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On the absolute harmonic summability of a double series related to a double Fourier series.

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Summary. - *The author defines absolute harmonic summability for double series and studies a problem relating absolute summability factors for double* FOURIER *series by harmonic means.*

1. DEFINITION 1. - A double series $\Sigma\Sigma U_{m,n}$ with the sequence of partial sums $\{s_{m,n}\}$ is said to be summable by harmonic means or summable $(H, 1, 1)$ if the sequence

$$t_{m,n} = \frac{1}{P_m P_n} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{s_{m-l,\, n-k}}{(l+1)(k+1)}.$$

$$\left(P_n = 1 + \frac{1}{2} + \dots + \frac{1}{n+1}\right),$$

tends to a finite limits as $(m, n) \to \infty$.

This is known [6]. It may be noted that this is a particular case $p_n = \frac{1}{n+1}$ of Nörlund summability of the double sequence as defined by HERRIOT [4].

DEFINITION 2. - A double series $\Sigma\Sigma U_{m,n}$ is said to be absolutely summable by harmonic means or summable $|H, 1, 1|$, if

$$\sum_m \sum_n | t_{m,n} - t_{m,n-1} - t_{m-1,n} + t_{m-1,n-1} | < \infty,$$

$$\sum_m | t_{m,n} - t_{m-1,n} | < \infty, \quad \sum_n | t_{m,n} - t_{m,n-1} | < \infty.$$

Compare TIMAN [7].

2. Suppose that the function $f(u, v)$ is integrable in the sense of LEBESGUE over the square $(-\pi, \pi;\ -\pi, \pi)$ and is periodic with period 2π in each variable. The double FOURIER series asso-

(*) Pervenuta alla Segreteria dell'U.M.I. il 22 giugno 1961.

ciated with the function $f(u, v)$ is

$$\sum_{1}^{\infty}\sum_{1}^{\infty} A_{m,n}(u, v) \ [5].$$

We write

$$\varphi(u, v) = \frac{1}{4}[f(x+u, y+v) + f(x+u, y-v)$$

$$+ f(x-u, y+v) + f(x-u, y-v)],$$

$$K_n(u) = \sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1}\right)\frac{\cos(n-\nu)u}{(n-\nu)\log^{\varepsilon}(n-\nu+1)},$$

$$h_1(u) = \sum_{\nu=0}^{n-1}\frac{\cos(n-\nu)u}{(\nu+1)(n-\nu)\log^{\varepsilon}(n-\nu+1)},$$

$$h_2(u) = \sum_{\nu=1}^{n-1}\frac{\cos(n-\nu)u}{\nu(\nu+1)(n-\nu)\log^{\varepsilon}(n-\nu+1)}.$$

VARSHNEY [9] has proved the following theorem:

THEOREM A. – Suppose $\Sigma A_n(t)$ be the FOURIER series of the function $f(t)$ which is integrable (L) and is periodic. Let

$$\int_0^t |\varphi(u)| \, du = O(t), \quad \text{as } t \longrightarrow 0,$$

then the series $\Sigma n^{-1}\{\log(n+1)\}^{-\varepsilon} A_n(t)$, $(\varepsilon > 0)$, at $t = x$, is absolutely harmonic summable, where

$$\varphi(t) = \frac{1}{2}[f(x+t) + f(x-t)].$$

We shall prove the following theorem:

THEOREM. – If

$$\varphi(u,\ v) \equiv \int_0^u ds \int_0^v \mid \varphi(s,\ t) \mid dt = O(uv),$$

(2.1)
$$\int_0^\pi dt \mid \int_0^u \varphi(s,\ t) ds \mid = O(u),$$

$$\int_0^\pi ds \mid \int_0^v \varphi(s,\quad) dt \mid = O(v),$$

as $u \to o,\ v \to o$, then the double series

(2.2)
$$\Sigma\ \Sigma \frac{A_{m,\,n}(u,\ v)}{mn\, \{\log (m+1) \cdot \log (n+1)\}^{\varepsilon}}, \qquad (\varepsilon > 0),$$

is sumable $\mid H,\ 1,\ 1 \mid$ at $u = x$ and $v = y$.

This theorem is a generalisation of Theorem A for double FOURIER series.

3. We shall require the following lemmas.

LEMMA 1. – Uniformly for $o < t < \pi$,

$$\left| \sum_n^m \frac{\sin kt}{k} \right| \leq A,$$

where m and n are any positive integers.

This is known [8, p. 440].

LEMMA 2. – If $o < t < \pi$, then

$$\left| \sum_{k=0}^{n} \frac{\cos (k+1)t}{k+1} \right| = O\left(1 + \log \frac{1}{t}\right).$$

This is known [3].

With the help of Lemmas 1 and 2, we deduce

LEMMA 3. – If $o < t < \pi$, then for all positive integers m & m',

$$\sum_{k=m}^{m'} \frac{\overset{\sin}{\cos} (n-k)t}{k+1} = O\left(1 + \log \frac{1}{t}\right).$$

LEMMA 4. - We have

$$(i) \qquad \sum_{k=0}^{\left[\frac{n}{2}\right]-2} \left| \Delta\left(\frac{(n+1)P_n-(k+1)P_k}{(n-k)\log^{\varepsilon}(n-k+1)}\right)\right| = O(\log^{1-\varepsilon} n), \qquad \varepsilon > 0,$$

$$(ii) \qquad \sum_{k=0}^{\left[\frac{n}{2}\right]-2} \left| \Delta\left(\frac{1}{(n-k)\log^{\varepsilon}(n-k+1)}\right)\right| = O\left(\frac{1}{n\log^{\varepsilon} n}\right) \qquad \varepsilon > 0,$$

$$(iii) \qquad \sum_{k=1}^{\left[\frac{n}{2}\right]-2} \left| \Delta\left(\frac{1}{k\log^{\varepsilon}(n-k+1)}\right)\right| = O\left(\frac{1}{\log^{\varepsilon} n}\right), \qquad \varepsilon > 0.$$

PROOF. - Result (i) is known [9]. For proving (ii) we observe that

$$\sum_{k=0}^{\left[\frac{n}{2}\right]-2} \left| \Delta\left(\frac{1}{(n-k)\log^{\varepsilon}(n-k+1)}\right)\right|$$

$$= O\left[\sum_{k=0}^{[n/2]-2} \frac{1}{(n-k)(n-k-1)\log^{\varepsilon}(n-k+1)}\right]$$

$$+ O\left[\sum_{k=0}^{[n/2]-2} \frac{1}{(n-k)^2\log^{1+\varepsilon}(n-k+1)}\right]$$

$$= O\left[\frac{1}{n^2}\sum_{0}^{\left[\frac{n}{2}\right]-2} \frac{1}{\log^{\varepsilon}(n-k+1)}\right]$$

$$= O[1/n\log^{\varepsilon} n].$$

Again

$$\sum_{k=1}^{\left[\frac{n}{2}\right]-2} \left| \Delta\left(\frac{1}{k\log^{\varepsilon}(n-k+1)}\right)\right|$$

$$= O\left[\sum_{1}^{[n/2]-2} \frac{1}{k^2\log^{\varepsilon}(n-k+1)}\right] + O\left[\sum_{1}^{[n/2]-2} \frac{1}{k(n-k+1)\log^{1+\varepsilon}(n-k+1)}\right]$$

$$= O\left[\frac{1}{\log^{\varepsilon} n}\sum_{1}^{\left[\frac{n}{2}\right]-2} \frac{1}{k^2}\right] + O\left[\frac{1}{n\log^{1+\varepsilon} n}\sum_{1}^{\left[\frac{n}{2}\right]-2} \frac{1}{k}\right]$$

$$= O(1/\log^{\varepsilon} n).$$

This proves the lemma.

LEMMA 5. - For $o \leq t \leq \pi$, $\varepsilon \geq o$, we have

$$K_n(t) = O(\log^2 n/n).$$

PROOF. - Since

$$\frac{\cos(n-\nu)t}{\log^{\varepsilon}(n-\nu+1)} = O(1) \text{ as } n \to \infty \text{ and } \frac{P_n}{\nu+1} \geq \frac{P_\nu}{n+1} \text{ for } n \geq \nu,$$

we have

$$| K_n(t) | \leq \sum_{\nu=0}^{n-1} \left| \frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right| \frac{1}{(n-\nu)}$$

$$= O\left[\sum_{\nu=0}^{n-1} \frac{P_n}{(\nu+1)(n-\nu)}\right] = O\left(\frac{\log^2 n}{n}\right).$$

This proves the lemma.

Proceding in this manner we can establish the following inequalities:

$$h_1(u) = O(\log n/n),\ h_2(u) = O(\log n/n),\quad \frac{d}{du} K_n(u) = O(\log^2 n),$$

$$\frac{d}{du} h_1\ (u) = O(\log n),\quad \frac{d}{du} h_2(u) = O(\log n).$$

LEMMA 6. - If $o < u < \pi$ and for every positive $\varepsilon \neq 1$,

$$K_n(u) = O\left\{ \log\left(\frac{r}{u}\right) \Big/ n \log^{\varepsilon - 1} n \right\},$$

$$h_1(u) = O\left\{ \log\left(\frac{r}{u}\right) \Big/ n \log^{\varepsilon} n \right\},$$

$$h_2(u) = O\left\{ \log\left(\frac{r}{u}\right) \Big/ n \log^{\varepsilon} n \right\},$$

where r is some fixed constant greater than π.

PROOF. - Result (*i*) is known [9]. We have

$$h_1(u) = \sum_{\nu=0}^{n-1} \frac{\cos(n-\nu)u}{(\nu+1)(n-\nu)\log^{\varepsilon}(n-\nu+1)}$$

$$= \sum_{\nu=0}^{\left[\frac{n}{2}\right]-1} + \sum_{\left[\frac{n}{2}\right]}^{n-1} = \Sigma_1 + \Sigma_2, \text{ say,}$$

so that by ABEL's transformation,

$$\Sigma_1 = \sum_{\nu=0}^{\left[\frac{n}{2}\right]-1} \frac{1}{(n-\nu)\log^{\varepsilon}(n-\nu+1)} \frac{\cos(n-\nu)u}{\nu+1}$$

$$= \sum_{\nu=0}^{[n/2]-2} \left\{ \Delta\left(\frac{1}{(n-\nu)\log^{\varepsilon}(n-\nu+1)}\right)\right\} \left\{ \sum_{k=0}^{\nu} \frac{\cos(n-k)u}{k+1} \right\}$$

$$+ \frac{1}{\left(\left[\frac{n}{2}\right]+1\right)\log^{\varepsilon}\left(\left[\frac{n}{2}\right]+2\right)} \cdot \left\{ \sum_{\nu=0}^{[n/2]-1} \frac{\cos(n-\nu)u}{\nu+1} \right\}$$

$$= O\left[\log\left(\frac{r}{u}\right) \Big/ n \log^{\varepsilon} n\right],$$

and

$$\Sigma_2 = \sum_{[n/2]}^{n-1} \frac{\cos(n-\nu)u}{(\nu+1)(n-\nu)\log^{\varepsilon}(n-\nu+1)}$$

$$= O\left(\frac{1}{n^2}\right) \sum_{[n/2]}^{n-1} \frac{1}{\log^{\varepsilon}(n-\nu+1)}\Bigg)$$

$$= O\left(\frac{1}{n\log^{\varepsilon} n}\right).$$

This proves the result (*ii*). Again

$$h_2(u) = \sum_{\nu=1}^{\left[\frac{n}{2}\right]-1} \frac{\cos(n-\nu)u}{n\nu(\nu+1)\log^{\varepsilon}(n-\nu+1)}$$

$$+\sum_{\nu=1}^{[n/2]-1}\frac{\cos(n-\nu)u}{n(\nu+1)(n-\nu)\log^{\varepsilon}(n-\nu+1)}$$

$$+\sum_{[n/2]}^{n-1}\frac{\cos(n-\nu)u}{\nu(\nu+1)(n-\nu)\log^{\varepsilon}(n-\nu+1)}$$

$$=\Sigma'+O\left\{\log\left(\frac{r}{u}\right)\Big/ n^2\log^{\varepsilon} n\right\}+O\left\{\frac{1}{n^3}\sum_{[n/2]}^{n-1}\frac{1}{\log^{\varepsilon}(n-\nu+1)}\right\}$$

where by ABEL's transformation,

$$\Sigma'=\frac{1}{n}\sum_{\nu=1}^{[n/2]-1}\frac{1}{\nu\log^{\varepsilon}(n-\nu+1)}\cdot\frac{\cos(n-\nu)u}{\nu+1}$$

$$=\frac{1}{n}\sum_{\nu=1}^{[n/2]-2}\left\{\Delta\left(\frac{1}{\nu\log^{\varepsilon}(n-\nu+1)}\right)\right\}\left\{\sum_{k=0}^{\nu}\frac{\cos(n-k)u}{k+1}\right\}$$

$$+\frac{1}{n}\cdot\frac{1}{\left(\left[\frac{n}{2}\right]-1\right)\log^{\varepsilon}\left(\left[\frac{n}{2}\right]+2\right)}\cdot\left\{\sum_{\nu=0}^{[n/2]-1}\frac{\cos(n-\nu)u}{\nu+1}\right\}$$

$$=O\left\{\log\left(\frac{r}{u}\right)\Big/ n\log^{\varepsilon} n\right\}.$$

This proves the lemma completely.

4. PROOF OF THE THEOREM. - Let $U_{m,n}$ be the (m, n) th term of the double series (2.2), then we have

$$t_{m,n}=\frac{1}{P_mP_n}\sum_{\mu=0}^{m}\sum_{k=0}^{n}\frac{s_{m-\mu,\,n-k}}{(\mu+1)(k+1)}$$

$$=\frac{1}{P_mP_n}\left[\sum_{k=0}^{n}\frac{(m+1)^{-1}s_{0,\,n-k}+m^{-1}s_{1,\,n-k}+\dots+s_{m,\,n-k}}{k+1}\right]$$

$$=\frac{1}{P_mP_n}\left[(m+1)^{-1}\sum_{k=0}^{n}\frac{s_{0,\,n-k}}{k+1}+m^{-1}\sum_{k=0}^{n}\frac{s_{1,\,n-k}}{k+1}+\dots\dots\right.$$

$$\left.\dots\dots+\sum_{k=0}^{n}\frac{s_{m,\,n-k}}{k+1}\right]$$

$$= \frac{1}{P_m P_n} [m+1)^{-1} \{ (n+1)^{-1} s_{0,0} + n^{-1} s_{0,1} + \dots + s_{0,n} \}$$

$$+ m^{-1} \{ (n+1)^{-1} s_{1,0} + n^{-1} s_{1,1} + \dots + s_{1,n} \}$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \{ (n+1)^{-1} s_{m,0} + n^{-1} s_{m,1} + \dots + s_{m,n} \}]$$

$$= \frac{1}{P_m P_n} [(m+1)^{-1} \{ P_n U_{0,0} + P_{n-1} U_{0,1} + \dots + U_{0,n} \}$$

$$+ m^{-1} \{ P_n U_{1,0} + P_{n-1} U_{1,1} + \dots + U_{1,n} \}$$

$$+ \dots\dots\dots\dots\dots\dots \quad \dots\dots\dots\dots\dots\dots$$

$$+ \{ P_n U_{m,0} + P_{n-1} U_{m,1} + \dots + U_{m,n} \}],$$

so that

$$t_{m,n} - t_{m,n-1} = \left[\sum_{\mu=0}^{m-1} \frac{1}{(\mu+1) P_m} \cdot \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) U_{m-\mu, n-\nu} \right],$$

and

$$t_{m,n} - t_{m,n-1} - t_{m-1,n} + t_{m-1,n-1}$$

$$= \frac{1}{P_m} \cdot \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) U_{m,n-\nu}$$

$$- \left[\frac{1}{(m+1) P_m P_{m-1}} \sum_{\mu=1}^{m-1} \frac{1}{\mu+1} \cdot \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) U_{m-\mu, n-\nu} \right]$$

$$- \left[\frac{1}{P_{m-1}} \sum_{\mu=1}^{m-1} \frac{1}{\mu(\mu+1)} \cdot \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) U_{m-\mu, n-\nu} \right].$$

For the double FOURIER series of $f(u, v)$ at $u = k$ and $v = l$, we have

$$A_{m,n}(u, v) = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi(u, v) \cos nu \cos mv du dv$$

so that

$$t_{m,n} - t_{m,n-1} - t_{m-1,n} + t_{m-1,n-1}$$

$$= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi(u, v) \left\{ \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\cos (n-k)u}{(n-k) \log^\varepsilon (n-k+1} \right\}$$

$$\left\{ \frac{\cos mv}{m P_m \log^\varepsilon (m+1)} \right\} du dv$$

$$- \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi(u, v) \left\{ \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\cos (n-k)u}{(n-k) \log^\varepsilon (n-k+1)} \right\}$$

$$\left\{ \frac{1}{(m+1) P_m P_{m-1}} \sum_{l=1}^{m-1} \frac{\cos (m-l)v}{(l+1)(m-l) \log^\varepsilon (m-l+1)} \right\} du dv$$

$$- \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi(u, v) \left\{ \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left(\frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\cos (n-k)u}{(n-k) \log^\varepsilon (n-k+1)} \right\}$$

$$\left\{ \frac{1}{P_{m-1}} \sum_{l=0}^{m-1} \frac{\cos (m-l)\, v}{l(l+1)(m-l) \log^\varepsilon (m-l+1)} \right\} du dv$$

$$= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi(u, v) \frac{K_n(u) \cos mv}{m P_n P_{n-1} P_m \log^\varepsilon (m+1)} du dv$$

$$- \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi(u, v) \frac{K_n(u) h_1(v)}{P_n P_{n-1} (m+1) P_m P_{m-1}} du dv$$

$$- \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi(u, v) \frac{K_n(u) h_2(v)}{P_n P_{n-1} P_{m-1}} du dv$$

$$= \frac{4}{\pi^2} [J_1 - J_2 - J_3], \text{ say.}$$

In order to show

$$\sum_m \sum_n | t_{m,n} - t_{m,n-1} - t_{m-1,n} + t_{m-1,n-1} | < \infty,$$

we have to prove that

$$\Sigma\Sigma\,|J_1| < \infty, \quad \Sigma\Sigma\,|J_2| < \infty, \quad \Sigma\Sigma\,|J_3| < \infty. \tag{4.1}$$

Now,

$$J_1 = \frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \cdot \int_0^\pi \int_0^\pi \varphi(u, v) K_n(u) \cos mv\,du\,dv$$

$$= \frac{1}{m \log^{1+\varepsilon} m \log^2 n} \left[\int_0^{1/n} \int_0^{1/m} + \int_{1/n}^{\pi} \int_0^{1/m} + \int_0^{1/n} \int_{1/m}^{\pi} + \int_{1/n}^{\pi} \int_{1/m}^{\pi} \right]$$

$$= J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4}, \text{ say.} \tag{4.2}$$

$$| J_{1,1} | = \frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \left| \int_0^{1/n} \int_0^{1/m} \varphi(u, v) K_n(u) \cos mv\,du\,dv \right|$$

$$= O\left(\frac{1}{mn \log^{1+\varepsilon} m} \cdot \int_0^{1/n} \int_0^{1/m} | \varphi(u, v) |\, du\,dv \right)$$

$$= O\left(\frac{1}{m^2 n^2 \log^{1+\varepsilon} m} \right). \tag{4.3}$$

$$J_{1,2} = \frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \cdot \int_{1/n}^{\pi} K_n(u) du \int_0^{1/m} \varphi(u, v) \cos mv\,dv$$

$$= \frac{1}{m \log^{1+\varepsilon} m \log^2 n} \left[\int_{1/n}^{\pi} K_n(u) du \left\{ \Phi_1\left(u, \frac{1}{m}\right) \cos 1 + \int_0^{1/m} \Phi_1(u, v) m \sin mv\,dv \right\} \right]$$

$$= J_{1,2,1} + J_{1,2,2},$$

where

$$\Phi_1(u, v) = \int_0^v \varphi(u, t)dt,$$

provided this integral exists [1], so that

$$J_{1,2,1} = \frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \int_{1/n}^{\pi} K_n(u)\, \Phi_1\left(u, \frac{1}{m}\right) \cos 1du$$

$$= O\left(\frac{1}{m \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n} \cdot \int_{1/n}^{\pi} \left| \Phi_1\left(u, \frac{1}{m}\right)\right| \log \frac{r}{u}\, du\right)$$

$$= O\left(\frac{1}{m^2 \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n}\right)$$

$$J_{1,2,2} = \frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \int_{1/n}^{\pi} K_n(u)du \int_0^{1/m} \Phi_1(u, v) m \sin mv dv$$

$$= O\left(\frac{1}{m \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n} \cdot \int_0^{1/m} m \mid \sin mv \mid dv \int_{1/n}^{\pi} \mid \Phi_1(u, v) \log \frac{r}{u}\, du\right)$$

$$= O\left(\frac{1}{m \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n} \cdot \int_0^{1/m} mv \mid \sin mv \mid dv\right)$$

$$= O\left(\frac{1}{m^2 \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n}\right).$$

Thus

$$\mid J_{1,2} \mid = O\left(\frac{1}{m^2 \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n}\right). \tag{4.4}$$

Again

$$J_{1,3} = \frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \cdot \int_{1/m}^{\pi} \cos mv dv \int_0^{1/n} \varphi(u, v) K_n(u) du$$

$$= \frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \cdot \left[\int_{1/m}^{\pi} \cos mv dv \left\{ \Phi_1\left(\frac{1}{n}, v\right) K_n\left(\frac{1}{n}\right) - \int_0^{1/n} \Phi_1(u, v) K'_n(u) du \right\} \right]$$

$$= J_{1,3,1} + J_{1,3,2}.$$

Now

$$J_{1,3,1} = O\left(\frac{\log (rn)}{m \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n} \cdot \int_{1/m}^{\pi} \left| \Phi_1\left(\frac{1}{n}, v\right) \right| dv\right)$$

$$= O\left(\frac{\log (rn)}{m \log^{1+\varepsilon} m \cdot n^2 \log^{1+\varepsilon} n}\right)$$

$$= O\left(\frac{1}{m \log^{1+\varepsilon} m \cdot n^2 \log^{\varepsilon} n}\right),$$

and

$$J_{1,3,2} = O\left(\frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \cdot \int_0^{1/n} | K'_n(u) | du \cdot \int_{1/m}^{\pi} | \Phi_1(u, v) | dv\right)$$

$$= O\left(\frac{1}{m \log^{1+\varepsilon} m \cdot \log^2 n} \cdot \int_0^{1/n} u | K'_n(u) | du\right)$$

$$= O\left(\frac{1}{m \log^{1+\varepsilon} m} \cdot \int_0^{1/n} u du\right)$$

$$= O\left(\frac{1}{n^2 m \log^{1+\varepsilon} m}\right),$$

therefore

(4.5) $$| J_{1,3} | = O\left(\frac{1}{n^2 m \log^{1+\varepsilon} m}\right).$$

Also,

$$J_{1,4} = \frac{1}{m \log^{1+\varepsilon} m \log^2 n} \cdot \int_{1/n}^{\pi} \int_{1/m}^{\pi} \varphi(u, v) K_n(u) \cos mv\, du\, dv$$

$$= O\left(\frac{1}{m \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n} \cdot \int_{\frac{1}{n}}^{\pi} \int_{\frac{1}{m}}^{\pi} | \varphi(u, v) | \log \frac{r}{u}\, du\, dv\right)$$

(4.6) $$= O\left(\frac{1}{m \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n}\right),$$

since by partial integration for double integral [2; 10] we observe that

$$\int_{\frac{1}{n}}^{\pi} \int_{\frac{1}{m}}^{\pi} | \varphi(u, v) | \log \frac{r}{u}\, du\, dv$$

is bounded at the points at which conditions (2.1) hold.

Combining (4.2), (4.3), (4.4), (4.5), and (4.6), we have

$$\Sigma \Sigma\, | J_1 | = O\left(\Sigma \Sigma \frac{1}{n^2 m^2 \log^{1+\varepsilon} m}\right) + O\left(\Sigma \Sigma \frac{1}{n^2 m \log^{1+\varepsilon} m}\right)$$

$$+ O\left(\Sigma \Sigma \frac{1}{m^2 \log^{1+\varepsilon} m \cdot n \log^{1+\varepsilon} n}\right)$$

(4.7) $$= O(1).$$

Now let us consider J_2.

$$J_2 = \frac{1}{m\log^2 m\log^2 n}\left\{\int_0^{1/n}\int_0^{1/m} + \int_{1/n}^{\pi}\int_0^{1/m} + \int_0^{1/n}\int_{1/m}^{\pi} + \int_{\frac{1}{n}}^{\pi}\int_{\frac{1}{m}}^{\pi}\right\}\varphi(u, v)K_n(u)h_1(v)\,du\,dv$$

(4.8) $$= J_{2,1} + J_{2,2} + J_{2,3} + J_{2,4}, \text{ say.}$$

$$J_{2,1} = \frac{1}{m\log^2 m\log^2 n}\int_0^{1/n}\int_0^{1/m}\varphi(u, v)K_n(u)h_1(v)\,du\,dv$$

$$= O\left(\frac{1}{nm^2\log m}\cdot\int_0^{1/n}\int_0^{1/m}|\varphi(u, v)|\,du\,dv\right)$$

(4.9) $$= O\left(\frac{1}{n^2m^3\log m}\right).$$

By partial integration as in $J_{1,2}$, we have

$$J_{2,2} = \frac{1}{m\log^2 m\log^2 n}\left[\int_{1/n}^{\pi} K_n(u)du \int_0^{1/m}\varphi(u, v)h_1(v)dv\right]$$

$$= \frac{1}{m\log^2 m\log^2 n}\left[\int_{\frac{1}{n}}^{\pi} K_n(u)du\left\{\Phi_1\left(u, \frac{1}{m}\right)h_1\left(\frac{1}{m}\right) - \int_0^{1/m}\Phi_1(u, v)h'_1(v)dv\right\}\right]$$

$$= J_{2,2,1} + J_{2,2,2}, \text{ say.}$$

$$J_{2,2,1} = O\left(\frac{\log(rm)}{n\log^{1+\varepsilon} n\cdot m^2\log^{2+\varepsilon} m}\cdot\int_{\frac{1}{n}}^{\pi}\left|\Phi_1\left(u, \frac{1}{m}\right)\right|\log\frac{r}{u}\,du\right)$$

$$= O\left(\frac{1}{n\log^{1+\varepsilon} n\cdot m^3\log^{1+\varepsilon} m}\right).$$

$$J_{2,2,2} = O\left(\frac{1}{m \log^2 m \cdot n \log^{1+\varepsilon} n} \cdot \int_0^{\frac{1}{m}} |h'_1(v)|\, dv \cdot \int_{\frac{1}{n}}^{\pi} |\Phi_1(u, v)| \log \frac{r}{u}\, du\right)$$

$$= O\left(\frac{1}{m \log^2 m \quad n \log^{1+\varepsilon} n} \cdot \int_0^{\frac{1}{m}} v \mid h'_1(v) \mid dv\right)$$

$$= O\left(\frac{1}{m \log m \cdot n \log^{1+\varepsilon} n} \cdot \int_0^{\frac{1}{m}} v dv\right)$$

$$= O\left(\frac{1}{m^3 \log m \cdot n \log^{1+\varepsilon} n}\right),$$

so that

$$(4.10) \quad J_{2,2} = O\left(\frac{1}{m^3 \log m \cdot n \log^{1+\varepsilon} n}\right).$$

Also

$$J_{2,3} = \frac{1}{m \log^2 m \log^2 n}\left[\int_{\frac{1}{m}}^{\pi} h_1(v) dv \int_0^{\frac{1}{n}} \varphi(u, v) K_n(u)\, du\right]$$

$$= \frac{1}{m \log^2 m \log^2 n}\left[\int_{\frac{1}{m}}^{\pi} h_1(v) dv \left\{ \Phi_1\left(\frac{1}{n}, v\right) K_n\left(\frac{1}{n}\right)\right.\right.$$

$$\left.\left. - \int_0^{\frac{1}{n}} \Phi_1(u, v) K'_n(u) du \right\}\right]$$

$$= J_{2,3,1} + J_{2,3,2}, \text{ say.}$$

$$J_{2,3,1}=O\left(\frac{\log(rn)}{m^2\log^{2+\varepsilon} m\cdot n\log^{1+\varepsilon} n}\cdot\int_{\frac{1}{m}}^{\pi}\left|\Phi_1\left(\frac{1}{n},v\right)\right|\log\left(\frac{r}{v}\right)dv\right)$$

$$=O\left(\frac{1}{m^2\log^{2+\varepsilon} m\cdot n^2\log^{\varepsilon} n}\right).$$

$$J_{2,3,2}=O\left(\frac{1}{m^2\log^{2+\varepsilon} m\cdot\log^2 n}\cdot\int_0 |K'_n(u)|\,du\int_{\frac{1}{m}}^{\pi}|\Phi_1(u,v)|\log\frac{r}{v}\,dv\right)$$

$$=O\left(\frac{1}{m^2\log^{2+\varepsilon} m\cdot\log^2 n}\cdot\int_0^{\frac{1}{n}} u\,|K'_n(u)|\,du\right)$$

$$=O\left(\frac{1}{m^2\log^{2+\varepsilon} m\cdot n^2}\right).$$

Thus

$$J_{2,3}=O\left(\frac{1}{m^2\log^{2+\varepsilon} m\cdot n^2}\right) \tag{4.11}$$

$$J_{2,4}=O\left(\frac{1}{n\log^{1+\varepsilon} n\cdot m^2\log^{2+\varepsilon} m}\cdot\int_{\frac{1}{n}}^{\pi}\int_{\frac{1}{m}}^{\pi}|\varphi(u,v)|\log\frac{r}{u}\log\frac{r}{v}\,du\,dv\right)$$

$$=O\left(\frac{1}{n\log^{1+\varepsilon} n\cdot m^2\log^{2+\varepsilon} m}\right), \tag{4.12}$$

since the integral

$$\int_{\frac{1}{n}}^{\pi}\int_{\frac{1}{m}}^{\pi}|\varphi(u,v)|\log\left(\frac{r}{u}\right)\log\left(\frac{r}{v}\right)du\,dv$$

is bounded.

With the help of (4.8), (4.9), (4.10), (4.11), and (4.12) we obtain

$$\Sigma\Sigma \mid J_2 \mid < \infty. \tag{4.13}$$

Similarly,

$$\Sigma\Sigma \mid J_3 \mid < \infty. \tag{4.14}$$

Collecting (4.1), (4.7), (4.13) and (4.14), we have

$$\Sigma\Sigma \mid t_{m,n} - t_{m,n-1} - t_{m-1,n} + t_{m-1,n-1} \mid < \infty.$$

Similarly it can be shown that

$$\Sigma \mid t_{m,n} - t_{m-1,n} \mid < \infty, \quad \Sigma \mid t_{m,n} - t_{m,n-1} \mid < \infty.$$

This completes the proof of the theorem.

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