BOLLETTINO UNIONE MATEMATICA ITALIANA

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.4, p. 445–448.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_4_445_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1961.

Note on quasi-orthogonal polynomials.

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Summary. - *A characterization is given of those sequences of quasi-orthogonal polynomials which form also Appell sets.*

A sequence of polynomials

$$q_n(x)(n=0,\ 1,\ 2, \ldots), \deg q_n(x) = n, \tag{Q}$$

is said to be quasi-orthogonal if there is an interval (a, b) and a non-decreasing function $\alpha(x)$ such that

$$\int_a^b x^m q_n(x) d\alpha(x) \begin{cases} = 0 \text{ if } 0 \leq m \leq n-2 \\ \neq 0 \text{ if } 0 \leq m = n-1 \\ \neq 0 \text{ if } 0 = m = n. \end{cases}$$

These polynomials have been studied by CHIHARA [2] (see the references given there) and by DICKINSON [3].

Recently, CARLITZ has given in [1] a charactertzation of those sequences of orthogonal polynomials

$$p_n(x\ (n=0,\ 1,\ 2, \ldots), \deg p_n(x) = n, \tag{P}$$

for which $p'_n(x) = np_{n-1}(x)$ for $n \geq 1$. In this note we shall do the same for sequences (Q) of quasi-orthogonal polynomials.

In [3], DICKINSON has shown that if (Q) is a sequence of quasi-orthogonal polynomials, with respect to the interval (a, b) and the non-decreasing function $\alpha(x)$, then there is a sequence of polynomials (P), orthogonal with respect to (a, b) and $\alpha(x)$, such that

$$p_n(x) = \sum_{i=0}^{n} T_i q_i(x) \qquad (n \geq 0), \tag{1}$$

where $T_n \neq 0$ for $n \geq 0$. We may assume that each $q_i(x)$ is monic (in particular, $q_0(x) = 1$) and that $T_0 = 1$. It was also shown by

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DICKINSON in [3] that there are constants b_n, c_n, d_n $(n = 0, 1, 2, ...)$ such that

$$(2) \qquad q_{n+1}(x) = (x + b_n)q_n(x) - c_n q_{n-1}(x) + d_n \sum_{i=0}^{n-2} T_i q_i(x)$$

for all n, with $d_0 = d_1 = 0$.

Now suppose that (Q) is a sequence of quasi-orthogonal polynomials and that

$$(3) \qquad q'_n(x) = nq_{n-1}(x) \qquad (n \geq 1).$$

From (2) and (3)

$$(n+1)q_n(x) = q_n(x) + n(x + b_n)q_{n-1}(x) - (n-1)c_n q_{n-2}(x)$$

$$+ d_n \sum_{i=1}^{n-2} iT_i q_{i-1}(x),$$

or

$$(4) \qquad q_n(x) = (x + b_n)q_{n-1}(x) - \frac{n-1}{n} c_n q_{n-2}(x)$$

$$+ \frac{d_n}{n} \sum_{i=0}^{n-3} (i+1) T_{i+1} q_i(x).$$

If we replace n by $n - 1$ in (2) and compare the result with (4), we see that we must have

$$d_n(i+1)T_{i+1} = nd_{n-1}T_i \qquad (i = 0, 1, ..., n-3).$$

For this we must have, of course, $n \geq 3$, This being so, we have

$$(5) \qquad d_n = nT_1^{-1}d_{n-1} \text{ and } d_n = n \frac{T_{n-2}}{(n-2)T_{n-3}} d_{n-1}.$$

If, for a given $k \geq 2$, $d_k = 0$, it follows from (5) that $d_k = 0$ for all k. In this case (2) becomes the three-tern recurrence relation

$$(6) \qquad q_{n+1}(x) = (x + b_n)q_n(x) - c_n q_{n-1}(x).$$

However, CARLITZ has shown in [1] that if the sequence (Q) satisfies (3) and (6), then (Q) is esseotially the sepuence of HERMITE polynomials. Thus, in this case, (Q) is not a sequence of quasi-orthogonal polynomials. Thus, we must have $d_k \neq 0$ for $k \geq 2$.

It now follows from (5) that for all $n \geq 0$,

$$\frac{T_{n-1}}{nT_n} = \frac{1}{T_1},$$

or

$$T_n = \frac{T_1}{n} T_{n-1} = \frac{T_1^2}{n(n-1)} T_{n-2} = \dots,$$

(6) $$T_n = \frac{T^n}{n!} \text{ where } T = T_1.$$

If we substitute from (7) into (1) we obtain

(8) $$p_n(x) = \sum_{i=0}^{n} \frac{T^i}{i!} q_i(x).$$

If we now differentiate (8) and use (3) we see that

$$p'_n(x) = T \sum_{i=0}^{n-1} \frac{T^i}{i!} q_i(x) = T p_{n-1}(x).$$

Now let $f_n(x) = n! T^{-n} p_n(x)$. Then we see immediately that the sequence $f_n(x)$, $n = 0, 1, 2, \dots$, is orthogonal with respect to the interval (a, b) and the non-decreasing function $\alpha(x)$, and furthermore, that $f'_n(x) = n f_{n-1}(x)$ for $n \geq 1$. Hence, by a theorem of SHOHAT [4] there are constants a and b, with $a \neq 0$, such that $f_n(x) = a^n H_n\left(\frac{x}{a} + b\right)$, where $H_n(x)$ is the n-th HERMITE polynomial (we use the notation used by SHOHAT, so that $H'_n(x) = nH_{n-1}(x)$ for all $n \geq 1$). Thus,

$$p_n(x) = \frac{T^n a^n}{n!} H_n\left(\frac{x}{a} + b\right).$$

From (8), we have $T^n q_n(x) = n![p_n(x) - p_{n-1}(x)]$ and so, finally, we obtain the desired characterization:

If (Q) *is a sequence of quasi-orthogonal polynomials, each of which is monic, such that* $q'_n(x) = nq_{n-1}(x)$ *for all* $n > 1$, *then* $q_0(x) = 1$, *and*

(9) $$q_n(x) = a^n H_n\left(\frac{x}{a} + b\right) - nT^{-1}a^{n-1}H_{n-1}\left(\frac{x}{a} + b\right) \qquad (n \geq 1),$$

where T, a, b *are constants,* $T \neq 0$, $a \neq 0$, *and* $H_n(x)$ *is the* n-th *Hermite polynomial.* Of course, if the sequence (Q) is given by (9), then it is a sequence of quasi-orthogonal polynomials and satisfies (3).

Now let us assume that (Q) is a sequence of quasi-orthogonal polynomials such that

$$q_n(x+1) - q_n(x) = nq_{n-1}(x) \qquad (n \geq 1). \tag{10}$$

If we now proceed as above, using (2) and (10), we again come to the conclusion that $p_n(x)$ is given by (8). Then.

$$p_n(x+1) - p_n(x) = Tp_{n-1}(x).$$

Hence, $f_n(x) = n!\, T^{-n} p_n(x)$, $n = 0, 1, 2, \dots$, is a sequence of orthogonal polynomials which satisfies (10). But, in this case, CARLITZ [1] has shown that, except for a constant factor, $f_n(x)$ is the n-th POISSON-CHARLIER polynomial. Thus, we have determined all sequences of quasi-ortogonal polynomials for which (10) holds.

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