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On the Solution of the Differential Equation $f(x, y, y^{(1)}, \dots, y^{(n)}) = 0.$

Nota di Smbat Abian e di Arthur B. Brown (a Flushing, N. Y.)

Sunto. - *Si indica un procedimento per applicare il metodo delle approssimazioni successive alla soluzione di una equazione implicita differenziale ordinaria d'ordine* n, *senza risolverla esplicitamente rispetto alla derivata* n*-esima. Sono date valutazioni dell'errore sul resto della approssimazione* m*-esima*

Summary. - *A procedure is given for applying the method of successive approximations to the solution of an implicit* n*th order ordinary differential equation, without solving explicitly for the* n*th derivative. Appraisals of the remainder error of the* m*th approximation are given.*

In this paper a procedure is given for applying the method of successive approximations to the solution of a differential equation of the type $f(x, y, y^{(1)}, \dots, y^{(n)}) = 0$, without solving it explicitly for $y^{(n)}$. In Theorem 1 below we state the existence and uniqueness of the solution under hypotheses weaker than those usually imposed. The statement of the theorem includes an outline of the procedure for constructing the solution. In Theorem 2 four appraisals of the remainder error of the mth approximation are given; two in terms of the original data, and two in terms of the mth and $(m-1)st$ approximating functions. The latter two appraisals are valid regardless of errors in computation through the $(m-1)st$ approximating function.

The results obtained in this paper generalize and extend those obtained by the authors for the case $n = 1$.

In what follows, unless otherwise stated, the index i runs from 0 to $n-1$, the index j runs from 1 to n, $y^{(0)}(x) \equiv y(x)$, $y^{(j)} \equiv \dfrac{d^j(y)}{dx^j}$ and $(x, (y), z) \equiv (x, y_0, \dots, y_{n-1}, z)$.

Theorem 1. - Let $f(x, y_0, \dots, y_{n-1}, z) \equiv f(x, (y), z)$ be a continuous real-valued function defined on the closed region $N \subset E^{n+2}$ determined by the relations

$$|x - a| \leq H, \qquad |y_i - b_i| \leq H_i, \qquad |z - c| \leq h_n,$$

where H, H_i and h_n are $n+2$ positive constants, and let there be $n+2$ non-negative constants M_i, D_1 and D_2, with $D_1 > 0$ and $D_2 > 0$, such that, for points belonging to N,

(1) $$| f(x, (y), z) - f(x, (\eta), z) | \leqq \sum_i M_i | y_i - \eta_i |,$$

(2) $$| f(a, (b), c) | < h_n D_1,$$

and, if $z \neq \zeta$,

(3) $$D_1 \leqq \frac{f(x, (y), z) - f(x, (y), \zeta)}{z - \zeta} \leqq D_2.$$

Then there exists $n+1$ positive constants $h \leqq H$ and $h_i \leqq H_i$ such that the differential equation

$$f(x, y, y^{(1)}, \dots, y^{(n)}) = 0$$

has a unique solution $y = Y(x)$ in the interval $| x - a | \leqq h$, with $Y^{(i)}(a) = b_i$, $| Y^{(i)}(x) - b_i | \leqq h_i$ and $| Y^{(n)}(x) - c | \leqq h_n$.

Furthermore, h and h_i can be chosen so that if we let k be any constant satisfying

(4) $$0 < k[h_n D_2 + | f(a, (b), c) |] < 2h_n,$$

and if for $(x, (y), z) \in N$ we define

(5) $$F(x, (y), z) \equiv z - kf(x, (y), z),$$

then for $m = 1, 2, \dots$ and $r = 0, 1, \dots, n$, the function $Y_m(x; r)$ is well defined and continuous on $| x - a | \leqq h$, where $Y_m(x; r)$ is determined as follows.

Let $Y_1(x; n)$ be any function continuous on $| x - a | \leqq h$, and satisfying

(6) $$| Y_1(x; n) - c | \leqq h_n,$$

and

(7) $$Y_m(x; n-j) = b_{n-j} + \int_a^x Y_m(t; n-j+1) dt,$$

(8) $$Y_{m+1}(x; n) = F[x, Y_m(x; 0), Y_m(x; 1), \dots, Y_m(x; n)].$$

Furthermore, if we define

$$(9) \qquad Y_m(x) = Y_m(x;\ 0), \qquad |x - a| \leqq h,$$

then

$$(10) \qquad Y(x) = \lim_{m \to \infty} Y_m(x), \qquad |x - a| \leqq h.$$

Before beginning the proof we observe that it is clear that the hypotheses are easily satisfied for a given point $(a, (b), c)$ if $f(x, (y), z)$ is of class $c^{(1)}$ in a neighborhood of $(a, (b), c)$, and $\frac{\partial f}{\partial z}(a, (b), c) > 0$. If $\frac{\partial f}{\partial z}(a, (b), c) < 0$, or in general if (3) is satisfied with D_1 and D_2 both negative, we can obtain a condition of the form (3) by changing the sign of f.

We shall prove the theorem with the help of the following five lemmas.

LEMMA 1. - Let

$$(11) \qquad B = \max(|1 - kD_2|, |1 - kD_1|), \text{ and } A_i = kM_i.$$

Then

$$(12) \qquad 0 \leqq B < 1,$$

$$(13) \qquad (1 - B)h_n - k\,|f(a, (b), c)| > 0,$$

and, for points belonging to N,

$$(14) \qquad |F(x, (y), z) - F(x, (\eta), \zeta)| \leqq \sum_i A_i\,|y_i - \eta_i| + B\,|z - \zeta|.$$

PROOF. - Since $D_2 > 0$, we see from (4) that $0 < kD_2 < 2$. From (3) we see that $D_1 \leqq D_2$. Since $D_1 > 0$, it follows that both $1 - kD_2$ and $1 - kD_1$ are less than 1 in absolute value. Hence B, as defined in (11), satisfies (12).

To prove (13), we first note that, since $D_1 \leqq D_2$, the only possible values for B are $1 - kD_1$ and $kD_2 - 1$. If $B = 1 - kD_1$, (13) follows from (2). If $B = kD_2 - 1$, (13) follows from (4). We turn now to the proof of (14).

For points belonging to N, if $z \neq \zeta$, then, since $k > 0$, we infer from (3) that

$$(15) \qquad 1 - kD_2 \leqq 1 - k\,\frac{f(x, (y), z) - f(x, (y), \zeta)}{z - \zeta} \leqq 1 - kD_1.$$

Obviously,

$$(16) \qquad F(x, (y)\ z) - F(x, (\eta), \zeta) = F(x, (y), z) - F(x, (\eta), z) + \\ + F(x, (\eta), z) - F(x, (\eta), \zeta),$$

which, by (5),

$$= | - k[f(x, (y), z) - f(z, (\eta), z)]| + \\ + |(z - \zeta) - k[f(x, (\eta), z) - f(x, (\eta), \zeta)]|.$$

Relation (14), including the case $z = \zeta$, is now seen to follow from (16), (1), (11) and (15). This completes the proof of Lemma 1.

LEMMA 2. - There exist $n+1$ positive constants $h \leqq H$ and $h_i \leqq H_i$ such that

$$(17) \qquad 0 \leqq B + \sum_i A_i \frac{h^{n-i}}{(n-i)!} < 1,$$

$$(18) \qquad h(|c| + h_n) \leqq h_{n-1},$$

$$(19) \qquad \sum_{r=1}^{j-1} |b_{n-j+r}| \frac{h^r}{r!} + \frac{h^j}{j!}(|c| + h_n) \leqq h_{n-j}, \quad j = 2, 3, \dots, n,$$

and

$$(20) \qquad |F(x, y, z) - c| \leqq h_n$$

for every point of the closed region $N' \subset N$ defined by

$$(21) \qquad |x - a| \leqq h, \qquad |y_i - b_i| \leqq h_i, \qquad |z - c| \leqq h_n.$$

PROOF. - From (13) and the continuity of $F(x, (y), z)$ we infer that positive constants $h \leqq H$ and $h_i \leqq H_i$ exist such that, for $|x - a| \leqq h$ and $|y_i - b_i| \leqq h_i$,

$$(22) \qquad |F(x, (y), c) - F(a, (b), c)| \leqq (1 - B)h_n - k|f(a, (b), c)|.$$

It is clear from (12) that h can be decreased in value so that (17), (18) and (19) are satisfied, and that (22) will remain valid. Further, if $(x, (y), z)$ satisfies (21), from the obvious inequality

$$|F(x, (y)\ z) - c| \leqq |F(x, (y), z) - F(x, (y), c)| + \\ + |F(x, (y), c) - F(a, (b), c)| + |F(a, (b), c) - c|,$$

and from (14), (21), (22) and (5), we infer that

$$| F(x, (y), z) - c | \leqq Bh_n + (1 - B)h_n = h_n .$$

Hence (20) is satisfied, and Lemma 2 is proved.

LEMMA 3. - If $U(x)$ and $V(x)$ are of class $C^{(n)}$ on $| x - a | \leqq h$, with

$$(23) \qquad U^{(i)}(a) = b_i , \quad | U^{(i)}(x) - b_i | \leqq h_i , \quad | U^{(n)}(x) - c | \leqq h_n ,$$

$$(24) \qquad V^{(i)}(a) = b_i , \quad | V^{(i)}(x) - b_i | \leqq h_i , \quad | V^{(n)}(x) - c | \leqq h_n ,$$

then

$$(25) \quad |F[x, V(x), V^{(1)}(x), \ldots, V^{(n)}(x)] - F[x, U(x), U^{(1)}(x), \ldots, U^{(n)}(x)] | \leqq$$

$$\leqq \left[B + \sum_i A_i \frac{| x - a |^{n-i}}{(n-i)!} \right] \left[\max_{t \in [a, x]} | V^{(n)}(t) - U^{(n)}(t) | \right] .$$

PROOF. - In view of (23) and (24) the arguments of F in (25) are coordinates of points belonging to $N' \subset N$. Hence we may apply Lemma 1, and from (14) we infer that

$$(26) \qquad | F[x, V(x), \ldots, V^{(n)}(x)] - F[x, U(x), \ldots, U^{(n)}(x)] | \leqq$$

$$\leqq [\sum_i A_i | V^{(i)}(x) - U^{(i)}(x) |] + B[\max_{t \in [a, x]} | V^{(n)}(t) - U^{(n)}(t) |].$$

Since by (23) and (24) $V^{(i)}(a) = U^{(i)}(a)$ we have, obviously,

$$V^{(i)}(x) - U^{(i)}(x) = \int_a^x [V^{(i+1)}(\xi) - U^{(i+1)}(\xi)] d\xi ,$$

application of which with i successively equal to $n-1$, $n-2, \ldots$ $\ldots, 1, 0$, in view of (26) and the fact that $[\max_{t \in [a, x]} | V^{(i)}(t) - U^{(i)}(t) |]$ is a monotone non-decreasing function of $| x - a |$, yields (25). Hence Lemma 3 is valid.

LEMMA 4. - Let $U(x; n)$ be a continuous function on $| x - a | \leqq h$ with

$$(27) \qquad | U(x; n) - c | \leqq h_n ,$$

and let us define, for $|x-a| \leqq h$,

$$U(x;\ n-j) = b_{n-j} + \int_a^x U(t;\ n-j+1)dt, \tag{28}$$

and

$$U(x) \equiv U(x;\ 0). \tag{29}$$

Then $U(x)$ is of class $C^{(n)}$ with

$$U^{(j)}(x) \equiv U(x;\ j) \tag{30}$$

and $U(x)$ satisfies (23) for $|x-a| \leqq h$.

PROOF. - From (28). by differentiation with respect to x, we find $U'(x;\ i) = U(x;\ i+1)$ and hence, in view of (29), relation (30) is satisfied and consequently $U(x)$ is of class $C^{(n)}$ and $U(x;\ n-j)$ is of class $C^{(j)}$.

The next step in the proof of Lemma 4 is to show that

$$|U(x;\ i) - b_i| \leqq h_i, \qquad |x-a| \leqq h. \tag{31}$$

By (27) and (28) we have

$$|U(x;\ n-1) - b_{n-1}| \leqq |x-a|(|c| + h_n), \tag{32}$$

which by (18) yields (31) for $i = n-1$.

From (32) we have

$$|U(x;\ n-1)| \leqq |b_{n-1}| + |x-a|(|c| + h_n),$$

and hence, using (28) with $j=2$, we obtain

$$|U(x;\ n-2) - b_{n-2}| \leqq \left| \int_a^x [\,|b_{n-1}| + |t-a|(|c|+h_n)]dt \right| =$$

$$= |b_{n-1}|\ |x-a| + \frac{|x-a|^2}{2}(|c| + h_n).$$

From (19) with $j=2$ we now infer that (31) is true for $i = n-2$.

It is easy to show, in similar fashion, for $j = 3, \ldots, n$, and for

$|x-a| \leq h$, that

$$|U(x;\, n \quad j) - b_{n-j}| < \sum_{r=1}^{j-1} |b_{n-j+r}| \frac{|x-a|^r}{r!} + \frac{|x-a|^j}{j!}(|c| + h_n),$$

and from (19) we infer that (31) is true for $i = n-3,\ n-2, \dots,$ .., 1, 0. This completes the proof of (31).

Relations (23) are now seen to follow from (28), (29), (30), (31) and (27). Hence Lemma 4 is proved.

LEMMA 5. - Let $U(x)$ satisfy the hypotheses of Lemma 4 and let us define, for $|x-a| \leq h$,

$$V(x;\, n) = F(x,\ U(x)\ U^{(1)}(x), \dots,\ U^{(n)}(x)], \tag{33}$$

$$V(x;\, n-j) = b_{n-j} + \int_a^x V(t;\, n-j+1)dt, \tag{34}$$

and

$$V(x) = V(x;\, 0). \tag{35}$$

Then $V(x)$ satisfies the hypotheses of Lemma 4, $U(x)$ and $V(x)$ satisfy the hipotheses of Lemma 3, and

$$V^{(j)}(x) = V(x;\, j). \tag{36}$$

Furthermore, if $W(x)$ is defined in terms of $V(x)$ in exactly the same way in which $V(x)$ is defined (in this lemma) in terms of $U(x)$, then for $|x-a| \leq h$

$$\max_{t \in [a,x]} |W^{(n)}(t) - V^{(n)}(t)| \leq \tag{37}$$
$$\leq [P_0(x-a)][\max_{t \in [a,x]} |V^{(n)}(t) - U^{(n)}(t)|],$$

where

$$P_0(t) = B + \sum_i A_i \frac{|t|^{n-i}}{(n-i)!}. \tag{38}$$

PROOF. - Since, by Lemma 4, $U(x)$ satisfies (23), we see from (33) and (20) that, for $|x-a| \leq h$, $|V(x;\, n) - c| \leq h_n$. Hence $V(x;\, n)$ satisfies the same hypotheses as $U(x;\, n)$ of Lemma 4. On comparing (34), (35) with (28), (29), we see that $V(x)$ is defined in terms of $V(x;\, n)$ in the same way in which $U(x)$ is defined in terms of $U(x;\, n)$. Hence the conclusion of Lemma 4 can be applied to $V(x)$, and we infer from (30) that (36) is satisfied, and from Lemma 4 that $U(x)$ and $V(x)$ satisfy the hypoteses of Lemma 3. It remains to prove (37).

From (36) with $j = n$ and from (33) we have

$$V^{(n)}(x) \equiv F[x, U(x), U^{(1)}(x), \dots, U^{(n)}(x)] \tag{39}$$

We then infer from the hypothesis on W that

$$W^{(n)}(x) \equiv F[x, V(x), V^{(1)}(x), \dots, V^{(n)}(x)]. \tag{40}$$

Since $U(x)$ and $V(x)$ satisfy the hipotheses of Lemma 3, we infer from that lemma and from (39), (40) and (38) that

$$| W^{(n)}(x) - V^{(n)}(x) | \leqq [P_0(x-a)][\max_{t \in [a,x]} | V^{(n)}(t) - U^{(n)}(t) |]. \tag{41}$$

Since the right member of (41) is a monotone non-decreasing function of $| x - a |$, we infer the truth of (37), and Lemma 5 is proved.

Now returning to the theorem, if we compare (6) with (27), (7) for $m = 1$ with (28), and (9) for $m = 1$ with (29), we see by (30) that

$$Y_1^{(j)}(x) = Y_1(x; j),$$

and hence (8) with $m = 1$ shows that $U(x) = Y_1(x)$, $Y_1(x) = Y_2(x)$, satisfy (33). In similar fashion, we can obtain easily that $U(x) = Y_1(x)$, $V(x) = Y_2(x)$, $W(x) = Y_3(x)$ satisfy the hypotheses of Lemma 5, and, by mathematical induction, that for $m \geqq 1$, $U(x) = Y_m(x)$, $V(x) = Y_{m+1}(x)$, $W(x) = Y_{m+2}(x)$ satisfy the hypotheses of Lemma 5, hence of lemma 4. Therefore

$$Y_m(x; j) = Y_m^{(j)}(x), \tag{42}$$

and $U(x) = Y_m(x)$ satisfies (23), that is,

$$Y_m^{(i)}(a) = b_i, \quad | Y_m^{(i)}(x) - b_i | \leqq h_i, \quad | Y_m^{(n)}(x) - c | \leqq h_n; \tag{43}$$

and in addition we infer from (41), for $|x - a| \leqq h$ and $r \geqq 2$, that

$$| Y_{r+1}^{(n)}(x) - Y_r^{(n)}(x) | \leqq [P_0(x-a)][\max_{t \in [x,a]} | Y_r^{(n)}(t) - Y_{r-1}^{(n)}(t) |]. \tag{44}$$

Let

$$P_j(t) = B\frac{|t|^j}{j!} + \sum_i A_i \frac{|t|^{n-i+j}}{(n-i+j)!}, \tag{45}$$

so that, in view of (38), for $j = 0, 1, \dots, n-1$,

$$\left| \int_a^x P_j(\xi - a) d\xi \right| = P_{j+1}(x-a). \tag{46}$$

In view of the equalities in (43) and the fact that the right member of (44) is a monotone non-decreasing function of $|x-a|$, we obtain from (44) and (46) by integration, with j successively equal to 1, 2, ..., n, for $m \geqq 2$, with $r = m$,

$$[\max_{t \in [a,x]} | Y_{m+1}^{(n-j)}(t) - Y_m^{(n-j)}(t) |] \leqq \tag{47}$$

$$\leqq [P_j(x-a)][\max_{t \in [a,x]} | Y_m^{(n)}(t) - Y_{m-1}^{(n)}(t) |].$$

From (44),

$$[\max_{|x-a| \leqq h} | Y_{m+1}^{(n)}(x) - Y_m^{(n)}(x) |] \leqq$$

$$\leqq [P_0(h)][\max_{|x-a| \leqq h} | Y_m^{(n)}(x) - Y_{m-1}^{(n)}(x) |].$$

From (38) and (17) we see that

$$0 \leqq P_0(h) < 1. \tag{48}$$

Hence the series $\sum_{m=1}^{\infty} | Y_{m+1}^{(n)}(x) - Y_m^{(n)}(x) |$ and the sequence $\{ Y_m^{(n)}(x) \}$ both converge uniformly on $|x-a| \leqq h$.

From (47),

$$[\max_{|x-a| \leqq h} | Y_{m+1}^{(n-j)}(x) - Y_m^{(n-j)}(x) |] \leqq$$

$$\leqq P_j(h)[\max_{|x-a| \leqq h} | Y_m^{(n)}(x) - Y_{m-1}^{(n)}(x) |].$$

By the WEIERSTRASS comparison test, we infer that the n sequences $\{ Y_m^{(i)}(x) \}$ converge uniformly on $|x-a| \leqq h$. Letting, as in (10),

$$Y(x) = \lim_{m \to \infty} Y_m^{(0)}(x) = \lim_{m \to \infty} Y_m(x),$$

we infer, by a well known theorem, that $\{ Y_m^{(j)}(x) \}$ converges uniformly to $Y^{(\nu)}(x)$. From (43) we infer that

$$(49)\qquad Y^{(\iota)}(a)=b_\iota, \quad | Y^{(\iota)}(x)-b_\iota | \leqq h_\iota, \quad | Y^{(n)}(x)-c | \leqq h_n .$$

From (8), (9), (42) and the continuity of F for $(x, (y), z) \in N'$, we have, for $| x-a | \leqq h$,

$$(50)\qquad Y^{(n)}(x) \equiv F[x, Y(x), Y^{(1)}(x), \dots , Y^{(n)}(x)].$$

From (5) we infer that

$$f[x, Y(x), Y^{(1)}(x), \dots , Y^{(n)}(x)] \equiv 0,$$

so that $y=Y(x)$ is a solution of the differential equation $f(x, y, y', \dots , y^{(n)})=0$, valid on $| x-a | \leqq h$ and satisfying (49).

To prove the uniqueness, let $U(x)$ be any function of class $C^{(n)}$ on $| x-a | \leqq h$ satisfying (23) and such that

$$f[x, U(x), U'(x), \dots , U^{(n)}(x)] \equiv 0.$$

Then by (5)

$$(51)\qquad U^{(n)}(x) \equiv F[x, U(x), \dots , U^{(n)}(x)].$$

In view of (50) and (51), we obtain from Lemma 3 and (38) the relation

$$[\max_{|x-a|\leqq h} | U^{(n)}(x)-Y^{(n)}(x) |] \leqq$$
$$\leqq [P_0(h)][\max_{|x-a|\leqq h} | U^{(n)}(x)-Y^{(n)}(x) |].$$

In view of (48), we infer that $U^{(n)}(x) \equiv Y^{(n)}(x)$, for $| x-a | \leqq h$. Since $U^{(\iota)}(a)=Y^{(\iota)}(a)$, n integrations yield the relation $U(x) \equiv Y(x)$. This completes the proof of Theorem 1.

We now give four appraisals of the remainder error.

THEOREM 2. - With $| x-a | \leqq h$ and $m \geqq 2$, if we define

$$(52)\qquad W_m(x)=\max_{t\in[a, x]} | Y_m^{(n)}(t)-Y_{m-1}^{(n)}(t) | ,$$

then

$$(53)\qquad | Y(x)-Y_m(x) | \leqq$$
$$\leqq \int_a^x \int_a^{t_n} \cdots \int_a^{t_2} \left\{ \frac{W_2(t_1)[P_0(t_1-a)]^{m-1}}{1-P_0(t_1-a)} \right\} dt_1 \dots dt_{n-1} dt_n ,$$

where $P_0(t)$ is given by (38);

$$(54) \qquad | Y(x) - Y_m(x) | \leqq \frac{| x - a |^n}{n!} \frac{W_2(x)[P_0(x-a)]^{m-1}}{1 - P_0(x-a)},$$

$$(55) \qquad | Y(x) - Y_m(x) | \leqq$$

$$\leqq \int_a^x \int_a^{t_n} \cdots \int_a^{t_2} \left[\frac{W_m(t_1) P_0(t_1 - a)}{1 - P_0(t_1 - a)} \right] dt_1 \dots dt_{n-1} dt_n ;$$

$$(56) \qquad | Y(x) - Y_m(x) | \leqq \frac{| x - a |^n}{n!} \frac{P_0(x-a) W_m(x)}{1 - P_0(x-a)}.$$

Furthermore, (55), (56) are valid regardless of all errors in computation through the calculation of $Y_{m-1}(x)$, provided only that $U = Y_{m-1}(x)$ is of class $C^{(n)}$ and satisfies (23), and that $Y_m(x)$ is obtained correctly from $Y_{m-1}(x)$.

PROOF. - Since the right member of (44) is a monotone non-decreasing function of $| x - a |$ we infer from (52) with $m = 2$, and (44) with $r = 2, 3, \dots, r_1$, that, for $r_1 \geqq 2$,

$$(57) \qquad \max_{t \in [a, x]} | Y_{r_1+1}^{(n)}(t) - Y_{r_1}^{(n)}(t) | \leqq [P_0(x-a)]^{r_1 - 1} W_2(x).$$

By (38) and (48) we see that $P_0(x - a) \leq P_0(h) < 1$. Since $\lim_{r \to \infty} Y_r^{(n)}(x) = Y^{(n)}(x)$, on applying (57) for $r_1 = m$, $m+1$, $m + 2, \dots$, we see by the formula for the sum of a geometric series that, for $m \geqq 2$,

$$| Y^{(n)}(x) - Y_m^{(n)}(x) | \leqq \frac{[P_0(x-a)]^{m-1} W_2(x)}{1 - P_0(x-a)}.$$

Since $Y^{(i)}(a) = Y_m^{(i)}(a) = b_i$, relation (53) follows on integrating n times. Relation (54) is an immediate consequence of (53) and the fact that the integrand of (53) is a monotone non-decreasing function of $| t_1 - a |$.

Relations (55) and (56) are proved from (44) in almost exactly the same way in which (53) and (54) were proved, but beginning with the relation (which follows from (52), (44) with $r = m$, $m+1$, $\dots$, r_1, and from the fact that the right member of (44) is a monotone non-decreasing function of $| x - a |$):

$$\max_{t \in [a, x]} | Y^{(n)}_{r_1+1}(t) - Y^{(n)}_{r_1}(t) | \leq [P_0(x - a)]^{r_1 - m + 1} W_m(x),$$

for $r_1 \geq m$. Taking $r_1 = m, m + 1, m + 2, \ldots$, we obtain (55) and (56) as in the proof of (53) and (54) above.

The final statement of Theorem 2 follows from the fact that appraisals (55) and (56) involve only $Y_m(x)$ and $Y_{m-1}(x)$, and that we can consider $Y_{m-1}(x)$ to be a new $Y_1(x)$.

The following theorem is of interest in connection with (53) and (54).

THEOREM 3. - A permissible choice of $Y_1(x; n)$ is given by

(58) $$Y_1(x; n) = F(x, b_0, b_1, \ldots, b_{n-1}, c).$$

If $Y_1(x; n)$ is so chosen, then

$$W_2(x) \leq Bh_n + \sum_i A_i h_i .$$

PROOF. - That $Y_1(x; n)$ can be chosen to equal $F(x, b_0, \ldots, b_{n-1}, c)$ follows from the continuity of F, on comparing (6) and (20).

By (52) with $m = 2$,

$$W_2(x) \leq \max_{|x-a| \leq h} | Y_2^{(n)}(x) - Y_1^{(n)}(x) | ,$$

which, by (8) and (42) with $m = 1$, and (58),

$$= \max_{|x-a| \leq h} | F[x, Y_1(x), \ldots, Y_1^{(n)}(x)] - F(x, b_0, \ldots, b_{n-1}, c) | ,$$

which, by (14) and (43), is

$$\leq \sum_i A_i h_i + Bh_n .$$

This completes the proof.

REFERENCE - For related ideas in a more general setting, cf. «Implicit functions and their differentials in general analysis», by T. H. HILDEBRANDT and LAWRENCE H. GRAVES, Trans. Amer. Math. Soc., Vol. 29 (1927), pp. 127-153.