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Zanichelli

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SEZIONE SCIENTIFICA

BREVI NOTE

Some arithmetic properties of a polynomial.

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Summary. - *Here some arithmetic properties of the polyomial* $e^{-sx}\left(\frac{1+x}{1-x}\right)^s$ *will be considered.*

KELISKY [1] has considered the properties of $\partial_n(Z)$ where

$$\left(\frac{1+x}{1-x}\right)^z = \sum_{n=0}^{\alpha} \partial_n(z)x^n$$

and

$$\partial_n(z) = \sum_{p=0}^{n} \binom{z}{n-p}\binom{Z+p-1}{p}$$

now

$$Y = e^{-sx}\left(\frac{1+x}{1-x}\right)^s$$

satisfies the differential equation

$$(1) \qquad (1-x^2)\frac{dy}{dx} = sy(1+x^2).$$

Let

$$(2) \qquad Y = e^{-sx}\left(\frac{1+x}{1-x}\right)^s = \sum_{0}^{\alpha} D_n(s)x^n.$$

Differentiating (2) and substituting the values of y we have

$$(3) \qquad (n+1)D_{n+1}(s) - (n-1)\ D_{n-1}(s) = s[(D_n(s) + D_{n-2}(s)].$$

(*) Pervenuta alla Segreteria dell'U. M. I. l'11 Settembre 1961.

Differentiating (3) we have

(4)
$$(n+1)D'_{n+1}(s) - (n-1)D'_{n-1}(s) = [(D_n(s) + D_{n-2}(s)] + + s[D'n(s) + D'_{n-2}(s)].$$

Again

$$e^{-sx} = \left(\frac{1-x}{1+x}\right)^s \sum_{n=0}^{\alpha} D_n(s)x_n =$$

$$= \sum_0^{\infty} \partial_n(s)(-1)^n x^n \sum_0^{\infty} D_n(s)x_n$$

Hence

(5) $$s^n = (-1)^n n!\,[D_n(s) - D_{n-1}(s)\partial_1(s) + D_{n-2}(s)\partial_2(s) \dots + (-1)^n \partial_n(s)].$$

Now the LEGENDRE's P_n (s)

$$= \sum_{r=0}^{P} \frac{(-1)^r(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!} s^{n-2r}$$

(6)
$$= \sum_{r=0}^{P} \frac{(-1)^{n+r}(2n-2r)!}{2^n r!\,(n-r)!} [D_{n-2r}(s) -$$

$$D_{n-2r-1}(s)\partial_r(s) + \dots + (-1)^n \partial_{n-2r}(s)]$$

where $p = n/2$ if n even otherwise $n - 1/2$.

Nov the BESSEL polynomial (EMIL GROSS wals (2))

$$Y_n = \sum_0^r \frac{(n+r)!}{(n-r)!\,r!} \left(\frac{s}{2}\right)^r$$

$$= \sum_{r=0}^{r} \frac{(n+r)!}{(n-r)!\,r!} \cdot \frac{1}{2r} (-1)^r r!\,[(D_r(s) - D_{r-1}(s)\partial_r(s)$$

(7)
$$\dots + (-1)^r \partial_r(s))]$$

$$= \sum_{r=0}^{r} \frac{(n+r)!}{(n-r)!\,2^r} (-1)^r [(D_r(s) - D_{r-1}(s)\partial_r(s) \dots + (-1)^r \partial_r(s)].$$

Now

$$\sum_0^{\alpha} D_n(r+s)x^n = e^{-(s+r)x} \left(\frac{1+x}{1-x}\right)^{r+s}$$

$$= \sum_0^{\alpha} D_n(s)x_n \sum_0^{\alpha} D_n(r)x_n$$

Hence

(8) $$D_n(r+s) = D_n(s) + D_{n-1}(s)D_1(r) + \dots D_n(r).$$

Again

$$\sum_0^\alpha D_n(s)x^n = e^{-sx}\left(\frac{1+x}{1-x}\right)^s = e^{-sx} \sum_0^\infty \partial_n(s)xx^n$$

Then

(9) $$D_n(s) = \partial_n(s) - s\partial_{n-1}(s) + \frac{s^2}{2!}\partial_{n-2}(s) \dots + (-1)^n \frac{s^n}{n!}.$$

Putting $x = e^{i\theta}$ in (2) we have

$$e^{-s \operatorname{Cos} \theta} (\operatorname{Cos} (s \operatorname{Sin} \theta) - i \operatorname{Sin} (s \sin \theta))\, i^s \operatorname{Cot}^s_{\frac{\theta}{2}}$$

$$= \sum_0^\infty D_n(s) \operatorname{Cos} n\theta + i \sum_0^\infty D_n(s) \operatorname{Sin} n\theta$$

If s is even integer

$$D_n(s) = \frac{(-1)^{\frac{s}{2}}}{\pi} \int_0^\pi e^{-s \operatorname{Cos} \theta} \operatorname{Cos} (s \operatorname{Sin} \theta) \operatorname{Cot}^s_{\frac{\theta}{2}} \operatorname{Cos} n\theta \, d\theta$$

(10) $$D_n(s) = \frac{(-1)^{\frac{s-1}{2}}}{\pi} \int_0^\pi e^{-s \operatorname{Cos} \theta} \operatorname{Sin} (s \operatorname{Sin} \theta) \operatorname{Cot}^s_{\frac{\theta}{2}} \cdot$$

$\cdot \operatorname{Sin} n\theta \, d\theta$ if $s = odd$ integr.

Differentiating (2) with respect to s we have

$$-xe^{-sx}\left(\frac{1+x}{1-x}\right)^s + e^{-sx}\left(\frac{1+x}{1-x}\right)^s \log\frac{1+x}{1-x}$$

$$= \sum_0^\alpha D'_n(s)x^n$$

Hence

(11) $$D'_{2n+1}(s) = D_{2n}(s) + 2\left[\frac{D_{2n-2}(s)}{3} + \frac{D_{2n-4}(s)}{5} + \frac{1}{2_{n+1}}\right]$$

(12) $$D'_{2n}(s) = D_{2n-1}(s) + 2\left[\frac{D_{2n-3}(s)}{3} + \frac{D_{2n-5}(s)}{5} \dots + \frac{D_1(s)}{2_{n-1}}\right].$$

REFERENCE

[1] R. Kelishy, *The numbers generated by exp (arc Tan x)*, « Duke Mathematical Journal », Vol. 26 (1959) pp. 569-581.

[2] Gross Wald, « E. Trans Amer. Math. », Soc. 71, 197 (1951).