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A generating function for the product of two ultraspherical polynomials.

Nota di Leonard Carlitz (Duke University, U.S.A)

Sunto. - *Si generalizza una funzione generatrice del prodotto di due polinomi di* Legendre, *dovuta a* L. C. Maximon *e si ottiene così una funzione generatrice del prodotto di due polinomi ultrasferici.*

Summary. - *A generating function for the product of two Legendre polynomials due to* L. C. Maximon *is generalized to yield a generating function for the product of two ultraspherical polynomials.*

1. In a recent paper Maximon [1] has proved the interesting formula

$$\sum_{n=0}^{\infty} z^n P_n(\cos\alpha) P_n(\cos\beta) \tag{1}$$

$$= \{1 - 2z\cos(\alpha+\beta) + z^2\}^{-\frac{1}{2}} \cdot F\left[\frac{1}{2}, \frac{1}{2}; 1; \frac{4z\sin\alpha\sin\beta}{1 - 2z\cos(\alpha+\beta) + z^2}\right] \qquad (|z| < 1),$$

where $P_n(x)$ is the Legendre polynomial. It may be of interest to note that (1) can be generalized to

$$\sum_{n=0}^{\infty} \frac{n!}{(2\nu)_n} z^n C_n^{\nu}(\cos\alpha) C_n(\cos\beta) \tag{2}$$

$$= \{1 - 2z\cos(\alpha+\beta) + z^2\}^{-\nu}$$

$$\cdot F\left[\nu, \nu; 2\nu; \frac{4z\sin\alpha\sin\beta}{1 - 2z\cos(\alpha+\beta) + z^2}\right] \qquad (|z| < 1),$$

where $C_n(x)$ is the ultraspherical polynomial defined by

$$(1 - 2xz + z^2)^{-\nu} = \sum_{n=0}^{\infty} z^n C_n^{\nu}(x). \tag{3}$$

To prove (2) we make use of the formula (see for example Watson [2])

$$\int_0^{\pi} \sin^{2\nu-1}\omega\, C_n^{\nu}(\cos\Omega) d\omega \tag{4}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)n!}{\Gamma\left(\nu+\frac{1}{2}\right)(2\nu)_n} C_n^\nu(\cos\alpha)C_n^\nu(\cos\beta) \qquad (R(\nu)>0).$$

where

$$\cos\Omega = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\omega.$$

It follows from (3) and (4) that

$$\pi^{\frac{1}{2}}\frac{\Gamma(\nu)}{\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{n=0}^{\infty}\frac{n!}{(2\nu)_n}z^n C_n^\nu(\cos\alpha)C_n^\nu(\cos\beta)$$

$$= \int_0^\pi \frac{\sin^{2\nu-1}\omega\,d\omega}{\{1-2z\cos\Omega+z^2\}^\nu}$$

$$= \int_0^\pi \frac{\sin^{2\nu-1}\omega\,d\omega}{\{1-2z[\cos(\alpha+\beta)+(1-\cos\omega)\sin\alpha\sin\beta]+z^2\}^\nu}$$

(on replacing ω by $\pi-\omega$)

$$= \{1-2z\cos(\alpha+\beta)+z^2\}^{-\nu}\int_0^\pi \frac{\sin^{2\nu-1}\omega\,d\omega}{\left\{1-\dfrac{4z\sin\alpha\sin\beta\sin^2\frac{1}{2}\omega}{1-2z\cos(\alpha+\beta)+z^2}\right\}^\nu}$$

$$= \{1-2z\cos(\alpha+\beta)+z^2\}^{-\nu}$$

$$2^{2\nu-1}\sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}\left(\frac{4z\sin\alpha\sin\beta}{1-2z\cos(\alpha+\beta)+z^2}\right)^n\int_0^\pi \sin^{2n+2\nu-1}\frac{1}{2}\omega\cos^{2\nu-1}\frac{1}{2}\omega\,d\omega.$$

The last integral is equal to

$$2\int_0^{\frac{\pi}{2}}\sin^{2n+2\nu-1}\Phi\cos^{2\nu-1}\Phi\,d\Phi = \frac{\Gamma(n+\nu)\Gamma(\nu)}{\Gamma(n+2\nu)}$$

Therefore

$$\pi^{\frac{1}{2}}\frac{\Gamma(\nu)}{\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{n=0}^{\infty}\frac{n!}{(2\nu)_n}z^n C_n^\nu(\cos\alpha)C_n^\nu(\cos\beta)$$

$$= \frac{2^{2\nu-1}\Gamma^2(\nu)}{\Gamma(2\nu)}\{1-2z\cos(\alpha+\beta)+z^2\}^{-\nu}$$

$$\sum_{n=0}^{\infty}\frac{(\nu)_n(\nu)_n}{n!(2\nu)_n}\left(\frac{4z\sin\alpha\sin\beta}{1-2z\cos(\alpha+\beta)+z^2}\right)^n$$

Since

$$\Gamma(2\nu) = 2^{2\nu-1}\pi^{-\frac{1}{2}}\Gamma(\nu)\Gamma\left(\nu + \frac{1}{2}\right),$$

(2) follows at once.

2. It is clear from (2) that

$$\sum_{n=0}^{\infty} \frac{n!}{(2\nu)_n} z^n C_n^\nu(\cos\alpha) C_n^\nu(\cos\beta)$$

$$= \sum_{r=0}^{\infty} \frac{(\nu)_r(\nu)_r}{r!\,(2\nu)_r} (4z \sin\alpha \sin\beta)^r (1 - 2z\cos(\alpha+\beta) + z^2)^{-\nu-r}$$

$$= \sum_{r=0}^{\infty} \frac{(\nu)_r(\nu)_r}{r!\,(2\nu)_r} (4z \sin\alpha \sin\beta)^r \sum_{s=0}^{\infty} z^s C_s^{\nu+r}(\cos(\alpha+\beta))$$

$$= \sum_{n=0}^{\infty} z^n \sum_{r=0}^{n} \frac{(\nu)_r(\nu)_r}{r!\,(2\nu)_r} (4 \sin\alpha \sin\beta)^r C_{n-r}^{\nu+r}(\cos(\alpha+\beta))$$

and therefore

$$\frac{n!}{(2\nu)_n} C_n^\nu(\cos\alpha) C_n^\nu(\cos\beta) \tag{5}$$

$$= \sum_{r=0}^{n} \frac{(\nu)_r(\nu)_r}{r!\,(2\nu)_r} (4 \sin\alpha \sin\beta)^r C_{n-r}^{\nu+r}(\cos(\alpha+\beta)).$$

This formula is evidently equivalent to (2).

In particular for $\beta = -\alpha$, (5) becomes

$$\frac{n!}{(2\nu)_n} (C_n^\nu(\cos\alpha))^2 = \sum_{r=0}^{n} \frac{(\nu)_r(\nu)_r}{r!\,(2\nu)_r} (-4\sin^2\alpha)^r \cdot C_{n-r}^{\nu+r}(1);$$

since

$$C_n^\nu(1) = \frac{(2\nu)_n}{n!},$$

the right member reduces to

$$\sum_{r=0}^{n} \frac{(\nu)_r(\nu)_r}{r!\,(2\nu)_r} \frac{(2\nu+2r)_{n-r}}{(n-r)!} (-4\sin^2\alpha)^r$$

$$= \frac{(2\nu)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r(\nu)_r(n+2\nu)_r}{r!\,(2\nu)_r\left(\nu+\frac{1}{2}\right)_r} \sin^{2r}\alpha.$$

Therefore we have

$$(C_n^\nu(x))^2 = \left(\frac{(2\nu)_n}{n!}\right)^2 {}_3F_2\left[\begin{matrix}-n,\ \nu,\ n+2\nu;\ 1-x^2\\ 2\nu,\ \nu+\frac{1}{2}\end{matrix}\right]. \tag{6}$$

This can also be obtained directly from (2) with $\beta = -\alpha$.

For $\nu = \frac{1}{2}$, (5) and (6) reduce to

$$P_n(\cos\alpha)P_n(\cos\beta) = \sum_{r=0}^{n} \frac{\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_r}{r!\ r!} (4\sin\alpha\sin\beta)^r C_{n-r}^{\frac{1}{2}+r}(\cos(\alpha+\beta)), \tag{7}$$

$$(P_n(x))^2 = {}_3F_2\left[\begin{matrix}-n,\ n+1,\ \frac{1}{2};\ 1-x^2\\ 1,\ 1\end{matrix}\right], \tag{8}$$

respectively.

If in (5) we take $\beta = \alpha$ we get another formula for $(C_n^\nu(x))^2$. On the other hand if we take $\beta = \frac{\pi}{2} - \alpha$, then we get

$$\frac{n!}{(2\nu)_n} C_n^\nu(\cos\alpha)C_n^\nu(\sin\alpha) = \sum_{r=0}^{n} \frac{(\nu)_r(\nu)_r}{r!\,(2\nu)_r} (2\sin 2\alpha)^r \cdot C_{n-r}^{\nu+r}(0).$$

Since

$$C^\nu_{2n+1}(0) = 0, \quad C^\nu_{2n}(0) = (-1)^n \frac{(\nu)_n}{n!},$$

this becomes

$$\frac{n!}{(2\nu)_n} C_n^\nu(\cos\alpha)C_n^\nu(\sin\alpha) = \sum_{2s\leq n} (-1)^s \frac{(\nu)_{n-s}(\nu)_{n-s}}{s!\,(n-2s)!\,(2\nu)_{n-2s}} (2\sin 2\alpha)^{n-2s}. \tag{9}$$

If we prefer, the right member of (9) can be exhibited as an ${}_4F_3$.

REFERENCES

[1] C. MAXIMON, *A generating function for the product of two Legendre polynomials*, « Norske Videnskabers Selskab Forhandlinger », vol. 29 (1957), pp. 82-86.

[2] G. N. WATSON, *Notes on generating functions of polynomials; (3) Polynomials of Legendre and Gegenbauer*, « Journal of the London Mathematical Society », vol. 8 (1933), pp. 289-292.