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Some integral equations satisfied by the complete elliptic integrals of the first and second kind.

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Some integral equations satisfied by the complete elliptic integrals of the first and second kind.

by LEONARD CARLITZ (a Durham, U. S. A.) (*)

Summary. - *It is shown that the complete elliptic integrals* K(k), E(k) *satisfy* (9) *and* (10) *below.*

WATSON [2] showed the bilinear generating function

$$A(x,\ y,\ t) = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) P_n(x) P_n(y) t^n \tag{1}$$

can be expressed in terms of the complete elliptic integrals $K(k)$, $E(k)$, where the modulus k is a rather complicated function of x, y, t. Since

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{\delta_{mn}}{n + \frac{1}{2}}$$

it follows at once from (1) that $A(x,\ y,\ t)$ satisfies the integral equation

$$\int_{-1}^{1} A(x,\ y,\ t) A(x,\ z,\ u) dx = A(y,\ z,\ tu). \tag{2}$$

Hence WATSON'S result implies the existence of a certain integral equation containing K and E.

In order to simplify this relation it is first necessary to simplify WATSON'S result. MAXIMON [1] has proved the formula

$$\sum_{n=0}^{\infty} P_n(\cos\alpha) P_n(\cos\beta) t^n \tag{3}$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 12 Giugno 1961.

$$=(1-2t\cos(\alpha+\beta)+t^2)^{-\frac{1}{2}}F\left[\frac{1}{2},\ \frac{1}{2};\ 1;\ \frac{4t\sin\alpha\sin\beta}{1-2t\cos(\alpha+\beta)+t^2}\right]$$

$$=\frac{2}{\pi}(1-2t\cos(\alpha-\beta)+t^2)^{-\frac{1}{2}}K(\mathrm{k}(\alpha,\ \beta,\ t)),$$

where

(4) $$k=k(\alpha,\ \beta,\ t)=\left\{\frac{4t\sin\alpha\sin\beta}{1-2t\cos(\alpha+\beta)+t^2}\right\}^{\frac{1}{2}}.$$

It will also be convenient to put

(5) $$T=T(\alpha,\ \beta,\ t)=(1-2t\cos(\alpha+\beta)+t^2)^{\frac{1}{2}}.$$

Clearly

$$A(\cos\alpha,\ \cos\beta,\ t)=t^{\frac{1}{2}}\frac{\partial}{\partial t}\left\{t^{\frac{1}{2}}\sum_{n=0}^{\infty}P_n(\cos\alpha)P_n(\cos\beta)t^n\right\}$$

$$=t^{\frac{1}{2}}\left(\frac{t^{\frac{1}{2}}K(k)}{T}\right),$$

where k and T are defined by (4) and (5). Now

$$t^{\frac{1}{2}}\frac{\partial}{\partial t}\left(\frac{t^{\frac{1}{2}}K(k)}{T}\right)=\frac{T\left(t\frac{\partial K}{\partial t}+\frac{1}{2}K\right)-tK\frac{\partial T}{\partial t}}{T^2}.$$

It is easily verified that

$$\frac{\partial k^2}{\partial t}=\frac{(1-t^2)k^2}{tT^2},$$

also

$$\frac{\partial T}{\partial t}=-\frac{\cos(\alpha+\beta)-t}{t}.$$

It follows that

$$t^{\frac{1}{2}}\frac{\partial}{\partial t}\left(\frac{t^{\frac{1}{2}}K(k)}{T}\right)=\frac{(1-t^2)k^2}{T^3}\frac{dK}{d(K^2)}+\frac{(1-t^2)K}{2T^3}$$

and therefore

$$A(\cos\alpha,\ \cos\beta,\ t) = \frac{2}{\pi}\,\frac{1-t^2}{T^3}\left(k^2\frac{dK}{d(k^2)} + \frac{1}{2}K\right). \tag{6}$$

Since

$$\frac{dK}{dk} = \frac{E - k'^2K}{kk'^2}, \quad \frac{dK}{d(k^2)} = \frac{E - k'^2K}{2k^2k'^2},$$

it follows that

$$k^2\frac{dK}{d(k^2)} + \frac{1}{2}K = \frac{E}{2k'^2}.$$

Thus (6) becomes

$$A(\cos\alpha,\ \cos\beta,\ t) = \frac{1-t^2}{\pi T^3}\,\frac{E(k)}{k'^2}. \tag{7}$$

WATSON's result

$$A(\cos\alpha,\ \cos\beta,\ t) = \frac{2E(k_1) - k_1'^2K(k_1)}{\pi k_1'^4}(1-t^2)\,\cdot$$

$$\cdot\left[\frac{1}{2}(1-2t\cos(\alpha+\beta)+t^2) + \frac{1}{2}(1-2t\cos(\alpha-\beta)+t^2)^{\frac{1}{2}}\right]^{-3},$$

where

$$k_1 = \frac{(1-2t\cos(\alpha+\beta)+t^2)^{\frac{1}{2}} - (1-2t\cos(\alpha-\beta)+t^2)^{\frac{1}{2}}}{(1-2t\cos(\alpha+\beta)+t^2)^{\frac{1}{2}} + (1-2t\cos(\alpha-\beta)+t^2)^{\frac{1}{2}}}$$

can be reconciled with (7) by making use of the transformation formula

$$E\left(\frac{2k^{\frac{1}{2}}}{1+k}\right) = \frac{1}{1+k}(2E(k) - k'^2K(k)). \tag{8}$$

Indeed it is easily verified that if

$$k^2 = \frac{4k_1}{(1+k_1)^2}$$

then

$$k'^2 = \left(\frac{1-k_1}{1+k_1}\right)^2 = \frac{1-2t\cos(\alpha-\beta)+t^2}{1-2t\cos(\alpha+\beta)+t^2},$$

so that

$$k^2 = \frac{4t\sin\alpha\sin\beta}{1-2t\cos(\alpha+\beta)+t^2}$$

in agrement with (4).

Substituting from (7) in (2) we get

$$(9) \qquad \int_0^\pi \frac{E(k(\alpha,\Phi,t))E(k(\beta,\Phi,u))}{k'^2(\alpha,\Phi,t)k'^2(\beta,\Phi,u)}$$

$$\cdot \frac{\sin\Phi\, d\Phi}{(1-2t\cos(\alpha+\Phi)+t^2)^{3/2}(1-2u\cos(\beta+\Phi)+u^2)^{3/2}}$$

$$= \frac{\pi(1-t^2u^2)}{(1-t^2)(1-u^2)}\,\frac{E(k(\alpha,\beta,tu))}{k'^2(\alpha,\beta,tu)(1-2tu\cos(\alpha+\beta)+t^2u^2)^{3/2}}.$$

Moreover if we put

$$B(x,y,t) = \sum_{n=0}^{\infty} P_n(x)P_n(y)t^n,$$

then it follows from (3) and (7) that

$$(10) \qquad \int_0^\pi K(k(\alpha,\beta,t))\,\frac{E(k(\beta,\Phi,u))}{k'^2(\beta,\Phi,u)}$$

$$\cdot \frac{\sin\Phi\, d\Phi}{(1-2t\cos(\alpha+\Phi)+t^2)^{\frac{1}{2}}(1-2u\cos(\beta+\Phi)+u^2)^{3/2}}$$

$$= \frac{\pi}{1-u^2}\,\frac{K(k(\alpha,\beta,tu))}{(1-2tu\cos(\alpha+\beta)+t^2u^2)^{\frac{1}{2}}}.$$

It may be of interest to note in connection with (7) the explicit formula:

$$\frac{E(k)}{k'^2} = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!\, n!} k^{2n}$$

$$= \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{n!\, n!}.$$

We note also that if $\cos\beta = t\cos\alpha$, (4) reduces to

$$k = \frac{2\sqrt{k_1}}{1 + k_1},$$

where

$$k_1 = \frac{t \sin\alpha}{\sqrt{1 - t^2 \cos^2}}.$$

MAKING use of (8) we find that (7) reduces to

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) P_n(x) P_n(xt) t^n = \frac{\sqrt{1 - x^2t^2}}{\pi(1 - t^2)} (2E(k) - k'^2 K(k)), \tag{11}$$

where the modulus now is

$$\frac{t\sqrt{1 - x^2}}{\sqrt{1 - x^2t^2}}.$$

We may compare (11) with the formula of GERONIMUS

$$\sum_{n=0}^{\infty} P_n(x) P_n(xt) t^n = \frac{2K}{\pi\sqrt{1 - x^2t^2}}$$

which is quoted by WATSON.

REFERENCES

[1] G. MAXIMON, *A generating function for the product of two Legendre polynomials*, « Norske Videnskabers Selskab Forkandlinger », vol. 20 (1957), pp. 82-86.

[2] G. N. WATSON, *Notes on generating functions of polynomials: (3) Polynomials of Legendre and Gegenbaur*, « Journal of the London Mathematical Society », vol. 8(1933), pp. 289-292.