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On the non-negativity of Green's functions.

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Sunto. - *Facendo uso di un problema variazionale associato, si mostra che le funzioni di* GREEN *di una classe di operatori di* STURM-LIOUVILLE *sono non-positive. Risultati di questa natura sono stati ottenuti da* ARONSZAJN *e* SMITH *facendo uso della teoria dei nuclei riproduttori.*

Summary. - *Using an associated variational problem, it is shown that the* GREEN'*s functions of a class of* STURM-LIOUVILLE *operators are non-positive. Results of this nature have been obtained by* ARONSZAJN *and* SMITH *using the theory of reproducing kernels.*

§ 1. Introduction.

Consider the inhomogeneous equation

$$u'' + q(x)u = f(x), \quad u(o) = u(1) = 0, \tag{1}$$

whose solution can be written in the form

$$u = \int_0^1 K(x, y)f(y)dy. \tag{2}$$

We wish to examine the sign of the kernel $K(x, y)$, which we shall call the GREEN's function of the equation, under a suitable assumption concerning $q(x)$.

This problem has been investigated by ARONSZAJN and SMITH, [1], using the theory of reproducing kernels and the result we shall obtain is a special case of a general result contained in their paper. Since, however, the method we shall use is so simple, we feel that it is worth noting. Similar results may be obtained for equations of the form

$$u_{xx} + u_{yy} + u_{zz} + q(x, y, z)u = f(x, y, z), \tag{3}$$

under corresponding assumptions, either by means of the method we present here, or as consequences of the general results of ARONSZAJN and SMITH.

§ 2. Statement of Results.

The result we shall demonstrate is

THEOREM. – *Let* q(x) *satisfy the condition*

(1) $$q(x) \leq \pi^2 - d, \ d > 0,$$

where π^2 *appears as the smallest characteristic value of the Sturm–Liouville problem*

(2) $$\begin{gathered} u'' + \lambda u = 0, \\ u(o) = u(1) = 0. \end{gathered}$$

Then

(3) $$K(x, y) \leq 0$$

for $0 \leq x, \ y \leq 1$.

§ 3. Discussion.

The condition in (2.1) asserts the negative definite nature of the quadratic form

(1) $$\int_0^1 u(u'' + q(x)u)dx = \int_0^1 [q(x)u^2 - u'^2]dx.$$

It will be clear from what follows that the truth of the theorem hinges upon this fact, which is also the basis of the abstract presentation contained in the paper cited.

The corresponding result for the equation of (1.3) is

THEOREM. – *Consider the Sturm–Liouville equation*

(2) $$\begin{gathered} u_{xx} + u_{yy} + u_{zz} + u = 0, \ u(x, y, z) \in D, \\ u = 0, \ (x, y, z) \in B, \end{gathered}$$

where B *is the boundary of the finite domain* D.

If $q(x, y, z) \leq \lambda_1 - d, \ d > 0$, *where* λ_1 *is the smallest characteristic value of the problem above, then the Green's function associated with the operator*

(3) $$u_{xx} + u_{yy} + u_{zz} + q(x, y, z)u$$

is non-positive.

The proof follows the same lines as that given for the one-dimensional case below. It is clear that a variety of boundary conditions can be imposed.

§ 4. Proof of Theorem.

Consider the problem of minimizing the inhomogeneous quadratic form

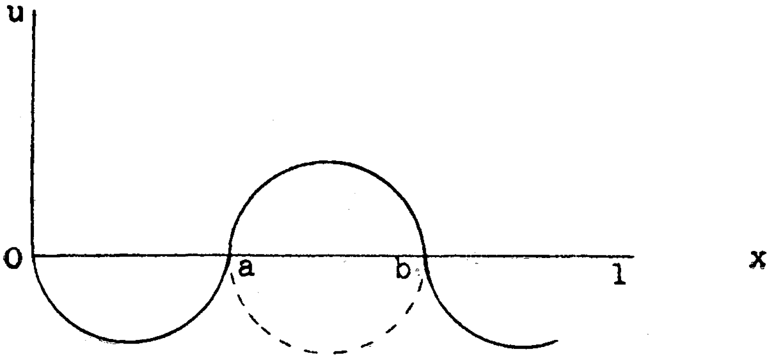
$$J(u) = \int_0^1 [u'^2 - q(x)u^2 + 2f(x)u]\, dx, \tag{1}$$

under the assumption of (2.1) concerning $q(x)$, over all function $u(x)$ which satisfy the conditions $u(o) = u(1) = 0$, and for which the integrals exist.

The positive definite nature of the quadratic terms ensures the existence of a minimum which is determined by the EULER equation, which is precisely (1.1).

A necessary and sufficient condition that $K(x, y)$ ben non-positive is that $u(x) \leq 0$ for $0 \leq x \leq 1$ whenever $f(x) \geq 0$ for $0 \leq x \leq 1$.

Suppose that the minimizing function, $u(x)$, possessed an interval $[a, b,]$ within which it was positive.



Consider now the new function obtained from $u(x)$ by retaining the values of $u(x)$ in the intervals where $u(x) \leq 0$ and replacing $u(x)$ by $-u(x)$ in intervals where $u(x) \geq 0$. This does not change the value of the quadratic terms and decreases the integral $2\int_0^1 f(x)u dx$. Hence we have a contradiction to the statement that $u(x)$ yielded the minimum of $J(u)$. This change introduces discontinuities in the derivative $u'(x)$ which do not affect the integrability of u'^2 and $q(x)u^2$.

REFERENCES

[1] ARONSZAJN, N. and K. T. SMITH, *A Characterization of Positive Reproducing Kernels, Application to Green's Functions Studies in Eigenvalue Problems,* Technical Report 15, U. of Kansas, 1956.