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On the absolute negative summability of double Fourier series.

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Summary. - *The author studies the absolute negative summability of double Fourier series from the known result of absolute convergence by using a Tauberian type result in double series.*

1. Definition [2]. - A double series $\Sigma\Sigma a_{m,n}$ is said to be absolutely summable (c, α, β) or summable $| C, \alpha, \beta |$, $(\alpha, \beta) > -1$, when

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} | \sigma_{m,n}^{\alpha,\beta} - \sigma_{m-1,n}^{\alpha,\beta} - \sigma_{m,n-1}^{\alpha,\beta} + \sigma_{m-1,n-1}^{\alpha,\beta} |$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\tau_{m,n}^{\alpha\,\beta}|}{mn} < \infty,$$

$$\sum_{m=1}^{\infty} | \sigma_{m,0}^{\alpha,\beta} - \sigma_{m-1,0}^{\alpha,\beta} | = \sum_{m=1}^{\infty} \frac{|\tau_m^{\alpha}|}{m} < \infty,$$

$$\sum_{n=1}^{\infty} | \sigma_{0,n}^{\alpha,\beta} - \sigma_{0,n-1}^{\alpha,\beta} | = \sum_{n=1}^{\infty} \frac{|\tau_n^{\beta}|}{n} < \infty,$$

where $\sigma_{m,n}^{\alpha,\beta}$ and $\tau_{m,n}^{\alpha,\beta}$ is the (c, α, β)-mean of $a_{m,n}$ and $mn\ a_{m,n}$ respectively, i.e.,

$$\tau_{m,n}^{\alpha,\beta} = (A_m^{\alpha} A_n^{\beta})^{-1} \sum_{k=0}^{n} \sum_{l=0}^{m} A_{m-l}^{\alpha-1} A_{n-k}^{\beta-1} \cdot kla_{l,k};$$

and

$$\tau_m^{\alpha} = (A_m^{\alpha})^{-1} \sum_{l=0}^{m} A_{m-l}^{\alpha-1} la_{l,0},$$

$$\sum_{0}^{\infty} A_m^{\alpha} x^m = (1 - x)^{-\alpha-1} \quad \text{for} \quad |x| < 1.$$

Let the function $f(x, y)$ be integrable in tha Lebesgue sense over the square $Q(-\pi, -\pi; \pi, \pi)$ is doubly periodic with period 2π in each variable. The double Fourier series associated with

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the function $f(x, y)$ in the complex form is

$$(1.1) \qquad \sum_{-\infty, -\infty}^{\infty, \infty} C_{m,n} e^{i(mx+ny)}$$

where

$$4C_{m,n} = \frac{1}{\pi^2} \iint_Q f(u, v) e^{i(mu+nv)} du dv.$$

We shall prove the following theorem:

THEOREM. - If $f(x, y)$ satisfies the conditions

$$| f(x_2, y) - f(x_1, y) | \leq K_1(y) | x_2 - x_1 |^{\alpha_1}$$
$$| f(x, y_2) - f(x, y_1) | \leq K_2(x) | y_2 - y_1 |^{\beta_1}$$
$$| f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1) + f(x_1, y_1) |$$
$$\leq K | x_2 - x_1 |^{\alpha} | y_2 - y_1 |^{\beta},$$

where α, β, α_1, β_1, are positive numbers less than 1, K is constant, $K_1(y)$, $K_2(y)$ are integrable fnnctions. Also the variation of f over $0 \leq x \leq 2\pi$, is an integrable function of y and conversly. Finally suppose f is of bounded variation in the sense CARATHEODORY [1] then the double FOURIER series (1.1) of $f(x, y)$ is

$$\left| C, -\frac{\delta}{2}, -\frac{\nu}{2} \right|$$

summable where $\delta = \min(\alpha, \alpha_1)$, and $\nu \min(\beta, \beta_1)$

2. We require the following lemma.

LEMMA. - If $f(x, y)$ satisfies the condition of the theorem then

$$\sum | C_{m,n} | (| m | + 1)^{\delta/2} (| n | + 1)^{\nu/2} < \infty.$$

This is known [3]

First we prove the theorem of Tauberian type for double series:

THEOREM A. - If $\sum_1^\infty \sum_1^\infty | U_{m,n} |$ converges, then

$\sum_1^\infty \sum_1^\infty m^{-\alpha} n^{-\beta} U_{m,n}$; $0 < \alpha < 1$, $0 < \beta < 1$, is summable $| C, -\alpha, -\beta |$.

PROF. - We denote by $\sigma_{m,n}^{\alpha,\beta}$ the (m, n) th CESARO mean of order $(\alpha, \beta, > -1)$ of the series

$$\sum_{m,n=1}^{\infty} x_{m,n}$$

Then

$$x_{m,n}^{-\alpha,-\beta} = \sigma_{m,n}^{-\alpha,-\beta} - \sigma_{m-1,n}^{-\alpha,-\beta} - \sigma_{m,n-1}^{-\alpha,-\beta} + \sigma_{m-1,n-1}^{-\alpha,-\beta}$$

$$= \frac{1}{mnA_m^{-\alpha}A_n^{-\beta}} \sum_{k=1}^{n} \sum_{l=1}^{m} A_{m-l}^{-\alpha-1} A_{n-k}^{-\beta-1} lk x_{l,k}$$

Putting $x_{m,n} = \dfrac{U_{m,n}}{A_m^{\alpha} A_n^{\beta}}$ and applying ABEL's transformation for double series, we have,

$$|x_{m,n}^{-\alpha,-\beta}| \leq \frac{1}{mnA_m^{-\alpha}A_n^{-\beta}} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} A_{m-l}^{-\alpha-1} A_{n-k}^{-\beta-1} lk \frac{|U_{l,k}|}{A_l^{\alpha} A_k^{\beta}}$$

$$+ \frac{1}{mA_m^{-\alpha}} \sum_{l=1}^{m-1} A_{m-l}^{-\alpha-1} l \frac{|U_{l,n}|}{A_l^{\alpha}}$$

$$+ \frac{1}{nA_n^{-\beta}} \sum_{k=1}^{n-1} A_{n-k}^{-\alpha-1} k \frac{|U_{m,k}|}{A_k^{\beta}} + 2\,|U_{m,n}|$$

$$\sum_{m=2}^{M+1} \sum_{n=2}^{N+1} |x_{m,n}^{-\alpha,-\beta}| \leq \sum_{p=1}^{N} \frac{-1}{(p+1)A_{p+1}^{-\beta}} \sum_{q=1}^{M} \frac{-1}{(q+1)A_{q+1}^{-\alpha}}$$

$$+ \sum_{k=1}^{p} \sum_{l=1}^{q} A_{q-l+1}^{-\alpha-1} A_{p-k+1}^{-\beta-1} \frac{kl\,|U_{l,k}|}{A_l^{\alpha} A_k^{\beta}}$$

$$+ \sum_{q=1}^{M} \frac{-1}{(q+1)A_{q+1}^{-\alpha}} \sum_{l=1}^{q} A_{q-l+1}^{-\alpha-1} \frac{l\,|U_{l,n}|}{A_l^{\alpha}},$$

$$+ \sum_{p=1}^{N} \frac{-1}{(p+1)A_{p+1}^{-\beta}} \sum_{k=1}^{p} A_{p-k+1}^{-\beta-1} \frac{k\,|U_{m,k}|}{A_k^{\beta}}$$

$$+ \sum_{n=2}^{N} \sum_{m=2}^{M} |U_{m,n}|$$

The first summation on the R.H.S. is

$$\leq \sum_{k=1}^{N} \sum_{l=1}^{M} \frac{lk\,|U_{l,k}|}{A_l^{\alpha} A_k^{\beta}} \sum_{p=k}^{N} \sum_{q=l}^{M} \frac{A_{p-k+1}^{-\beta-1} A_{q-l+1}^{-\alpha-1}}{(p+1)(q+1) A_{p+1}^{-\beta} A_{q+1}^{-\alpha}}$$

where

$$\sum_{p=k}^{N} \frac{-A_{p-k+1}^{-\beta-1}}{(p+1) A_{(p+1)}^{-\beta}} = \sum_{L=1}^{N-k+1} \frac{-A_L^{-\beta-1}}{(k+L) A_{k+L}^{-\beta}}$$

$$\leq \frac{1}{k^{1-\beta}} \sum_{L=1}^{N-k+1} (-A_L^{-\beta-1}) \leq \frac{1}{k^{1-\beta}} \sum_{L=1}^{\infty} -A_L^{-\beta-1}$$

since $(1-x)^{\beta} = \sum_{L=0}^{\infty} A_L^{-\beta-1} x^L$ and then $\sum_{L=0}^{\infty} A_L^{-\beta-1} = 0$, we have

$$\sum_{m=2}^{M-1} \sum_{n=2}^{N-1} |x_{m,n}^{-\alpha,-\beta}| \leq \sum_{l=1}^{M} \sum_{k=1}^{N} \frac{lk}{l^{\alpha} k^{\beta}} \frac{|U_{l,k}|}{l^{1-\alpha} k^{1-\beta}}$$

$$\leq \sum_{l=1}^{M} \sum_{k=1}^{N} |U_{l,k}|.$$

This proves the result.

PROOF OF THE THEOREM. - Combining the lemma and theorem *A* we get the theorem immediately

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