Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **12** (1957), n.3, p. 414–417.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1957_3_12_3_414_0>

A q-analog of a formula of Toscano.

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Sunto. - *Si ottiene una formula, analoga ad un'altra relativa ai polinomi di* HERMITE *stabilita da* TOSCANO [4], *nel caso di* q-*polinomi.*

Summary. - *The paper contains a* q-*analog of a formula for* HERMITE *polynomials recently proved by* TOSCANO [4].

TOSCANO [4, formula (12)] has proved the interesting identity

$$(1)\qquad \sum_{r=-m}^{m}(-1)^r\binom{2m}{m-r}H_{n+r}(x)\,H_{n-r}(x)=$$

$$=\frac{(2m)!\,(n-m)!}{m}\sum_{k=m}^{n}\binom{k-1}{m-1}\frac{H^2_{n-k}(x)}{(n-k)!}\qquad(1\leq m\leq n),$$

where $H_n(x)$ is the HERMITE polynomial defined by

$$e^{xt-\frac{1}{2}t^2}=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}.$$

The polynomial [2]

$$(2)\qquad H_n(x,\,q)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}x^r,$$

where

$$(3)\qquad \begin{bmatrix}n\\r\end{bmatrix}=\frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-r+1})}{(1-q)(1-q^2)\dots(1-q^r)},\qquad \begin{bmatrix}n\\0\end{bmatrix}=1,$$

has properties analogous to those of the HERMITE polynomials. In the present note we show that

$$(4)\qquad \sum_{r=-m}^{m}(-1)^r\begin{bmatrix}2m\\m-r\end{bmatrix}q^{\frac{1}{2}r(r+1)}H_{n+r}(x,\,q)\,H_{n-r}(x,\,q)=$$

$$=\frac{(q)_{2m}(q)_{n-m}}{(q)_m}\sum_{k=m}^{n}\begin{bmatrix}k-1\\m-1\end{bmatrix}q^{(n-k)m}x^k\frac{H^2_{n-k}(x,\,q)}{(q)_{n-k}}\,(1\leq m\leq n),$$

where

$$(q)_m = (1-q)(1-q^2)\dots(1-q^m),\ (q)_0 = 1,$$
$$(a)_m = (1-a)(1-aq)\dots(1-aq^{n-1}),\ (a)_0 = 1.$$

To prove (4), we shall make use of the identities [2, formulas (1.7), (1.8)]

$$H_m(x,q)\,H_n(x,q) = \sum_{r=0}^{\min(m,n)} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} (q)_r x^r H_{m+n-2r}(x,q),$$

$$H_{m+n}(x,q) = \sum_{r=0}^{\min(m,n)} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} (q)_r q^{\frac{1}{2}r(r-1)} x^r \cdot$$
$$\cdot H_{m-r}(x,q)\,H_n(x,q).$$

Thus

$$H_{n+r}(x,q)\,H_{n-r}(x,q) = \sum_s \begin{bmatrix} n+r \\ s \end{bmatrix} \begin{bmatrix} n-r \\ s \end{bmatrix} (q)_s x^s H_{2n-2s}(x,q) =$$

$$= \sum_s \begin{bmatrix} n+r \\ s \end{bmatrix} \begin{bmatrix} n-r \\ s \end{bmatrix} (q)_s x^s \sum_j (-1)^j \begin{bmatrix} n-s \\ j \end{bmatrix} \begin{bmatrix} n-s \\ j \end{bmatrix} (q)_j q^{\frac{1}{2}j(j-1)} x^j \cdot$$
$$\cdot H^2_{n-s-j}(x,q) =$$

$$= \sum_{k=0}^{n} x^k H^2_{n-k}(x,q) \sum_{s+j=k} (-1)^j \begin{bmatrix} n+r \\ s \end{bmatrix} \begin{bmatrix} n-r \\ s \end{bmatrix} \begin{bmatrix} n-s \\ j \end{bmatrix} \begin{bmatrix} n-s \\ j \end{bmatrix} (q)_s (q)_j q^{\frac{1}{2}j(j-1)}$$

so that

$$(5)\qquad \sum_{r=-m}^{m} (-1)^r \begin{bmatrix} 2m \\ m-r \end{bmatrix} q^{\frac{1}{2}r(r+1)} H_{n+r}(x,q)\,H_{n-r}(x,q) =$$

$$= \sum_{k=0}^{n} x^k H^2_{n-k}(x,q) \sum_{s+j=k} (-1)^j \begin{bmatrix} n-s \\ j \end{bmatrix} \begin{bmatrix} n-s \\ j \end{bmatrix} (q)_s (q)_j q^{\frac{1}{2}j(j-1)} \cdot$$
$$\cdot \sum_{r=-m}^{m} (-1)^r q^{\frac{1}{2}r(r+1)} \begin{bmatrix} 2m \\ m-r \end{bmatrix} \begin{bmatrix} n+r \\ s \end{bmatrix} \begin{bmatrix} n-r \\ s \end{bmatrix}.$$

Now the sum on the extreme right is equal to

$$(-1)^m q^{2mn-2ms+\frac{1}{2}m(m+1)} \left\{ \frac{(q)_{n-m}}{(q)_s (q)_{n+m-s}} \right\}^2 \cdot$$
$$\cdot \sum_{r=0}^{2m} (-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix} (q^{n-m+1})_r (q^{-n-m+s})_r \cdot$$

$$\cdot (q^{n-m+1})_{2m-r} (q^{-n-m+s})_{2m-r} \, q^{\frac{1}{2}r(2m-r+1)} =$$

$$= (-1)^m q^{2mn-2ms+\frac{1}{2}m(m+1)} \left\{ \frac{(q)_{n-m}}{(q)_s (q)_{n+m-s}} \right\}^2 \cdot$$

$$\cdot (q^{n-m+1})_m (q^{-n-m+s})_m (q^{m+1})_m (q^{-m+s+1})_m =$$

$$= q^{m(n-s)} \frac{(q)_{2m} (q)_n (q)_{n-m}}{(q)_m (q)_s (q)_{s-m} (q)_{n-s} (q)_{n+m-s}},$$

where we have used the following formula of JACKSON [3, formula (2)]:

$$\sum_{r=0}^{2m} (-1)^r \begin{bmatrix} 2m \\ r \end{bmatrix} (a)_r (b_r) (a)_{2m-r} (b)_{2m-r} \, q^{\frac{1}{2}r(2m-r+1)} = \tag{6}$$

$$= (a)_m (b)_m (q^{m+1})_m (abq^m)_m \,.$$

Thus the right member of (5) becomes

$$\frac{q^{mn} (q)_{2m} (q)_n (q)_{n-m}}{(q)_m} \sum_{k=m}^{n} x^k H^2_{n-k}(x, q) \cdot \tag{7}$$

$$\cdot \sum_{s-j=k} (-1)^j \begin{bmatrix} n-s \\ j \end{bmatrix} \begin{bmatrix} n-s \\ j \end{bmatrix} \frac{(q)_j \, q^{-ms+\frac{1}{2}j(j-1)}}{(q)_{s-m} (q)_{n-s} (q)_{n+m-s}}.$$

The inner sum on the right of (7) is equal to

$$\frac{q^{-mk}}{(q)_{n-k} (q)_{n+m-k} (q)_{k-m}} \sum_{j=0}^{k-m} \frac{(q^{n-k+1})_j (q^{-k+m})_j}{(q)_j (q^{n+m-k+1})_j} q^{kj} =$$

$$= q^{-mk} \begin{bmatrix} k-1 \\ m-1 \end{bmatrix} \frac{1}{(q)_n (q)_{n-k}},$$

by the q-analog of GAUSS' Theorem [1, p. 68].

Thus (7) becomes

$$\frac{(q)_{2m} (q)_{n-m}}{(q)_m} \sum_{k=m}^{n} q^{(n-k)m} \begin{bmatrix} k-1 \\ m-1 \end{bmatrix} x^k \frac{H^2_{n-k}(x, q)}{(q)_{n-k}},$$

substituting in the right number of (5), it is clear that we have proved (4).

If we define $G_n(x, q)$ by means of

$$G_n(x, q) = H_n(x, q^{-1}),$$

then corresponding to (4) we get

$$\sum_{r=-m}^{m} (-1)^r \begin{bmatrix} 2m \\ m-r \end{bmatrix} q^{\frac{1}{2}r(r-1)} \quad G_{n+r}(x, q)\, G_{n-r}(x, q) = \tag{8}$$

$$= q^{-m} \frac{(q)_{2m}(q)_{n-m}}{(q)_m} \sum_{k=m}^{n} (-1)^k \begin{bmatrix} k-1 \\ m-1 \end{bmatrix} q^{\frac{1}{2}k(k+1-2n)} x^k \cdot \frac{G^2_{n-k}(x, q)}{(q)_{n-k}} .$$

In particular for $m = 1$, (4) and (8) become

$$H_n^2(x, q) - H_{n+1}(x, q)\, H_{n-1}(y, q) = \tag{9}$$

$$= (1-q)(q)_{n-1} \sum_{k=1}^{n} q^{n-k} x^k \frac{H^2_{n-k}(x, q)}{(q)_{n-k}} ,$$

$$G_n^2(x, q) - G_{n+1}(x, q)\, G_{n-1}(x, q) = \tag{10}$$

$$= q^{-1}(1-q)(q)_{n-1} \sum_{k=1}^{n} (-1)^k q^{\frac{1}{2}k(k+1-2n)} \quad \frac{G^2_{n-k}(x, q)}{(q)_{n-k}} ,$$

respectively, while for $m = n$ we get

$$\sum_{r=-n}^{n} (-1)^r \begin{bmatrix} 2n \\ n-r \end{bmatrix} q^{\frac{1}{2}r(r+1)} \quad H_{n+r}(x, q)\, H_{n-r}(x, q) = \frac{(q)_{2n}}{(q)_n} x^n . \tag{11}$$

$$\sum_{r=-n}^{n} (-1)^r \begin{bmatrix} 2n \\ n-r \end{bmatrix} q^{\frac{1}{2}r(r-1)} \quad G_{n+r}(x. q)\, G_{n-r}(x, q) = \tag{12}$$

$$= (-1)^n q^{-\frac{1}{2}n(n+1)} \frac{(q)_{2n}}{(q)_n} x^n .$$

It is not difficult to verify (9) and (10) directly using the recurrences satisfied by $H_n(x, q)$ and $G_n(x, q)$.

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