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On the Spectra of Group Commutators (*).

Nota di C. R. PUTNAM (a Lafayette - Indiana)

Summary. - *There are obtained results on the location of the spectrum of* $ABA^{-1}B^{-1}$ *in case* A *commutes with* $AB - BA$.

1. In this paper all operators A, B, ... are bounded (linear) on a HILBERT space. Let $sp(A)$ denote the spectrum of A. It was shown independently by KLEINECKE [4] and SHIROKOV [7] that if

$$AC = CA, \tag{1}$$

where C denotes the commutator

$$C = AB - BA, \tag{2}$$

then $sp(C)$ consists of 0 only. In case A^{-1} and B^{-1} exist (that is, if 0 fails to belong to $sp(A)$ and $sp(B)$) one can consider the commutator D defined by

$$D = ABA^{-1}B^{-1} \tag{3}$$

and raise the question whether (1) implies

$$spD = 1 \text{ only.} \tag{4}$$

It was shown in [6] that the answer is affirmative in case A has a logarithm commuting with every operator which commutes with A, that is, if

$$A = e^E, \quad AX = XA \Rightarrow EX = XE \qquad (X \text{ arbitrary}). \tag{5}$$

It is known [2] that not every nonsingular operator possesses a logarithm and, fact (*loc. cit.*), that there exist nonsingular operators which do not even possess square roots. On the positive side however, it is known that if A is nonsingular, so that O belongs to the open complement of $sp(A)$ and if, in addition, O belongs to the unbounded component in the canonical decomposition of this

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open set, then A does have a logarithm E (WINTNER [8]) which satisfies (5) (cf. the remark in section 3 of [6]). This holds, for instance, if A is a nonsingular finite matrix or if A is nonsingular and differs from some multiple of the unit operator by a completely continuous operator. Whether every operator possessing some logarithm necessarily possesses some (possibly different) logarithm satisfyng (5) is apparently not known; cf. section 4 of [6].

Whether (1) alone is sufficient to ensure (4) will remain undecided. In this paper some facts will be ascertained concerning the set $sp(D)$ if (1) and something less than (5) are assumed. First, it is to be noted that (2) implies

$$F \equiv CA^{-1}B^{-1} = D - I \tag{6}$$

and $CB^{-1}A^{-1} = I - D^{-1}$. On using (1), it seen that

$$I - D^{-1} = A(D - I)A^{-1} \tag{7}$$

and hence

$$sp(D^{-1}) = 2 - sp(D). \tag{8}$$

Consequently $sp((I + F)^{-1}) = 1 - sp(F)$. Thus it follows that if λ belongs to F so also does $\lambda/(1 \pm \lambda)$. Successive applications of this formula lead to the result that

$$\lambda_n = \lambda/(1 + n\lambda), \quad n = 0, \pm 1, \pm 2, \dots, \tag{9}$$

belongs to $sp(F)$ whenever λ does. If $\lambda \neq 0$ belongs to $sp(F)$, that is, if $sp(D)$ contains some value other than 1, then necessarily $sp(D)$ contains an infinity of distinct values; in particular, as was noted by HERSTEIN [3], relation (4) surely holds in case A and B are finite matrices. (This result also follows from [6] of course since (5) must then hold).

A slightly different method for obtaining (9) is as follows. As a consequence of (1), relation (2) can be generalized to

$$nA^{n-1}C = A^nB - BA^n \tag{2_n}$$

for $n = 0, 1, 2, \dots$ and hence also for $n = -1, -2, \dots$; cf. HALMOS [1], p. 192.

Since A^n commutes with $A^{n-1}C$, corresponding to (7) one has

$$(7_n) \qquad I - D_n^{-1} = A^n(D_n - I)A^{-n},$$

where D_n is defined by

$$(3_n) \qquad D_n = A^nBA^{-n}B^{-1}.$$

Just as before, λ in $sp(F)$ implied $\lambda/(1\pm\lambda)$ is in $sp(F)$, it now follows that λ in $sp(F)$ implies λ_n, defined by (9), is in $sp(F)$.

If $\lambda \neq 0$, then the linear fractional transformation

$$(10) \qquad w = \lambda/(1 + z\lambda)$$

maps the real axis into the circle (or line, if λ is real) containing λ and tangent to the real axis at the origin. It is an easy consequence of this observation and the fact that λ_n of (9) is in $sp(F)$ whenever λ is, that

(i) *If* (1) *holds and if* D *is unitary, then* sp(D) $= 1$ *only, that is*, D $=$ I.

Another result is the following:

(ii) *If* (1) *holds and if* $\|CA^{-1}B^{-1}\| < 2$, *or even, if the spectral radius of* F *is less than* 2, *then* $\lambda = 1$ *is the only real point in* sp(D).

In order to prove (ii), suppose the assertion is false, so that there exists some real $\lambda \neq 0$ in $sp(F)$. It will be clear from the proof that there is no loss of generality in assuming $\lambda > 0$. Next, choose the (negative) integer $n = n(\lambda)$ so that for some δ, $0 < \delta < 1$, $-\delta\lambda = 1 + (n-1)\lambda < 0 < 1 + n\lambda = (1-\delta)\lambda$. (That $\lambda \neq -1/n$ for $n = \pm 1, \pm 2, \ldots$ follows from (9) and the fact that $sp(F)$ is a bounded set.) It is seen from (9) that $\lambda_{n-1} = -1/\delta$ and $\lambda_n = 1/(1-\delta)$ belong to $sp(F)$.

But $\lambda_n - \lambda_{n-1} \geqq 4$ and hence the spectral radius of F is not less than 2, in contradiction with the hypothesis. This completes the proof of (ii).

By condition (11_n) will be meant that for a positive integer n

the operator A possesses an n–th root, denoted by $A^{1/n}$ commuting with all operators with commute with A, thus

$$(11_n)\qquad A=(A^{1/n})^n,\quad AX=XA \Rightarrow A^{1/n}X=XA^{1/n}.$$

(It follows from [1] that an operator may have at least a finite number of n–th roots and not have a logarithm). It is easy to generalize (2_n) when (1) and (11_n) hold, for some fixed n, to the following

$$(2_t)\qquad tA^{t-1}C=A^tB-BA^t,$$

where t is a rational number with denominator n. (It is understood of course that $A^{m/n}=(A^{1/n})^m$). Corresponding to (3_n) and (7_n) one now has

$$(3_t)\qquad D_t=A^tBA^{-t}B^{-1}$$

and

$$(7_t)\qquad I-D_t^{-1}=A^t(D_t-I)A^{-t}.$$

In view of (6), relation (2^t) can be written also as

$$(12_t)\qquad tF\equiv tCA^{-1}B^{-1}=D_t-I.$$

Each of the last four formula lines holds whenever (1) and (11_n) hold with the understanding that t is a rational number with denominator n.

The following theorem is an obvious corollary of (ii) by virtue of (12_t) and (6).

(iii) *If* (1) *and* (11_n) *hold for some positive integer* n *for which the spectral radius of* F *is less than* 2n, *then* $\lambda=0[\lambda=1]$ *is the only real point in* [sp(F) sp(D)].

Next, there will be proved:

(iv) *Suppose that* (1) *holds and that* (11_{n_k}) *holds for a sequence of positive integers* $n_k\to\infty$. *Let* α *denote the spectral radius of* $F=CA^{-1}B^{-1}$. *Then if* $\alpha>0$, *the spectrum of* F *is contained in the set consisting of the two circular disks* $|z-i\alpha/2|\leqq\alpha/2$ *and* $|z+i\alpha/2|\leqq\alpha/2$. *Moreover the entire boundary of at least one of these circles is contained in* sp(F).

In order to prove (iv), note that (2_t), (3_t), (7_t) and (12_t) now hold for a dense set of rationals, namely those with denominators n_k. Corresponding to the derivation of (9) using (2_n), (3_n) and (7_n) one obtains in a similar fashion the result that

$$\lambda_t = \lambda/(1 + t\lambda) \tag{9_t}$$

belongs to $sp(F)$ whenever λ does. Here t belongs to the dense set of rationals referred to above. Since $sp(F)$ is closed it follows that λ_t of (9_t) is in $sp(F)$ for all real t. Referring again to the transformation (10) it is seen that if $\lambda \neq 0$ is in $sp(F)$ (hence, by (iii), λ cannot be real), then the image of the real axis, namely the circle containing λ and tangent to the real axis at the origin, belongs to $sp(F)$. If, in addition, λ is at the distance α (the spectral radius of F) from the origin then necessarily $\lambda = \pm i\alpha$, and the assertion of (iv) follows. This completes the proof.

REMARK. – In case condition (5) holds (as was assumed in [6]), then the proof of [6] shows essentially that (2_t), (3_t), (7_t) and (12_t) can be obtained for all complex t; relation (10) would then imply (with $t = z$) that $sp(F)$ is unbounded, a contradiction, whenever it contains a number $\lambda \neq 0$.

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