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On Jacobi polynomials.

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Sunto. - *Several results involving the Jacobi polynomials are obtained. in particular some formulas stated by Feldheim are proved.*

1. In the notation of Szégo [5, Chapter 4] the JACOBI polynomial may be defined by

$$P_n^{(\alpha,\beta)}(x) = \sum_{n=r}^{n} \binom{n+\alpha}{n-r}\binom{n+\beta}{r}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r}. \tag{1.1}$$

Thus P_n is a polynomial of degree n, not only in x, but also in each of α, β. Moreover it is an immediate consequence of (1.1) that

$$\Delta_\alpha P_n^{(\alpha,\beta)}(x) = \frac{x+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{1.2}$$

$$\Delta_\beta P_n^{(\alpha,\beta)}(x) = \frac{x-1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{1.3}$$

where

$$\Delta_\alpha f(\alpha) = f(\alpha+1) - f(\alpha).$$

Also we have [5, p. 62]

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{1.4}$$

In view of (1.2) and (1.3) it is natural to apply formulas of finite differences to P_n; indeed TOSCANO [6] has derived some interesting results from the formula

$$F(-n, \gamma+\delta; \gamma; x) = \frac{(-1)^n\Gamma(\gamma)}{x^\gamma\Gamma(\gamma+\delta)} \Delta_\gamma^n \frac{\Gamma(\gamma+\delta)x}{\Gamma(\gamma)}. \tag{1.5}$$

In the present note we discuss some simple results of a different nature.

2. In the first place we observe that if $f_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n in x that satisfies

$$\frac{d}{dx} f_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) f_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{2.1}$$

and we put

$$f_n^{(\alpha,\beta)}(x) = \sum_{n=0}^{n} \binom{n}{r} C_r(\alpha, \beta, n) \left(\frac{x-1}{2}\right)^r,$$

where $C_r(\alpha, \beta, n)$ is indipendent of x, then (2.1) yields the recurrence

$$nC_{r+1}(\alpha, \beta, n) = (n+\alpha+\beta+1) C_r(\alpha+1, \beta+1, n-1). \tag{2.2}$$

Repeated application of (2.2) leads to

$$C_r(\alpha, \beta, n) = \frac{(n-r)!\,(n+\alpha+\beta+1)_r}{n!} C_0(\alpha+r, \beta+r, n-r). \tag{2.3}$$

If we assume also that

$$\Delta_\beta f_n(x) = \frac{x-1}{2} f_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{2.4}$$

we find that

$$\Delta_\beta C_r(\alpha, \beta, n) = C_{r-1}(\alpha+1, \beta+1, n-1)$$

and in particular

$$C_0(\alpha, \beta+1, n) = C_0(\alpha, \beta, n).$$

Hence if C_0 is a polynomial in β it is independent of β; we may write

$$C_0(\alpha, \beta, n) = C_0(\alpha, *, n).$$

Thus (2.3) implies

$$f_n^{(\alpha,\beta)}(x) = \sum_{r=0}^{n} \binom{n+\alpha+\beta+r}{r} C_0(\alpha+r, *, n-r) \left(\frac{x-1}{2}\right)^r. \tag{2.5}$$

Suppose now that in addition

$$f_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}; \tag{2.6}$$

then (2.5) gives

$$C_0(\alpha, *, n) = \binom{n+\alpha}{n}$$

and we get

$$(2.7)\qquad f_n^{(\alpha,\beta)}(x)=\sum_{n=0}^{n}\binom{n+\alpha}{n-r}\binom{n+\alpha+\beta+r}{r}\left(\frac{x-1}{2}\right)^r$$

$$=\binom{n+\alpha}{n}F\left(-n,\,n+\alpha+\beta+1;\,\alpha+1,\,\frac{1-x}{2}\right).$$

Hence $f_n^{(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(x)$. Thus *if* $\mathrm{f}_n^{(\alpha,\beta)}(x)$ *is a polynomial in* x *and* β, *of degree* n *in* x *that satisfies* (2.1), (2.4) *and* (2.6), *then* $\mathrm{f}_n^{(\alpha,\beta)}(x)=\mathrm{P}_n^{(\alpha,\beta)}(x)$. Similarly *if* $\mathrm{f}_n^{(\alpha,\beta)}(x)$ *is a polynomial in* x *and* α, *of degree* n *in* x, *that satisfies* (2.1) *and*

$$\Delta_\alpha f_n^{(\alpha,\beta)}(x)=\frac{x+1}{2}f_{n-1}^{(\alpha+1,\beta+1)}(x),\qquad f_n^{(\alpha,\beta)}(-1)=(-1)^n\binom{n+\beta}{n},$$

then $\mathrm{f}_n^{(\alpha,\beta)}(x)=\mathrm{P}_n^{(\alpha,\beta)}(x)$.

Incidentally, for the LAGUERRE polynomial

$$L_n^{(\alpha)}(x)=\sum_{n=0}^{n}\binom{n+\alpha}{n-r}\frac{(-x)^r}{r!},$$

we have the following result. *If* $\mathrm{f}_n^{(\alpha)}(x)$ *is a polynomial in* x *of degree* n *that satisfies*

$$\frac{d}{dx}f_n^{(\alpha)}(x)=-f_{n-1}^{(\alpha+1)}(x),\qquad f_n^{(\alpha)}(0)=\binom{n+\alpha}{n},$$

then $\mathrm{f}_n^{(\alpha)}(x)=\mathrm{L}_n^{(\alpha)}(x)$.

3. It follows from (1.2) and (1.3) that

$$(3.1)\qquad P_n^{(\alpha+1,\beta)}(x)-P_n^{(\alpha,\beta+1)}(x)=P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

$$(3.2)\qquad P_n^{(\alpha+1,\beta)}(x)-2P_n^{(\alpha,\beta)}(x)+P_n^{(\alpha,\beta+1)}(x)=xP_{n-1}^{(\alpha+1,\beta+1)}(x).$$

Repeated application of (3.1) gives

$$(3.3)\qquad \sum_{r=0}^{k}(-1)^r\binom{k}{r}P_n^{(\alpha+k-r,\,\beta+r)}(x)=P_{n-k}^{(\alpha-k,\,\beta+k)}(x)$$

and in particular

$$(3.4)\qquad \sum_{r=0}^{k}(-1)^r\binom{k}{r}P_n^{(\alpha-r,\,\beta+r)}(x)=\begin{cases}1 & (k=n)\\ 0 & (k>n).\end{cases}$$

It is also evident from (1.2) and (1.3) that

$$\Delta_\alpha^r P_n^{(\alpha,\,\beta)}(x) = \left(\frac{x+1}{2}\right)^r P_{n-r}^{(\alpha+r,\,\beta+r)}(x),$$

$$\Delta_\alpha^r P_n^{(\alpha,\,\beta)}(x) = \left(\frac{x-1}{2}\right)^r P_{n-r}^{(\alpha+r,\,\beta+r)}(x),$$

and more generally

$$\Delta_\alpha^r \Delta_\beta^s P_n^{(\alpha,\,\beta)}(x) = \left(\frac{x+1}{2}\right)^r \left(\frac{x-1}{2}\right)^s P_{n-r-s}^{(\alpha+r+s,\,\beta+r+s)}(x). \tag{3.5}$$

In particular when $r+s=n$, (3.5) becomes

$$\left(\frac{x+1}{2}\right)^r \left(\frac{x-1}{2}\right)^s = \Delta_\alpha^r \Delta_\beta^s P_{r+s}^{(\alpha,\,\beta)}(x), \tag{3.6}$$

which implies

$$x^n = \sum_{r=0}^{n} \binom{n}{r} \Delta_\alpha^r \Delta_\beta^{n-r} P_n^{(\alpha,\,\beta)}(x). \tag{3.7}$$

It is clear that the formula

$$f(\alpha+\mu) = \sum_r \binom{\mu}{r} \Delta^r f(\alpha)$$

implies

$$P_n^{(\alpha+\mu,\,\beta)}(x) = \sum_{r=0}^{n} \binom{\mu}{r} \left(\frac{x+1}{2}\right)^r P_{n-r}^{(\alpha+r,\,\beta+r)}(x), \tag{3.8}$$

$$P_n^{(\alpha,\,\beta+\nu)}(x) = \sum_{s=0}^{n} \binom{\nu}{s} \left(\frac{x-1}{2}\right)^s P_{n-s}^{(\alpha+s,\,\beta+s)}(x), \tag{3.9}$$

and more generally

$$P_n^{(\alpha+\mu,\,\beta+\nu)}(x) = \sum_{r,\,s=0}^{n} \binom{\mu}{r}\binom{\nu}{s} \left(\frac{x-1}{2}\right)^r \left(\frac{x-1}{2}\right)^s P_{n-s}^{(\alpha+s,\,\beta+s)}(x),$$

$$= \sum_{k=0}^{n} P_{n-k}^{(\alpha+k,\,\beta+k)}(x) \sum_{r+s=0} \binom{\mu}{r}\binom{\nu}{s} \left(\frac{x+1}{2}\right)^r \left(\frac{x-1}{2}\right)^s;$$

using (1.1), this reduces to

$$P_n^{(\alpha+\mu,\,\beta+\nu)}(x) = \sum_{k=0}^{n} P_k^{(\mu-k,\,\nu-k)}(x)\, P_{n-k}^{(\alpha+k,\,\beta+k)}(x). \tag{3.10}$$

Incidentally it is evident from (1.4) that

$$(3.11)\qquad P_n^{(\alpha,\beta)}(x+y)=\sum_{r=0}^{n}\binom{n+\alpha+\beta+r}{r}2^{-r}y^r P_{n-r}^{(\alpha+r,\,\beta+r)}(x).$$

Returning to (3.2), which we rewrite as

$$P_n^{(\alpha+1,\beta)}(x)+P_n^{(\alpha,\beta+1)}(x)=2P_n^{(\alpha,\beta)}(x)+xP_{n-1}^{(\alpha+1,\,\beta+1)}(x),$$

it is easily verified that this implies

$$(3.12)\qquad \sum_{r=0}^{n}\binom{k}{r}P_n^{(\alpha+r,\,\beta+k-r)}(x)=\sum_{r=0}^{n}\binom{k}{r}2^{k-r}x^r P_{n-r}^{(\alpha+r,\,\beta+r)}(x).$$

4. In (1.1) replace α, β by $\alpha-n$, $\beta-n$, respectively. Then we have

$$\sum_{n=0}^{\infty}t^nP_n^{(\alpha-n,\,\beta-n)}(x)=\sum_{n=0}^{\infty}t^n\sum_{r=0}^{n}\binom{\alpha}{n-r}\binom{\beta}{r}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r}$$

$$=\sum_{r,\,s=0}^{\infty}\binom{\alpha}{s}\binom{\beta}{r}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^s t^{r+s}$$

$$(4.1)\qquad =\left(1+\frac{x+1}{2}t\right)^{\alpha}\left(1+\frac{x-1}{2}t\right)^{\beta}.$$

Again we have

$$\sum_{n=0}^{\infty}t^nP_n^{(\alpha,\,\beta-n)}(x)=\sum_{n=0}^{\infty}t^n\sum_{r+s=n}\binom{\alpha+n}{s}\binom{\beta}{r}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^s$$

$$=\sum_{r=0}^{\infty}\binom{\beta}{r}\left(\frac{x-1}{2}\right)^r t^r\sum_{s=0}^{\infty}\binom{\alpha+r+s}{s}\left(\frac{x+1}{2}\right)^s t^s$$

$$=\sum_{r=0}^{\infty}\binom{\beta}{r}\left(\frac{x-1}{2}\right)^r t^r\left(1-\frac{x+1}{2}t\right)^{-\alpha-r-1}$$

$$=\left(1-\frac{x+1}{2}t\right)^{-\alpha-1}\sum_{r=0}^{\infty}\binom{\beta}{r}\left(\frac{x-1}{2}t\right)\left(1-\frac{x+1}{2}t\right)^{-r}$$

$$=\left(1-\frac{x+1}{2}t\right)^{-\alpha-1}\left(1+\frac{\frac{x-1}{2}t}{1-\frac{x+1}{2}t}\right)^{\beta}$$

so that

$$(4.2)\qquad \sum_{n=0}^{\infty} t^n P_n^{(\alpha,\, \beta-n)}(x) = (1-t)^{\beta}\left(1 - \frac{x+1}{2}t\right)^{-\alpha-\beta-1}.$$

Similarly we have

$$(4.3)\qquad \sum_{n=0}^{\infty} t^n P_n^{(\alpha-n,\, \beta)}(x) = (1+t)^{\alpha}\left(1 - \frac{x+1}{2}t\right)^{-\alpha-\beta-1}.$$

The formula (4.2) is due to FELDHEIM [4, p. 120].

By means of (4.1) most of the results of § 3 can be proved rapidly. For example we have

$$\sum_{n=0}^{\infty} t^n P_n^{(\alpha+\mu-n,\, \beta+\nu-n)}(x) = \left(1 + \frac{x+1}{2}t\right)^{\alpha+\mu}\left(1 + \frac{x-1}{2}t\right)^{\beta+\nu}$$

$$= \sum_{r=0}^{\infty} t^r P_r^{(\alpha-r,\, \beta-r)}(x) \sum_{s=0}^{\infty} t^s P_s^{(\eta-s,\, \nu-s)}(x),$$

$$P_n^{(\alpha+\eta-,\, \beta+\nu-n)}(x) = \sum_{r=0}^{n} P_r^{(\alpha-r,\, \beta-r)}(x) P_{n-r}^{(\mu-n+r,\, \nu-n+r)}(x),$$

which is equivalent to (3.10). Use of (4.2) and (4.3) suggests various additional formulas. In particular (4.2) implies

$$\sum_{n=0}^{\infty} t^n P_n^{(\alpha+\mu,\, \beta-n)}(x) = \left(1 - \frac{x+1}{2}t\right)^{-\mu} \sum_{n=0}^{\infty} t^n P_n^{(\alpha,\, \beta-n)}(x),$$

which gives

$$(4.4)\qquad P_n^{(\alpha+\mu,\, \beta)}(x) = \sum_{r=0}^{n} \binom{\mu+r-1}{r}\left(\frac{x+1}{2}\right)^r P_{n-r}^{(\alpha,\, \beta+r)}(x).$$

Similarly (4.3) leads to

$$(4.5)\qquad P_n^{(\alpha,\, \beta+\nu)}(x) = \sum_{r=0}^{n} \binom{\nu+r-1}{r}\left(\frac{x-1}{2}\right)^r P_{n-r}^{(\alpha+r,\, \beta)}(x).$$

It is of course easy to prove (4.4) and (4.5) by means of repeated application of (1.2) and (1.3). The formula

$$(4.6)\qquad P_n^{(\alpha+\mu,\, \beta+\nu)}(x) = \sum_{r+s\le n} \binom{\mu+r-1}{r}\binom{\nu+s-1}{s}\left(\frac{x+1}{2}\right)^r\left(\frac{x-1}{2}\right)^s \cdot P_{n-r-s}^{(\alpha+r,\, \beta+s)}(x)$$

includes both (4.4) and (4.5); it may be compared with (3.10).

We also note the formula

$$(4.7) \qquad P_n^{(\alpha+\mu,\,\beta)}(x) = \sum_{r=0}^{n} \binom{\mu}{r} P_{n-r}^{(\alpha+r,\,\beta+\mu)}(x),$$

which follows easily from (4.3); for $\mu = 1$, (4.7) reduces to (3.1).

5. When $\alpha = \beta$, (4.1) becomes

$$(5.1) \qquad \sum_{n=0}^{\infty} t^n P_n^{(\alpha-n,\,\alpha-n)}(x) = \left(1 + \frac{x+1}{2}t\right)^{\alpha}\left(1 + \frac{x-1}{2}t\right)^{\alpha}.$$

We recall that

$$(5.2) \qquad \sum_{n=0}^{\infty} t^n P_n^{(\lambda)}(x) = (1 - 2xt + t^2)^{-\lambda},$$

where [5, p. 80]

$$(5.3) \qquad P_n^{(\lambda)}(x) = \frac{(2\alpha+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\,\alpha)}(x) \qquad \left(\lambda = \alpha + \frac{1}{2}\right).$$

Now the right member of (5.1) is equal to

$$\left(1 + \frac{2xu}{(x^2-1)^{1/2}} + u^2\right)^{\alpha} \qquad \left(u^2 = \frac{x^2-1}{4}t^2\right).$$

Thus (5.1) becomes

$$\sum_{n=0}^{\infty} t^n P_n^{(\alpha-n,\,\beta-n)}(x) = \sum_{n=0}^{\infty} (-1)^n u^n P_n^{(-\alpha)}\left(\frac{x}{(x^2-1)^{1/2}}\right),$$

so that

$$P_n^{(\alpha-n,\,\beta-n)}(x) = (-1)^n 2^n (x^2-1)^{n/2} P_n^{(-\alpha)}\left(\frac{x}{(x^2-1)^{1/2}}\right).$$

But by (5.3)

$$P_n^{(\alpha-n,\,\beta-n)}(x) = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{(\alpha-n)(\alpha-n\ 1)\ldots(\alpha-2n+1)} P_n^{\left(\alpha-n+\frac{1}{2}\right)}(x).$$

We have therefore

$$(5.4) \qquad (-1)^n 2^n (x^2-1)^{n/2} P_n^{(-\alpha)}(x(x^2-1)^{-1/2})$$

$$= \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{(\alpha-n)(\alpha-n-1)\ldots(\alpha-2n+1)} P_n^{\left(\alpha-n+\frac{1}{2}\right)}(x),$$

which is equivalent to a formula of TRICOMI [3, p. 178]. We note that (5.4) can also be proved by means of

$$P_n^{(\lambda)}(x) = (2\lambda)_n \sum_{2r \leq n} \frac{x^{n-2r}(x^2-1)^r}{2^{2r} \cdot r!\,(n-2r)!\left(\lambda+\frac{1}{2}\right)_r},$$

which is a consequence of (5.2). As a particular case of (5.4) we mention ($\alpha = -1/2$)

$$(5.5)\quad (-1)^n 2^n (x^2-1)^{n/2} P_n(x(x^2-1)^{-1/2}) \frac{1 \cdot 3 \ldots (2n-1)}{(2n+1)(2n+3)\ldots(4n-1)} P_n^{(-n)}(x).$$

Other formulas involving the ultraspherical polynomials are readily obtained. For example (3.10) implies

$$(5.6)\qquad P_n^{(\alpha+\beta,\,\alpha+\beta)}(x) = \sum_{r=0}^{n} P_r^{(\alpha-r,\,\beta-r)}(x) P_{n-r}^{(\beta+r,\,\alpha+r)}(x),$$

while (3.8) and (3.9) give

$$(5.7)\qquad P_n^{(\beta,\,\beta)}(x) = \sum_{r=0}^{n} \binom{\beta-\alpha}{r}\left(\frac{x+1}{2}\right)^r P_{n-r}^{(\alpha+r,\,\beta+r)}(x),$$

$$(5.8)\qquad P_n^{(\alpha,\,\alpha)}(x) = \sum_{r=0}^{n} \binom{\alpha-\beta}{r}\left(\frac{x-1}{2}\right)^r P_{n-r}^{(\alpha+r,\,\beta+r)}(x).$$

Also it follows from (4.1) that

$$(5.9)\qquad P_n^{(\alpha,\,\beta)}(x) = \sum_{r=0}^{n} \binom{\alpha-\beta}{r}\left(\frac{x+1}{2}\right)^r P_{n-r}^{(\beta+r,\,\beta+r)}(x)$$

$$= \sum_{r=0}^{n} \binom{\beta-\alpha}{r}\left(\frac{x-1}{2}\right)^r P_{n-r}^{(\alpha+r,\,\alpha+r)}(x).$$

6. Returning to (1.4), it is clear that the polynomial

$$(6.1)\qquad f_n^{(\alpha,\,\beta)}(x) = \frac{2^n}{(\alpha+\beta-n+1)_n} \cdot P_n^{(\alpha-n,\,\beta-n)}(x)$$

satisfies

$$\frac{d}{dx} f_n^{(\alpha,\beta)}(x) = f_{n-1}^{(\alpha,\beta)}(x); \tag{6.2}$$

in other words $\{ f_n^{(\alpha,\beta)}(x) \}$ constitute an Appell set of polynomials. Using (2.7) we find that

$$\sum_{n=0}^{\infty} t^n f_n^{(\alpha,\beta)}(x) = e^{(x-1)t}\, {}_1F_1(-\alpha;\ -\alpha-\beta;\ 2t), \tag{6.3}$$

from which the property (6.2) is immediate. Also (3.11) follows at once. (6.3) may compared with the formula

$$\sum_{n=0}^{\infty} \frac{2^n t^n}{(\alpha+1)_n} P_n^{(\alpha,\beta-n)}(x) = e^{(x+1)t}\, {}_1F_1(-\beta;\ \alpha+1;\ (1-x)t) \tag{6.4}$$

found by FELDHEIM [4, p. 120]. Incidentally it follows from (6.4) that the polynomials

$$g_n^{(\alpha,\beta)}(x) = \frac{(x+1)^n}{(\alpha+1)_n} P_n^{(\alpha,\beta-n)}\left(\frac{x-1}{x+1}\right) \tag{6.5}$$

also form an Appell set.

When $\alpha=\beta$, it follows from [7, p. 104]

$$e^{-t}\, {}_1F_1(-\alpha;\ -2\alpha;\ 2t) = {}_0F_1\left(\frac{1}{2}-\alpha;\ \frac{1}{4}t^2\right)$$

$$= \Gamma\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}t\right)^{\frac{1}{2}+\alpha} I_{-\frac{1}{2}-\alpha}(t)$$

that

$$\sum_{n=0}^{\infty} t^n f_n^{(\alpha,\alpha)}(x) = e^{xt}\,\Gamma\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}t\right)^{\frac{1}{2}+\alpha} I_{-\frac{1}{2}-\alpha}(t). \tag{6.6}$$

By means of (6.3) and (6.4) we can derive certain formulas involving products of JACOBI polynomials. We shall make use of [2, p. 120, (42), (43)]; (6.8) had been found earlier by BAILEY [1]

$$(6.7)\qquad {}_1F_1(a;\ c;\ u+v) = \sum_{n=0}^{\infty} \frac{(a)_r(c-a)_r}{r!\,(c+r-1)_r(c)_{2r}}$$

$$\cdot (uv)^r\,{}_1F_1(a+r;\ c+2r;\ u)\,{}_1F_1(a+r;\ c+2r;\ v),$$

$$(6.8)\qquad {}_1F_1(a;\ c;\ u)\,{}_1F_1(a;\ c;\ v)$$

$$= \sum_{n=0}^{\infty} (-1)^r \frac{(a)_r(c-a)_r}{r!\,(c)_r(c)_{2r}} (uv)^r\,{}_1F_1(a+r;\ c+2r;\ u+v).$$

Thus by (6.4) and (6.7)

$$\sum_{=0}^{\infty} \frac{2^k(u+v)^k}{(\alpha+1)_k} P_k^{(\alpha,\,\beta-k)}(x) = e^{(x+1)(u+v)}\,{}_1F_1(-\beta;\ \alpha+1;\ (1-x)(u+v))$$

$$= \sum_{r=0} \frac{(-\beta)_r(\alpha+\beta+1)_r}{r!\,(\alpha+r)_r(\alpha+1)_{2r}} (1-x)^{2r}(uv)^r$$

$$\cdot \sum_{m=0}^{\infty} \frac{2^m u^m}{(\alpha+2r+1)_m} P_m^{(\alpha+2r,\,\beta-r-m)}(x) \sum_{n=0}^{\infty} \frac{2^n v^n}{(\alpha+2r+1)_n} P_n^{(\alpha+2r,\,\beta-r-n)}(x),$$

which yields after a little manipulation

$$(6.9)\qquad \binom{m+n}{m} P_{m+n}^{(\alpha,\,\beta-m-n)}(x)$$

$$= \frac{(\alpha+1)_m(\alpha+1)_n}{(\alpha+1)_{m+n}} \sum_{r=0}^{\min(m,\,n)} (-1)^r \binom{\beta}{r} \frac{(\alpha+1)_{2r}(\alpha+\beta+1)_r}{(\alpha+r)_r(\alpha+m+1)_r(\alpha+n+1)_r}$$

$$\cdot \left(\frac{1-x}{2}\right)^{2r} P_{m-r}^{(\alpha+2r,\,\beta-m)}(x)\,P_{n-r}^{(\alpha+2r,\,\beta-n)}(x).$$

Similarly by means of (6.8) we obtain

$$(6.10)\qquad P_m^{(\alpha,\,\beta-m)}(x)\,P_n^{(\alpha,\,\beta-n)}(x)$$

$$= \frac{(\alpha+1)_m(\alpha+1)_n}{(\alpha+1)_{m+n}} \sum_{r=0}^{\min(m,\,n)} \binom{\beta}{r}\binom{m+n-2r}{m-r} \frac{(\alpha+\beta+1)_r}{(\alpha+1)_r} \left(\frac{1-x}{2}\right)^{2r}$$

$$\cdot P_{m+n-2r}^{(\alpha+2r,\,\beta-m-r)}(x).$$

The formulas (6.9) and (6.10) were stated without proof by FELDHEIM [4, p. 134].

Finally using (6.3) together with (6.7) and (6.8) we get in like manner

$$(6.11)\qquad \binom{m+n}{m} P_{m+n}^{(\alpha-m-n,\,\beta-m-m)}(x)$$

$$= \frac{[(-\alpha-\beta)_{m+n}}{(-\alpha-\beta)_m(-\alpha-\beta)_n} \sum_{r=0}^{\min(m,n)} (-1)^r \binom{\beta}{r}$$

$$\cdot \frac{(-\alpha)_r(-\alpha-\beta)_{2r}}{(-\alpha-\beta+r-1)_r(-\alpha-\beta+m)_r(-\alpha-\beta+n)_r}$$

$$\cdot P_{m-r}^{(\alpha-m,\,\beta-m)}(x)\, P_{n-r}^{(\alpha-n,\,\beta-n)}(x),$$

$$(6.12)\qquad P_m^{(\alpha-m,\,\beta-m)}(x)\, P_n^{(\alpha-n,\,\beta-n)}(x)$$

$$+ \frac{(-\alpha-\beta)_m(-\alpha-\beta)_n}{(-\alpha-\alpha)_{m+n}} \sum_{r=0}^{\min(m,n)} \binom{\beta}{r}\binom{m+n-2r}{m-r} \frac{(-\alpha)_r}{(-\alpha-\beta)_r}$$

$$\cdot P_{m+n-2r}^{(\alpha-m-n+r,\,\beta-m-n+r)}(x).$$

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