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A proof and extension of Brouwer's fixed point theorem for the closed 2-cell.

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Summary. - *The main result in this paper is Corollary* 3. *according to which* if a continuous map f of a closed 2-cell E into Euclidean plane $R^2 \supset E$ maps the boundary of E into E then f leaves at least one point fixed.

A proof is given here (2) of the following extension of BROUWER'S fixed point theorem for a closed circular disc. No use is made of the formal techniques of Topology. The results in this paper will later be extended and generalized in various ways.

THEOREM. - *Let* Z *be a closed circular disc with circumference* C *in a Euclidean plane* R^2 *in which a positive sense for measurement of angles has been assigned, and* f *a continuous map of* Z *into* R^2 *which leaves no point of* C *fixed. If there exists a point* z *inside* C *and a constant angle* α *such that for no point* $c \in C$ *is* α *an angle from the vector* $\overline{c, f(c)}$ *to the vector* $\overline{z, c}$ *then* f *leaves at least one point fixed.*

It is clear that BROUWER'S theorem for the closed 2-cell, which is equivalent to the assertion that a continuous map of Z into itself has a fixed point, is an easy consequence of the theorem. In fact, it is enough to take $\alpha = 0$ and let z be any point inside C. Also, we note that in the above theorem, f does not necessarily map. Z into Z as it is required in the classical case.

Before proving the theorem, we state the following special cases.

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COROLLARY 1. – *Let* Z *be a closed circular disc in a Euclidean plane* R^2, *with center* O *and circumference* C, *and* f *a continuous map of* Z *into* R^2 *which leaves no point of* C *fixed and such that*

i) *for no point* $c \in C$ *is the direction from* c *to* f(c) *the same as the direction from* O *to* c;

or

ii) *for no point* $c \in C$ *is the direction from* c *to* f(c) *the same as the direction from* c *to* O.

Then f *leaves at least one point fixed.*

The corollary is obtained from the theorem by taking z at O and taking $\alpha = 0$ in Case *i*) and $\alpha = \pi$ radians in Case *ii*).

An immediate consequence of Corollary 1 with hypothesis *i*) is

COROLLARY 2. – *If a continuous map* f *of the closed circular disc* Z *into* $R^2 \supset Z$ *maps the circumference* C *of* Z *into* Z, *then* f *leaves at least one point fixed.*

By virtue of the SCHOENFLIES theorem, modified to apply to a JORDAN curve and its exterior, Corollary 2 implies the following result, which has weaker hypotheses than the classical BROUWER fixed point theorem.

COROLLARY 3. – *If a continuous map* f *of a closed* 2-*cell* E *into* $R^2 \supset E$ *maps the boundary of* E *into* E, *then* f *leaves at least one point fixed.*

PROOF OF THE THEOREM. – In what follows, a given fixed directed axis X as initial direction is assumed for measurement of angles in R^2. Also, an angle and its radian measure will be denoted by the same symbol. The parameters t, s are real and range over the closed interval [0,1]. A *continuous vector* shall mean a continuous vector function of t in R^2. Continuous vectors are denoted here by $U(t)$, $V(t)$, $\Phi(t, s)$, etc.

Let U(t) be a continuous vector with length $|U(t)| \neq 0$. If $\widehat{U(t)}$ denotes an angle from the X direction to the direction of $U(t)$. such that $0 \leq \widehat{U(0)} < 2\pi$ and $\widehat{U(t)}$ is continuous, it is clear that $\widehat{U(t)}$ is thus uniquely determined, single-valued and continuous. The notation $\widehat{U(t)}$ will henceforth be used only when $|U(t)| \neq 0$, $U(t)$ is continuous, and with the stated conventions on continuity of $U(t)$ and value of $\widehat{U(0)}$.

LEMMA. – *Let* α *be a real constant and* U(t), V(t) *two continuous vectors, with* $|U(t)| \neq 0$, $|V(t)| \neq 0$, *such that*

$$\widehat{U(1)} = \widehat{U(0)} + 2m\pi, \qquad \widehat{V(1)} = \widehat{V(0)} + 2n\pi, \qquad (m, n \text{ integers}) \tag{1}$$

and, for every integer k and every $t \in [0,1]$,

$$\widehat{V(t)} - \widehat{U(t)} \neq \alpha + 2k\pi. \tag{2}$$

Then

$$\widehat{U(1)} - \widehat{U(0)} = \widehat{V(1)} - \widehat{V(0)}. \tag{3}$$

PROOF. – By (1)

$$\widehat{V(1)} - \widehat{U(1)} = \widehat{V(0)} - \widehat{U(0)} + 2(n - m)\pi. \tag{4}$$

Hence, if $m \neq n$, we see that

$$\underset{t}{\text{maximum}}\,[\widehat{V(t)} - \widehat{U(t)}] - \underset{t}{\text{minimum}}\,[\widehat{V(t)} - \widehat{U(t)}] \geq 2\pi.$$

Conseqcently, for some integer k and some t, the continuous function $\widehat{V(t)} - \widehat{U(t)}$ must assume the value $\alpha + 2k\pi$, contrary to (2). Therefore $m = n$. and (4) implies (3).

Continuing with the proof of the theorem, suppose now that f leaves no point of Z fixed. As t varies from 0 to 1, let the point $c(t)$ describe C once at a uniform rate in the positive sense. so that $c(0) = c(1)$. Then. from the hypotheses of the theorem we see easily that the two vectors $\overline{c(t), f(c(t))} \equiv F(t)$ and $\overline{z, c(t)} \equiv G(t)$ satisfy the hypotheses of the lemma, and therefore can be taken respectively as the vectors $U(t)$, $V(t)$ of the lemma. But obviously, $\widehat{G(1)} - \widehat{G(0)} = 2\pi$. Hence, by (3), we must also have

$$\widehat{F(1)} - \widehat{F(0)} = 2\pi. \tag{5}$$

Since there is no fixed point, there exists a circumference $C_1 \subset Z$ with center at z, so small that C_1 and $f(C_1)$ are contained in different half-planes into which R^2 is separated by some straight line. For $t \in [0, 1]$, let $L(t)$ be the constant vector of lenght 1 in either of of the two directions on that straight line. Let the line segment joining z to $c(t)$ intersect C_1 at the point $c_1(t)$. Moreover, as s varies from 0 to 1, let the point $c(t, s)$ traverse the line segment joining $c(t)$ to $c_1(t)$ at a uniform rate so that $c(t, 0) \equiv$

$\equiv c(t)$ and $c(t, 1) \equiv c_1(t)$. This determines a deformation on Z of C into C_1.

For fixed s, the vector $\overline{c(t, s), f(c(t, s))} \equiv \Phi(t, s)$ is a continuous vector with lenght $\neq 0$. Furthermore, it is clear that $c(0, s) = c(1, s)$. Hence

$$\widehat{\Phi(1, s)} - \widehat{\Phi(0, s)} = 2k(s)\pi, \tag{6}$$

where $k(s)$ is an integer-valued function of s. Also, it is obvious that $\Phi(t, 0) \equiv F(t)$, so that by (5) we have

$$\widehat{\Phi(1, 0)} - \widehat{\Phi(0, 0)} = 2\pi. \tag{7}$$

Now, for $s_1, s_2, t \in [0, 1]$, let $A(s_1, s_2, t)$ be the smallest non-negative angle formed by $\Phi(t, s_1)$ and $\Phi(t, s_2)$. Since f is continuous and leaves no point fixed, we infer that A is continuous, hence uniformly continuous, in s_1, s_2, t. Therefore, given $\varepsilon > 0$, there corresponds $\delta > 0$ such that if $|s_1 - s_2| < \delta$, then

$$|A(s_1, s_2, t) - A(s_1, s_1, t)| < \varepsilon.$$

Taking $\varepsilon \leq \pi$ and noting that $A(s_1, s_1, t) = 0$, we have $A(s_1, s_2, t) < \pi$. Hence, in view of (6), $U(t) = \Phi(t, s_1)$, $V(t) = \Phi(t, s_2)$ satisfy the hypotheses of the lemma with $\alpha = \pi$. Therefore, by (3),

$$\widehat{\Phi(1, s_1)} - \widehat{\Phi(0, s_1)} = \widehat{\Phi(1, s_2)} - \widehat{\Phi(0, s_2)},$$

from which we conclude that $\widehat{\Phi(1, s)} - \widehat{\Phi(0, s)}$ is constant. From (7), we see that the constant value is 2π, and hence, taking $s = 1$, we have

$$\widehat{\Phi(1, 1)} - \widehat{\Phi(0, 1)} = 2\pi. \tag{8}$$

On the other hand, since C_1 and $f(C_1)$ are separated by a line parallel to the constant vector $L(t)$, the continuous vector $\Phi(t, 1) \equiv \overline{c(t, 1), f(c(t, 1))}$ never has the same direction as $L(t)$. Hence, in view of (8) and the constantness of the vector $L(t)$, the lemma with $\alpha = 0$ is applicable to the two continuous vectors $\Phi(t, 1)$ and $L(t)$, yielding

$$\widehat{\Phi(1, 1)} - \widehat{\Phi(0, 1)} = \widehat{L(1)} - \widehat{L(0)} = 0,$$

which contradicts (8). Thus, f has at least one fixed point. and the theorem is proved.