simplified.*

Let the series

$$\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

have radius of convergence 1. For $\theta > 0$, a sequence of intervals (p_h, q_h) on the positive real axis, with $p_h \to \infty$ and $q_h > (1+\theta)p_h$, is a θ-sequence of $H - O$ (HADAMARD-OSTROWSKI) gaps for (1) provided the condition

$$\limsup |a_n|^{1/n} < 1 \tag{2}$$

is satisfied for the indices n that fall into the intervals (p_h, q_h). It is a θ-sequence of $F - P$ (FABRY-PÓLYA) gaps provided the indices n that fall into the intervals (p_h, q_h) can be divided into two infinite sets such that (2) holds for the first set but not for the second, and such that the number ν_h of indices n in (p_h, q_h) which belong to the second set satisfies the condition

$$\nu_h = o(q_h - p_h). \tag{3}$$

In a recent paper [1], G. RICCI defined Λ, the order of $H - O$ lacunarity of the series (1), to be the supremum of the values θ for which (1) possesses a θ-sequence of $H - O$ gaps (with the special provision that $\Lambda = 0$ if no θ-sequence of $H - O$ gaps exists); similarly, he defined the order Λ^* of $F - P$ lacunarity. And by means of explicit constructions, he proved the following theorem. *To every pair of values* Λ *and* Λ^* *in the closed interval* $[0, \infty]$ *there corresponds a series* (1) *whose orders of* $H - O$ *and* $F - P$ *lacunarity are* Λ *and* Λ^*, *respectively*.

It is the purpose of this note to present a simpler example of a series (1) with the desired properties. Suppose first that Λ and Λ^* lie in the open interval $(0, \infty)$. Let

$$p_h = (2h)!\,,\ q_h = (1+\Lambda)p_h \quad (h \text{ odd, greater than } \Lambda),$$
$$p_h = (2h)!\,,\ q_h = (1+\Lambda^*)p_h \quad (h \text{ even, greater than } \Lambda^*).$$

(*) This note was written under contract DA 20-018-ORD-13585 between the Office of Ordnance Research of the U. S. Army and the Engineering Research Institute of the University of Michigan.

If the index n does not fall into one of the intervals (p_h, q_h), let $a_n = 1$. If n falls into one of the intervals (p_h, q_h) (h odd), let

$$a_n = \left\{ 1 - \frac{(q_h - n)(n - p_h)}{(q_h - p_h)^2} \right\}^n . \tag{4}$$

If n falls into one of the intervals (p_h, q_h) (h even), let

$$a_n = 0 \quad (n \text{ not a perfect square}), \tag{5}$$

$$a_n = 1 \quad (n \text{ a perfect square}). \tag{6}$$

Obviously, for every θ in $(0, \Lambda)$ the series (1) thus defined has a θ-sequence of $H - O$ gaps, and for every θ in $(0, \Lambda^*)$ it has a θ-sequence of $F - P$ gaps. Suppose, on the other hand, that for some fixed value θ_1 the series (1) has a θ_1-sequence of $H - O$ gaps (r_k, s_k). Because of (6), all except finitely many of the intervals (r_k, s_k) fall into intervals $(p_h\ q_h)$ whose indices h are odd; and because of (2) and (4), they satisfy the condition.

$$\limsup s_k/r_k < \lim q_h/p_h = 1 + \Lambda.$$

Therefore $\theta_1 < \Lambda$.

Similarly, suppose that for some fixed θ_2 the series (1) has a θ_2-sequence of $F - P$ gaps (r_k, s_k). Again, all except finitely many of the gaps (r_k, s_k) fall entirely or almost entirely into gaps (p_h, q_h). Also, there exists a sequence of special indices n which fall into intervals (r_k, s_k) and for which $\lim a_n^{1/n} = 1$. If infinitely many of these special indices fall into odd-numbered gaps (p_h, q_h). then it follows from (4) that the number μ_k of indices « of the second set » in (r_k, s_k) can not satisfy the condition $\mu_k = o(s_k - r_k)$ analogous to (3). Therefore infinitely many of the gaps (r_k, s_k) lie entirely or almost entirely in even-numbered gaps (p_h, q_h), and therefore $\theta_2 \leq \Lambda^*$.

It remains to modify the construction for the cases where one or both of the values Λ and Λ^* does not lie in the open interval $(0, \infty)$. If $\Lambda = 0$, we omit the odd-numbered gaps (p_h, q_h) from our construction; if $\Lambda^* = 0$, we omit the even-numbered gaps. If $\Lambda = \infty$, we replace the condition $q_h = (1 + \Lambda)p_h$ by $q_h = hp_h (h = 1, 3, 5, \dots)$; and we make a similar provision for the case where $\Lambda^* = \infty$. The remainder of the proof for these exceptional cases is trivial.

REFERENCE

[1] G. Ricci, *Prolungabilità e ultraconvergenza delle serie di potenze. Modulazione del margine delle lacune*. Rend. Mat. e Appl. (5) 14 (1955), 602-632.