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On the local uniqueness of the initial value problem of the differential equation $d^n x/dt^n = f(t, x)$.

Nota di AUREL WINTNER (Baltimore, U. S. A.)

Sunto. - *The uniqueness theorem of* NAGUMO-PERRON *is extended to the initial value problem of the differential problem mentioned in the title.*

Let R be a rectangle of the form $(0 < t \leqq a, -b \leqq x \leqq b)$ and, if R_0 denotes the (t, x)-set which results if the point $(0, 0)$ is adjoined to R, let $f(t, x)$ be a real-valued function which is continuous and bounded on R_0. Then, if $t^0 > 0$ is sufficiently small, the differential equation

$$(1) \qquad dx/dt = f(t, x)$$

and the initial condition $x(0) = 0$ possess at least one solution $x = x(t)$ on the interval $0 \leqq t \leqq t^0$. In order that this solution be unique on *some* interval $0 \leqq t \leqq t_0$, where $0 < t_0 \leqq t^0$, it is sufficient to assume that $f(t, x)$ satisfies the inequality

$$(2) \qquad | f(t, x') - f(t, x'') | \leqq |x' - x''| / t$$

whenever (t, x') and (t, x'') are in R.

This uniqueness theorem of NAGUMO (cf., e. g., [1], pp, 97-98, and, for an interesting refinement due to LEVY, [2], pp. 46-47) was extended by PERRON to systems, that is, for the case in which the x and f of (1) are vectors with real components (cf. [1], pp. 139-141). Hence, if x and f are scalars and $f(t, x)$ satisfies the preceding conditions but (1) is replaced by

$$(3) \qquad d^2x/dt^2 = f(t, x),$$

then, corresponding to every real constant c, there will exist a sufficiently small $t_0 = t_0(c) > 0$ having the property that (3) and the initial condition $(x(0), x'(0)) = (0, c)$, where $x'(t) = dx(t)/dt$, will possess just one solution $x(t)$ on the interval $0 \leqq t \leqq t_0$. In fact, if (3) is written as a (binary) vectorial system (1), then one component of the vectorial f will be a linear function, while its other component will be the (scalar) f of (3). But it turns out that the trivial nature of one of the components induces the possibility of a substantial improvement of the conditions (2) imposed on the (scalar) f of (3).

The possibility of such an improvement was suggested by a dynamical consideration. In fact, if (3) is interpreted as the equation of motion of a particle (of unit mass) moving along the x–axis, then it is natural to expect that the factor $1/t$, occurring in the restriction (2) of the variation of the (non-conservative) force $f(t, x)$, can be improved to a factor of the order of $1/t^2$. It will be shown that this is actually the case, since the uniqueness assertion for (3) remains true if the denominator t is replaced by $t^2/2$ in (2). The corresponding result for a differential equation of n–th order is as follows:

If f(t, x) *is real-valued and continuous on* R_0 *and if*

$$|f(t, x') - f(t, x'')| \leqq n! \, |x' - x''|/t^n$$

holds whenever (t, x′) *and* (t, x″) *are in* R, *then, corresponding to any set of real constants* $c_1, \ldots, c_{n-1}$, *there exists a positive* t_0 *having the property that the differential equation*

$$D^n x = f(t, x) \qquad (D = d/dt) \tag{5}$$

and the initial condition

$$x(0) = 0, \quad Dx(0) = c_1, \ldots, D^{n-1}(0) = c_{n-1} \tag{6}$$

cannot possess more than one solution $x = x(t)$ *on the interval* $0 \leqq t \leqq t_0$.

It will be clear from the proof that the theorem remains true if x, f and $c_0 = 0$, $c_1, \ldots, c_{n-1}$ are vectors.

Let $x = x'(t)$ and $x = x''(t)$ be two (possibly identical) solutions $x(t)$ of (5) and (6) on $0 \leqq t \leqq t_0$, where $t_0 > 0$. Then both $x'(t)$ and $x''(t)$ have continuous n–th derivatives (since $f(t, x)$ is continuous at $(t, x) = (0, 0)$ also), and $D^k x'(0) = D^k x''(0)$ holds not only for $k = 0, 1, \ldots, n-1$ but for $k = n$ also, as seen by placing $t = 0$ in (5). Hence, if $u(t) = x'(t) - x''(t)$, and if $v(t)$ denotes 0 or $u(t)/t^n$ according as $t = 0$ or $0 < t \leqq t^0$, then $v(t)$ is continuous for $0 \leqq t \leqq t^0$.

Next, if $x(t)$ is any solution of (5) on $0 \leqq t \leqq t^0$, then

$$x(t) = p(t) + (n-1)!^{-1} \int_0^t (t-s)^{n-1} f(s, x(s)) ds,$$

where $p(t)$ is that polynomial of degree $n - 1$ (at most) which is determined by the n initial conditions $D^k p(0) = D^k x(0)$, where $k < n$. Hence $p(t)$ is the same for $x(t) = x'(t)$ as for $x(t) = x''(t)$. It follows therefore, by subtraction, that $x'(t) - x''(t) = u(t)$ satisfies the inequality

$$|u(t)| \leqq n \int_0^t (t-s)^{n-1} |u(s)| s^{-n} ds,$$

since (4) is assumed. In view of the definition of $v(t)$, this inequality can be written in the form

$$|v(t)| \leqq nt^{-n}\int_0^t (t-s)^{n-1}|v(s)|\,ds \tag{7}$$

(if $t \neq 0$).

The balance of the proof is substantially the same as in the case, $n=1$, of NAGUMO. In fact, as pointed out above, $v(t)$ is continuous for $0 \leqq t \leqq t^0$ and vanishes at $t=0$. Hence the same is true of $w(t)$, if $w(t)$ denotes 0 or the maximum of $|v(s)|$ for $0 < s \leqq t$ according as $t=0$ or $t>0$ (but $t \leqq t^0$). But the assertion is the existence of a sufficiently small positive $t_0 (\leqq t^0)$ having the property that $x'(t) - x''(t)$ vanishes identically on the interval $0 \leqq t \leqq t_0$. Since $w(t)$ denotes the maximum of $|x'(s) - x''(s)|/s^n$ on the interval $0 \leqq s \leqq t$ if $t>0$, and since $w(+0) = w(0) = 0$, it follows that it is sufficient to prove that $w(t) = \text{const.}$ holds on $0 \leqq t \leqq t_0$ if $t_0 > 0$ is small enough.

Suppose, if possible, that $w(t) = \text{const.}$ is false on $0 \leqq t \leqq t_0$ for every choice of $t_0 > 0$. Then, on the one hand, $w(0)=0$ implies that $w(t) > 0$ holds for every positive $t (\leqq t^0)$ and, on the other hand, (7) shows that, since $w(t)$ is monotone,

$$|v(t)| < nt^{-n}w(t)\int_0^t (t-s)^{n-1}ds$$

holds for every positive $t (\leqq t^0)$. Since

$$\int_0^t (t-s)^{n-1}ds = \int_0^t s^{n-1}ds = t^n/n,$$

this can be written in the form

$$|v(t)| < w(t), \text{ where } 0 < t \leqq t^0. \tag{8}$$

But it is clear from the definition of the (continuous) function w (in terms of the continuous function v) that (8) is possible only if

$$w(t^*) < w(t), \text{ where } 0 < t \leqq t^0, \tag{9}$$

holds for some point $s = t^* = t^*(t)$ of the interval $0 < s \leqq t$, a point at which the maximum of $|v(s)|$ on $0 < s \leqq t$ becomes $w(t^*)$ (in fact, $w(t) > w(0) = 0$ if $t > 0$). But since $w(t)$ is monotone, it is clear that $w(t^*) = w(t)$. Hence (9) contains a contradiction.

REFERENCES

[1] E. KAMKE, *Differentialgleichungen reeller Funktionen*, Leipzig, 1930.
[2] P. LEVY, *Processus stochastiques et mouvement Brownien*, Paris, 1948.