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The number of points on certain cubic surfaces over a finite field.

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Sunto. - *Explicit formulas are found for the number of solutions in a finite field of the equation* $ax^2+by^2+cz^2+2a'x+2b'y+2c'z=2dxyz+e$.

In a recent paper in this Bollettino, L. A. ROSATI [4] has determined the number of points on certain classes of cubic surfaces over a finite field $GF(q)$. Geometric characterizations are freely used. The present writer has proved [1] by a simple method that the number of solutions x, y, z of the equation

$$ax^2+by^2+cz^2=2dxyz+e \qquad (abcd \neq 0) \tag{1}$$

is given by the formula

$$N=q^2+1+q\,\{\Psi(a)+\Psi(b)+\Psi(c)+\Psi(e)\}\cdot\Psi(d^2e-abc), \tag{2}$$

where q is odd and $\Psi(a)=+1,\ -1,\ 0$ according as a is a square, a non-square or zero of $GF(q)$. It is easily verified that the number of *points* on the surface

$$ax^2t+by^2t+cz^2t=2dxdz+et^3$$

is $N+3q$. For some similar results see [2], [3]. ROSATI has shown that the number of points on

$$axt^2+byt^2+czt^2+xyz+2et^3=0 \qquad (abc \neq 0) \tag{3}$$

is given by

$$q^2+3q+1+q\Psi(e^2+abc),$$

where again q is odd; this is equivalent to the result that the number of solutions of

$$ax+by+cz+xyz+2e=0 \tag{4}$$

is equal to

$$q^2+1+q\Psi(e^2+abc). \tag{5}$$

It may be of interest to point out that it is not difficult to find the number of solution of

$$ax^2+by^2+cz^2+2a'x+2b'y+2c'z=2dxyz+e. \tag{6}$$

There are a number of cases. If $c \neq 0$, the equatiou (6) may be written in the form

$$(cz-dxy+c')^2=f(x,\ y),$$

where

(7) $f(x, y) = (d^2x^2 - bc)y^2 - 2(c'dx + b'c)y - (acx^2 + 2a'cx + c'^2 - ce).$

Thus the number of solutions of (6) is the same as the number of solutions x, y, w of the equation

$$w^2 = f(x, y). \tag{8}$$

Clearly this number is equal to

$$\sum_{x, y} \{ 1 + \Psi(f(x, y)) \} = q^2 + \sum_{x, y} \Psi(f(x, y)),$$

where the sums are over all x, $y \in GF(q)$.

We shall make use of the easily proved formula

$$\sum_y \Psi(y^2 + a) = \begin{cases} q - 1 & (a = 0) \\ -1 & (a \neq 0). \end{cases} \tag{9}$$

Let

$$S = \sum_{x, y} \Psi(f(x, y)) \tag{10}$$

and put $S = S_1 + S_2$, where

$$S_1 = \sum_{\substack{x, y \\ d^2x^2b \neq c}} \Psi(f(x, y)), \qquad S_2 = \sum_{\substack{x, y \\ d^2x^2b = c}} \Psi(f(x, y)). \tag{11}$$

To evaluate S_1 let

$$g(x) = (c'dx + b'c)^2 + (d^2x^2 - bc)(acx^2 + 2a'cx + c'^2 - ce), \tag{12}$$

so that

$$\begin{aligned} S_1 &= \sum_{d^2x^2 \neq bc} \Psi(d^2x^2 - bc) \sum_z \Psi \left\{ z^2 - \frac{g(x)}{(d^2x^2 - bc)^2} \right\} \\ &= (q - 1) \sum_{g(x)=0} \Psi(d^2x^2 - bc) - \sum_{g(x) \neq 0} \Psi(d^2x - bc) \\ &= q \sum_{g(x)=0} \Psi(d^2x^2 - bc) - \sum_x \Psi(d^2x^2 - bc) \\ &= q \sum_{g(x)=0} \Psi(d^2x^2 - bc) - k, \end{aligned} \tag{13}$$

where $k = q - 1$ or -1 according as $bc = 0$ or $\neq 0$.

As for S_2 there are several possibilities. If the polynomials $d^2x^2 - bc$ and $c'dx + bc$ are relatively prime, that is if

$$\delta = cd^2(bc'^2 - b'^2c) \neq 0,$$

it follows at once that

$$S_2 = 0 \qquad (\delta \neq 0). \tag{14}$$

However, when $\delta = 0$, there are a number of cases to consider. We assume that $cd \neq 0$. If $b' = c' = 0$, then

$$S_2 = q \sum_{d^2x^2 = bc} \Psi(-(acx^2 + 2a'cx - ce)), \tag{15}$$

while

$$S_2 = q\Psi(ab'^2c^3 - 2a'b'c'd + c'^4d^2 - cc'^2d^2e) \qquad (c' \neq 0). \tag{16}$$

It therefore follows from (10), (11) and (13) that when $cd \neq 0$, the number of solutions of (6) is given by

$$N = q^2 + qT - k + S_2, \tag{17}$$

where

$$T = \sum_{g(x)=0} \Psi(d^2x^2 - bc), \tag{18}$$

$k = q - 1$ or -1 according as $b = 0$ or $\neq 0$, and S_2 is determined by (14), (15) and (16).

In particular, when $a' = b' = c' = 0$, we find that $(cd \neq 0)$

$$N = q^2 + q(\Psi(a) + \Psi(b) + \Psi(c) + \Psi(e))\Psi(e) - q + 1 \tag{19}$$

if $a = 0$, $b \neq 0$, or $a \neq 0$, $b = 0$, while for $a = b = 0$ we have

$$N = \begin{cases} q^2 + q\Psi(ce) - q + 1 & (e \neq 0) \\ 2q^2 - 2q + 1 & (e = 0). \end{cases} \tag{20}$$

The case $ab \neq 0$ is covered by (2).

A word may be added about the equation

$$z^2 = x \{ ay^2 + (bx + c)y + dx^2 + 2ex + f \} \qquad (a \neq 0). \tag{21}$$

By means of a linear transformation we may remove the term $(bx + c)y$. Then we find that the number of solutions of

$$z^2 = x \{ ay^2 + dx^2 + ex + f \} \tag{22}$$

is equal to

$$q^2 + q\Psi(a) \sum_{dx^2+2ex+f=0} \Psi(x). \tag{23}$$

It will be clear how equations somewhat more general than (21) can be treated in the same way.

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[4] L. A. Rosati, *Sul numero dei punti di una superficie cubica in uno spazio lineare finito*, « Bollettino della Unione Matematica Italiana », series 3, vol. 11 (1956), pp. 412-418.