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Zanichelli

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On the ratio $f(t + cf^{-\alpha}(t))/f(t)$. [1]

Nota di PHILIP HARTMAN (a Baltimore)

Summary. - *The condition* (1) $f(t+cf^{-\alpha}(t))/f(t) \to 1$ *as* $t \to \infty$, *its differentiated form* (2) $f'(t)/f^{1+\alpha}(t) \to 0$ *as* $t \to \infty$ *and its integrated form* (3) *log* $(f(u)/f(v))/(1+\int_u^v f^{\alpha}(s)ds) \to 0$ *as* $u,v \to \infty$ *frequently occur in the asymptotic integration theory of* $d^2x/dt^2 \pm f(t)x = 0$. *The results proved imply that* (3) *and* $\int^{\infty} f^{\alpha}(t)dt = \infty$ *are equivalent to* (1).

Limit relations of the type

(1) $$f(t + cf^{-\alpha}(t))/f(t) \to 1 \text{ as } t \to \infty$$

or its "differentiated" form

(2) $$f'(t)/f^{1+\alpha}(t) \to 0 \text{ as } t \to \infty$$

or its "integrated" form

(3) $$\underset{u<v<\infty}{\text{l.u.b}} \mid \log(f(u)/f(v)) \mid / (1 + \int_u^v f^{\alpha}(s)ds) \to 0 \text{ as } u \to \infty$$

concern the "regularity of growth" of $f(t)$. They occur frequently in the theory of the asymptotic integration of the differential equations

($4_\pm$) $$x'' \pm f(t)x = 0.$$

For example, if $f(t)$ is positive,

(5) $$f(t) > 0,$$

and continuous for $t \geqq 0$, WIMAN [8] has shown that (2), or even (1), with $\alpha = 1/2$, implies that if $x = x(t) \not\equiv 0$ is a solution of (4_+)

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and $N(T)$ is the number of zeros of $x(t)$ on $0 < t < T$, then

$$\pi N(T) \sim \int_0^T f^{1/2}(t)\,dt \text{ as } T \to \infty. \tag{6}$$

(For a simple proof of (6) under the assumption (2) with $\alpha = 1/2$, see [3], p. 642) Actually, when (5) holds, the relation (3) for $\alpha = 1/2$ is necessary and sufficient for the validity of (6); cf. [5], p. 10. A similar example concerns (4_-): when (5) holds, (3) with $\alpha = 1/2$ is necessary and sufficient for (4_-) to have a pair of solutions $x = x_\pm$ satisfying

$$\pm x_\pm'(t) \sim f^{1/2}(t)\, x_\pm(t) \text{ as } t \to \infty. \tag{7}$$

This follows from [2] after the change of variables $ds = f^{1/2}(t)dt$ in (4_-).

Other examples of the occurrence of (1), (2) or (3) in connection with $(4_\pm)$ can be found in [8], [1], [4], [6] and [7], pp. 52-60.

The object of this note is to prove a theorem concerning the "equivalence" of (1) and its integrated form (3).

(*) *Let* f(t) *be a positive continuous function for* $t \geqq 0$. *Let* m(t) *be a positive continuous function for* $t > 0$ *such that, for some non-negative constants* C *and* γ,

$$[m(v)/m(u)]^{\pm 1} \leqq C(v/u)^\gamma \text{ for } 0 < u < v < \infty. \tag{8}$$

Necessary and sufficient for

$$\operatorname*{l.u.b.}_{u < v < \infty} |\log f(u)/f(v)| \,/\, \left(1 + \int_u^v m(f(s))ds\right) \to 0 \text{ as } u \to \infty \tag{9}$$

is that

$$\int^\infty m(f(s))ds = \infty \tag{10}$$

and that, uniformly on every bounded c-set,

$$f(t + c/m(f(t)))/f(t) \to 1 \text{ as } \min\,(t, t + c/m(f(t))) \to \infty. \tag{11}$$

Note that there is no assumption of monotony on $f(t)$ in (*). Condition (8) holds (with $C = 1$) if $m(t)$ is continuously differentiable and $|m'(t)/m(t)| \leqq \gamma/t$; for example, if $m(t) = t^{\pm\gamma}$.

The proof of (*) is suggested by some arguments of [5] and of [6].

Proof of (*). It will first be shown that (9) implies (10). Suppose, if possible, that (9) holds, but that (10) fails to hold. Then (9) implies that lim log $f(t)$, as $t \to \infty$, exists as a (finite) number. Hence,

$\lim f(t)$, as $t \to \infty$, exists and is a positive (finite) number. Since $m(t) > 0$ for $t > 0$, it follows that (10) holds. This contradiction proves (10).

It will now be shown that (11) holds uniformly on bounded c-sets. Let $u, v > 0$ and put

$$h(u,v) = f(u)/f(v) \tag{12}$$

and

$$\varepsilon(u,v) = |\log h(u,v)| / (1 + |\int_u^v m(f(s))ds|). \tag{13}$$

By virtue of (9),

$$\varepsilon(u,v) \to 0 \text{ as } \min(u,v) \to \infty. \tag{14}$$

Let p,q denote (arbitrary) points of the t-interval $I = I_{uv}$, with endpoints u and v, such that

$$f(p) = \max_I f(t) \text{ and } f(q) = \min_I f(t). \tag{15}$$

Obviously,

$$\min(u,v) \to \infty \text{ implies } \min(p,q) \to \infty. \tag{16}$$

Put $H(u,v) = f(p)/f(q)$, so that

$$\log H(u,v) = \varepsilon(p,q)(1 + |\int_p^q m(f(s))ds|) \tag{17}$$

and

$$|\log h(u,v)| \leqq \log H(u,v). \tag{18}$$

For a fixed v and c, let

$$u = v + c/m(f(v)). \tag{19}$$

It is assumed that v, c are such that $u > 0$.

Since $|q-p| \leqq |v-u| = |c|/m(f(v))$,

$$|\int_p^q m(f(s))ds| \leqq |q-p| \max m(f(s)) \leqq |c| \max m(f(s))/m(f(v)),$$

where the max refers to the s-interval with endpoints p,q. By (8), the factor of $|c|$ is not greater than $C \max (f(s)/f(v))^\gamma \leqq C\, H^\gamma(u,v)$.

Consequently

$$\left| \int_p^q m(f(s))ds \right| \leqq C|c| H^{\gamma}(u,v).$$

It follows from (14), (16) and (17) that

$$\log H(u,v)/(1 + C|c| H^{\gamma}(u,v)) \to 0 \text{ as } \min(u,v) \to \infty.$$

Consequently, as $\min(u,v) \to \infty$,

(20) $$\text{either } H(u,v) \to 1 \text{ or } H(u,v) \to \infty$$

uniformly on every bounded c-set, consistent with $\min(u,v) \to \infty$. (For example, for the first alternative in (20), this means that if ε, M are given positive numbers, then there exists a number $T = T(\varepsilon, M)$ such that $|H(u,v) - 1| < \varepsilon$ whenever v and c are such that $|c| < M$ and $\min(u,v) > T$). When $c = 0$, then $f(p) = f(q) = f(v)$ and $H(u,v) = 1$. Consequently, the first alternative in (20) holds uniformly on bounded c-sets, as $\min(u,v) \to \infty$. In view of (12) and (19), it follows from (18) that (11) holds uniformly on bounded c-sets.

There remains to prove the converse implication, that is, that (10) and the limit relation (11), uniformly on bounded c-sets, imply (9). The assumption concerning (11) implies that, for $\varepsilon > 0$, there exists a T_ε such that

(21) $$|\log(f(u)/f(v))| < \varepsilon \text{ if } \min(u,v) > T_\varepsilon \text{ and } |u - v| \leqq 1/m(f(v)).$$

The assumption (10) implies that if $v > 0$, there exists a unique positive number $b = b(v)$ satisfying

(22) $$\int_v^{v+b} m(f(s))\, ds = 1.$$

By the continuity of $f(s)$ and of $m(t)$, there is a number p such that

(23) $$b(v)\, m(f(p)) = 1 \text{ and } v \leqq p \leqq v + b(v).$$

Applications of (21) to the two cases $(u,v) = (u,p)$ and $(u,v) = (v,p)$ give

(24) $$|\log(f(u)/f(v))| < 2\varepsilon \text{ if } v > T_\varepsilon \text{ and } v \leqq u \leqq v + b(v),$$

since $|u - p| \leqq b(v) = 1/m(f(p))$, $|v - p| \leqq b(v) = 1/m(f(p))$ and $\min(u,p) \geqq v$, $\min(v,p) = v$.

For a given $t > T_\varepsilon$, define a sequence of numbers $q_0(t), q_1(t) \ldots$

by placing $q_0(t)=t$ and if $q_0, \ldots, q_{k-1}$ have been defined, put $q_k = q_{k-1} + b(q_{k-1})$. Then $q_k > q_{k-1}$,

$$\int_{q_{k-1}}^{q_k} m(f(s))\, ds = 1 \qquad \text{for } k = 1, 2, \ldots, \tag{25}$$

and

$$| \log (f(u)/f(q_{k-1})) | < 2\varepsilon \text{ if } q_{k-1} < u \leqq q_k . \tag{26}$$

Clearly, (25) implies that $q_k \to \infty$ as $k \to \infty$.

If $u > t$, then there is a unique positive integer k such that $q_{k-1} < u \leqq q_k$. Consequently,

$$| \log (f(u)/f(t)) | < 2k\varepsilon \text{ and } 1 + \int_t^u m(f(s))\, ds \geqq 1 + (k-1) = k .$$

Thus

$$\log (f(u)/f(t)) | / (1 + \int_t^u m(f(s)ds) < 2\varepsilon \text{ if } u > t > T_\varepsilon .$$

This is equivalent to the assertion (9). Hence (*) is proved.

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