Aurel Wintner

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On applications of the Schwarzian derivative in the real domain.

Nota di AUREL WINTNER (Baltimore, U. S. A.)

Sunto. - *By making use of the formal circumstance that the Schwarzian (associated with a homogeneous, linear differential equation of second order) can be written in the form of a Riccatian, it is shown that certain earlier results of the author, concerning disconjugacy on the one hand and the existence of totally monotone solutions on the other hand, can be transferred from* RICCATI'*s resolvent to* SCHWARZ'*s resolvent in the real field.*

1. If $f(t)$ is a regular function on a simply-connected domain J of the complex t-plane, then the classical connection (RIEMANN; SCHWARZ, POINCARÉ) between the *schlicht* conformal transformations of J, the linear differential equation

(1) $$x'' + f(t)x = 0,$$

and its « Schwarzian resolvent » (KLEIN)

(2) $$[s] = 2f(t),$$

where

(2 bis) $$[s] = s'''/s' - 3s''^2/2s'^2,$$

is as follows [1]:

On the one hand, the quotient

(3) $$s(t) = x(t)/y(t)$$

of any two (linearly independent) solutions, $x = x(t)$ and $y = y(t)$, of (1), and only a function $s(t)$ of the form (3), is a solution of the non-linear differential equation (2) (which implies that the general solution of (2) results from any given solution $s = s(t)$ of (2) by a projective transformation

(4) $$s \rightarrow (as + b)/(cs + d), \qquad ad - bc \neq 0,$$

where the three integration constants, required by the order of (2), are represented by $a : b : c : d$). On the other hand, a solution (and so, in view of (4), every solution) of (2) is a *schlicht* function $s(t)$ on J if and only if the differential equation (1) happens to have the property that none of its solutions $x = x(t)$ has more than one zero on J. In the nomenclature customary today in the real field (cf. [4], p. 228), the latter property of (1) is the disconjugate character of (1) on J. Here and in the sequel, the solution $x(t)$ of (1) which vanishes identically is not considered to be a solution of (1).

In the real field (where J becomes an interval), various criteria are known for the disconjugate character of (1) on J. But it is easy to see that the above-quoted classical connections, being of a substantially formal nature, can be transcribed to the real field (to this end, it is only necessary to replace *schlicht* behavior with what in § 2 will be called ∞-*monotony*). This suggested the following considerations in which, however, the main point will be a striking formal parallelism between the Schwarzian resolvent, (2), of (1), with (3), and the Riccatian resolvent,

$$r' + r^2 = -f(t), \tag{5}$$

of (1), with

$$r(t) = x'(t)/x(t); \tag{6}$$

cf. (5) and (6) with (7) and (8) below. Owing to this formal parallelism, which not even for the complex field seems to have been fully exploited in the literature (even though it was observed by KLEIN and, undoubtedly, already by LIE; cf. [5], p. 154), it will be possible to obtain for (2), (3) results corresponding to the results of [7] and [6] on (5), (6).

2. On the line of the real variable t, let J be a (*bounded or unbounded*) interval which will be assumed to be *open*, and let $s(t)$ be any real-valued function, distinct from $+\infty$ and $-\infty$ at the t-values considered, which is defined and continuous at every point t with the possible exception of one point, say $t = t_0$, of J. If there is an excepted t_0, denote by J_1 and J_2 the open intervals (with J_1 to the left of J_2) into which J is divided by t_0, and call the function ∞-*monotone on* J if it satisfies the following three conditions: $s(t)$ is strictly monotone on J_1 as well as on J_2; no value s attained on J_1 is attained on J_2; finally, either $s(t_0 - 0) = +\infty$ and $s(t_0 + 0) = -\infty$ or $s(t_0 - 0) = -\infty$ and $s(t_0 + 0) = +\infty$.

Clearly, a real projective transformation can introduce a point t_0 but preserves the ∞–monotony of a given $s(t)$ on J. The same preservation holds for any given degree ($= n$) of continuous differentiability (class C^n) if the C^n–character of $s(t)$ is defined to be the existence of a continuous n–th derivative on $J_1 + J_2$ or on J according as the ∞–monotone function $s(t)$ does or does not have an excepted point t_0 on J.

Let $f(t)$ be a real–valued, continuous function on J, consider only real–valued solutions of the corresponding differential equations (1), (2) (and (5), of order 2, 3, and 1 respectively), and debar the constant 0 as a solution of (1). Then it is readily seen from the definition of ∞–monotony and from the proof of the classical connection, referred to in § 1, that one half of that connection can be transcribed to the present case as follows:

(i) *Some (or, since the rule* (4) *is valid, every) solution* s(t) *of* (2) *is ∞–monotone on* J *if and only if* (1) *is disconjugate on* (1).

The remaining half, (ii) below, of the real version of the classical connection must be approached with caution, since, if $f(t)$ is just continuous (class C^0), then all solutions, $x(t)$ and $s(t)$, of (1) and (2) will be just of class C^2 and C^3 respectively, whereas (the differential operator (2 bis) being of third order) the solutions of (2) do not appear to be representable in the form (3). But this appearance is misleading:

(ii) *A function (on* J *or on* $J_1 + J_2$*) is a solution of* (2) *if and only if it is representable as the quotient* (3) *of two, linearly independent, solutions of* (1).

The apparent paradox is eliminated if it is observed that, although neither the numerator nor the denominator of (3) is of class C^3 (when $f(t)$ is just continuous), their quotient (3) is of class C^3 by necessity (*). This is readily verified from their C^2–character and from the fact that their Wronskian is a non–vanishing constant.

(*) The situation is the same (i. e., (3) must be of class C^3) even if (1) is generalized to

$$x'' + g(t)x' + f(t)x = 0, \tag{1*}$$

where $g(t)$, like $f(t)$, is just continuous on J. In fact, the Wronskian of the numerator and the denominator of (3) is then a non–vanishing constant multiple of exp $G(t)$, where $G(t)$ is the indefinite integral of $-g(t)$ and, therefore, a function of class C^1.

What *is* a complication when (1*) is not in its normal form (1) is that (2)

3. It is seen from (2 bis) that (2) can be written in the form

$$u' + u^2 = -f(t), \tag{7}$$

where

$$u = -\frac{1}{2} s''/s'. \tag{8}$$

In fact, $\frac{1}{2}[s] = u' + u^2$ is an identity in t by virtue of (8) and (2 bis) alone (i. e., *without* the involvement of a linear differential equation (1) or a coefficient function f). In other words, *if* $\{\ \}$ *denotes the* RICCATI *operator*,

$$\{\ \} = {}' + {}^2 \tag{9}$$

(so that (5) and (7) become $\{r\} + f(t) = 0$ and $\{u\} + f(t) = 0$ respectively), *then the definition* (2 bis) *of* LAGRANGE'*s operator is equivalent to*

$$2[\ ] = \{L'\} - 3L, \quad \text{where} \quad L = (\log)' \tag{10}$$

(in fact, $u = -\frac{1}{2} Ls'$, by (8), while (6) means that $r = Lx$).

4. Needless to say, the same proviso is needed for the formulation (7) of (2), with (8) and (3), as for the formulation (5) of (1),

must then be generalized to

$$[s] = 2f(t) - g'(t) - g^2(t)/2 \tag{2*}$$

(cf., e. g., [5], p. 142), whereas the right-hand side of (2*) is meaningless when $g(t)$ is just continuous. But this can be helped, either by reducing (1*) to the normal form (1) (which is possible, since the relevant multiplier, being the function exp $G(t)/2$, is of class C^1) or else, without any change of the independent — or, for that matter, of the dependent — variable, by transcribing (2*) into the interval relation

$$\int_v^w [s(t)]dt = \int_v^w \{2f(t) - g^2(t)/2\}\, dt - \{g(w) - g(v)\}, \tag{2* bis}$$

where (v, w) is any pair of points on J.

with (6), since only such t-intervals must be considered as are free of zeros of the respective denominators, $s'(t)$ and $x(t)$, of (8) and (6). But the proviso needed for (7)-(8) turns out to be the precise analogue of the trivial proviso needed for (5)-(6):

(iii) *If* f(t) *is real-valued and continuous on* J, *and if* $x = x(t)$, $x = y(t)$ *is an arbitrary pair of linearly independent solutions of* (1), *then no solution* $u = u(t)$ *of the formulation* (7)-(8) *of* (2) *becomes infinite at more than one point* $t_0 = t_0(u)$ *of* J *if and only if* (1) *is disconjugate on* J.

(iii*) *If* (1) *is disconjugate on* J, *then it is always possible to choose the linearly independent solutions* x(t), y(t) *of* (1) *in* (3) *in such a way that both the denominator* ($= y$) *of the solution* (3) *of* (2) *and the denominator* ($= s'$) *of the corresponding solution* (8) *of* (7) *will stay positive* ($\neq 0$) *on the whole of* J.

For, since the Wronskian of $x(t)$ and $y(t)$ is a non-vanishing constant, (8) and a differentiation of (3) show that the zeros of $y(t)$ are identical with the infinities $u(t)$. Hence (iii) follows from (i)-(ii), and (iii*) from the existence theorem (Jacobi) which in [7] was used as the central fact concerning a differential equation (1) disconjugate on an open interval J.

5. Since (7) is the same differential equation as (5), except that (6) is replaced, *via* (3), by (8), it follows from (iii) that the results of [7] on the disconjugacy of (1), results which were there there obtained on the basis of the (trivial) r-analogue of the u-criterion (iii), can now be expressed in terms of the solutions of (2).

In particular, the result of [7], pp. 375-376, on the case in which (1) is non-oscillatory (for large positive t) and has a coefficient function for which the improper integral

$$\int^{\infty} f(t)dt = \lim_{T\to\infty} \int^{T} f(t)dt \quad \text{exists}$$

(it need not converge absolutely), can now be interpreted as properties of the solutions (8) of the formulation (7) of (2), rather than (*loc. cit.*) as properties of the solutions (6) of the formulation (5) of (1).

6. As another application of the same principle of transfer, the following result on a priori integrations of (2) by means of LAPLACE transforms of non-negative mass distributions will now be proved:

(iv) *For large positive* t, *let* f(t) *be a function of class* C^∞ *satisfying the* HAUSDORFF–BERNSTEIN *conditions*

$$(11) \qquad (-D)^n f(t) \geqq 0 \quad \text{for} \quad n = 0,\ 1,\ \ldots \qquad (D = ')$$

and rendering the differential equation (1) *non-oscillatory. Then there exist a sufficiently large* t_0 *and a solution* s(t) *of* (2) *in such a way that, on the half-line* $(t_0,\ \infty)$, *the function* s(t) *results by the two consecutive quadratures assigned by*

$$(12) \qquad -D^2 \log s(t) = \int_0^\infty e^{-tv} d\mu(v),$$

where $\mu(v)$ *is a certain non-decreasing function* $(d\mu \geqq 0)$ *on the closed half-line* $0 \leqq v < \infty$, *and the convergence of the integral* (12) *on the half-line* $(t_0,\ \infty)$ *is part of the assertion.* (*From such a particular solution* s(t), *the general solution of* (2) *results by a projective transformation* (4)).

The assumptions imposed by (iv) on $f(t)$ are not contradictory. For instance, if $f(t) = (2t)^{-2}$, then (11) is satisfied on $(0, \infty)$, and (1) possesses the pair of non-oscillatory solutions $x = t^{1/2}$, $y = t^{1/2} \log t$ (so that, according to (3) and (4), the general solution $s(t)$ of (2) is an arbitrary projective transform of $\log t$). More generally, if $f(t) = (C/t)^2$, where C is a positive constant, then (1) is non-oscillatory if and only if C does not exceed $\frac{1}{2}$ (cf., e. g., [4], p. 233), whereas (11) is satisfied for every C. The proof of (iv) proceeds as follows:

It was proved in [6] that if $f(t)$ satisfies (11) for large t and is such as to render (1) non-oscillatory, then there exist a sufficiently large t_0 and a solution $x(t)$ of (1) in such a way that the corresponding RICCATI ratio (6) will be representable on $(t_0,\ \infty)$ as a definite integral (12) in which $\mu(v)$, where $0 \leqq v < \infty$, is a certain non-decreasing function. But (6) is a solution of (5). On the other hand, (5) is the same differential equation as (7). Since (7) is identical with (2) by virtue of (8), this proves (iv). The (positive) factor $\frac{1}{2}$ in (8) can be thought of as absorbed by the $d\mu\ (\geqq 0)$ of (12).

REFERENCES

[1] In the classical writings, this connection is (sometimes tacitly) combined with what eventually became DARBOUX's criterion (involving the image of the boundary of the domain J) for a *schlicht* mapping. The above-mentioned formulation of the classical fact (recently rediscovered, and used so as to supply sufficient criteria for *schlicht* behavior in general, by NEHARI [2], p. 545 and pp. 49-50), when applied to the particular case of *schlicht* triangle functions, was generalized by FELIX KLEIN to « oscillation theorems », which deal with a self-overlapping triangle and, correspondingly, replace a recourse to DARBOUX's criterion by what corresponds to it in case of an arbitrary *Windungszahl*; cf. [3].

[2] Z. NEHARI, *The Schwarzian derivative and schlicht functions*, « Bulletin of the American Mathematical Society », vol. 55 (1949), pp. 545-551, and *Univalent functions and linear differential equations*, « Lectures on Functions of a Complex Variable », Ann. Arbor, 1955, pp. 49-60; cf. also pp. 214-215 and Lemma 2 and Lemma 3 (and the earlier results of G. M. GOLUSIN and M. SCHIFFER, referred to in connection with those lemmas) in a paper of A. RÉNYI, *On the geometry of conformal mapping*, « Acta Scientiearum Mathematicarum » (Szeged), vol. 12 (1950), pp. 214-222. As I observed some time ago, NEHARI's results become quite understandable (and, correspondingly, the proofs can be reduced considerably; cf. P. HARTMAN and A. WINTNER, *On linear. second order differential equations in the unit circle*, « Transactions of the American Mathematical Society, vol. 78 (1955), 493-495), if it is noticed that what is involved is precisely the distortion factor of the non-euclidean line element ds.

[3] F. KLEIN, *Gesammelte mathematische Abhandlungen*, vol. 2, pp. 551-567, or [5], pp. 211-249.

[4] L. BIEBERBACH, *Einführung in die Theorie der Differentialgleichungen im reellen Gebiet*, 1956, pp. 228-233.

[5] F. KLEIN, *Vorlesungen über die hypergeometrische Funktion*, ed. 1933

[6] A. WINTNER, *A priori Laplace transformations of linear differential equations*, « American Journal of Mathematics », vol. 71 (1949), pp. 587-594.

[7] A. WINTNER, *On the non-existence of conjugate points*, ibid., vol. 73 (1951), pp. 368-380.