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Some Tauberian conditions for summability (R', p) of trigonometrical series.

Nota di B. N. SAHNEY (a Sagar, India) (*)

Summary. - *Tauberian conditions for summability* (R', 1) *has been obtained by Sunouchi. These results generalise those of Szász. Sunouchi-Hirukawa have extended those of the first author. The author here invetigates the Tauberian conditions for summability* (R', p) *of which the results of Sunouchi and Sunouchi-Hirukawa are particular cases.*

1. The series $\sum_{\nu=1}^{\infty} a_\nu$ is said to be summable (R', p) to the sum zero if

$$F(t) = t \sum_{\nu=1}^{\infty} s_\nu \left(\frac{\sin \nu t}{\nu t}\right)^p \tag{1.1}$$

converges to the sum zero as $t \to o$, where s_n is the n — th partial sum of the series $\sum_{\nu=1}^{\infty} a_\nu$ and p is an integer (finite) such that $p \geq 1$.

Summability $(R', 1)$ was initially studied by SZÁSZ [1]. The result due to SZÀSZ was extended by SUNOUCHI [2] as follows:

THEOREM - S. If

$$S_n \equiv \sum_{\nu=1}^{n} s_\nu = o(n^{(1-\delta)}) \tag{1.2}$$

and

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}) \tag{1.3}$$

then the series $\sum_{\nu=1}^{\infty} a_\nu$ is summable $(R', 1)$ to the sum zero as $t \to o$ for $o < \delta < 1$.

(*) Pervenuta alla Segreteria dell'U. M. I. il 16 ottobre 1961.

Generalising this result HIRUKAWA and SUNOUCHI [3] proved the following theorem:

THEOREM - H-S. Let s_n^β be the (c, β) sum of the series $\sum_{\nu=1}^{n} a_\nu$. If

$$s_n^\beta = o(n^{\beta(1-\delta)}) \tag{1.4}$$

and

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}) \tag{1.5}$$

where $o < \delta < 1$, $0 \leq \beta$, then the series $\sum_{\nu=1}^{\infty} a_\nu$ is summable $(R', 1)$ to the sum zero as $t \to 0$.

Yano [4] has further extended these results on the Tauberian conditions for summability $(R', 1)$. He has also obtained analogous results for summability $(R, 1)$.

The object here is to discuss the summability (R', p) of the series $\sum_{\nu=1}^{\infty} a_\nu$ for all $p \geq 1$ (finite integer). We shall prove the following theorem of which Theorem-S and Theorem H-S will be particular cases for $p = 1$, $\beta = 1$ and $p = 1$, $v = \beta(1 - \delta)$ respectively.

THEOREM A. - *Let* S_n^β *be the* (c, β) *sum of the series* $\sum_{\nu=1}^{\infty} a_\nu$ *and*

$$S_n^\beta = o(n^r) \tag{1.6}$$

and

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}) \tag{1.7}$$

where $\beta > r > o$, $r + 1 > p$, $\delta = p(\beta - r)/(\beta + 1 - p)$ *and* $0 < \delta < 1$, *then the series* $\sum_{\nu=1}^{\infty} a_\nu$ *is summable* (R', p) *to the sum zero for all finite, positive integral values of* p.

2. We shall require the following lemmas in the sequel.

LEMMA 1. We write

$$\Phi(t) = \left(\frac{\sin t}{t}\right)^p \tag{2.1}$$

then

$$\left(\frac{d}{dt}\right)^k \Phi(t) \equiv \Phi^{(k)}(t) = O(1), \text{ as } t \to 0 \tag{2.2}$$

and

$$\Phi^{(k)}(t) = O(t^{-p}), \text{ as } t \to \infty. \tag{2.3}$$

This is due to KANNO [5].

LEMMA 2. - If

$$\sum_{n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}) \tag{2.4}$$

then

$$\bar{S}_n^0 \equiv \bar{s}_n \equiv \sum_{1}^{\infty} |a_\nu| = O(n^\delta), \tag{2.5}$$

$$S_n^0 \equiv s_n \equiv \sum_{1}^{\infty} a_\nu = O(n^\delta) \tag{2.6}$$

and

$$\sum_{\nu=n}^{\infty} \frac{|s_\nu|}{\nu^2} = O(n^{-(1-\delta)}). \tag{2.7}$$

This is due to SUNOUCHI (See SUNOUCHI [2]).

3. *Proof of the theorem.* We write

$$t \sum_{\nu=1}^{\infty} s_\nu \Phi(\nu t) = t\Big(\sum_{1}^{n} + \sum_{n+1}^{\infty}\Big) s_\nu \Phi(\nu t)$$

$$= \Phi_1 + \Phi_2, \text{ say.} \tag{3.1}$$

Now by Lemma 2, we have

$$\sum_{\nu=n}^{\infty} \left| \frac{s_\nu}{\nu^p} - \frac{s_{\nu+1}}{(\nu+1)^p} \right| = \sum_{\nu=n}^{\infty} \left| \left\{ \frac{s_\nu - s_{\nu+1}}{\nu^p} + \left(\frac{1}{\nu^p} - \frac{1}{(\nu+1)^p}\right) s_{\nu+1} \right\} \right|$$

$$\leq A \sum_{n}^{\infty} \frac{|a_\nu|}{\nu^p} + B \sum_{n}^{\infty} \frac{|s_\nu|}{\nu(\nu+1)^p}$$

$$= O\left(\frac{1}{n^{p-\delta}}\right), \text{ by Lemma 2.}$$

Moreover, by ABEL's transformation

$$t\sum_{\nu=n}^{\infty} s_\nu \Phi(\nu t) = t\sum_{\nu=n}^{\infty} \frac{s_\nu (\sin \nu t)^p}{(\nu t)^p} \tag{3.2}$$

$$= O(t^{-p}) \sum_{\nu=n}^{\infty} \left| \frac{s_\nu}{\nu^p} - \frac{s_{\nu+1}}{(\nu+1)^p} \right|$$

$$+ O(t^{-p})\left(\frac{1}{n^{p-\delta}}\right)$$

which is convergent for all $t \neq 0$. The fact is evident if $t = 0$
Now for a given positive ε, we write

$$n = (\varepsilon t)^{-\rho} \text{ where } \frac{p}{p-\delta} = \rho = \frac{\beta + 1 - p}{r + 1 - p}. \tag{3.3}$$

Estimating now Φ_2, we have

$$\Phi_2 = t\sum_{n+1}^{\infty} s_\nu \Phi(\nu t) \tag{3.4}$$

$$= t\sum_{n+1}^{\infty} \frac{s_\nu}{\nu^p} \cdot \left(\frac{\sin \nu t}{t}\right)^p$$

$$= O(t^{-p}) \left\{ \sum_{n+1}^{\infty} \left| \frac{s_\nu}{\nu^p} - \frac{s_{\nu+1}}{(\nu+1)^p} \right| + \frac{|s_n|}{n^p} \right\}$$

$$= O(t^{-p} n^{-(p-\delta)}), \text{ by } (3.2),$$

$$= O(t^{-p} (\varepsilon t)^{\rho(p-\delta)}), \text{ by } (3.3)$$

$$= O(\varepsilon^p), \text{ by } (3.3).$$

Next

$$\Phi_1 = t\sum_{\nu=1}^{n} s_\nu \Phi(\nu t) \tag{3.5}$$

$$= -t\sum_{\nu=1}^{n-1} \int_{\nu}^{\nu+1} s(x)\Phi(xt)dx$$

$$= -t\int_0^n s(x)\Phi(xt)dx.$$

Now we take an integer $k \geq 1$ from (1.6), (2.6) in Lemma 2 and by RIESZ's convexity Theorem (See KANNO [5]), we have

$$(3.6) \qquad \begin{cases} s_n^{\mu} = o\left(n^{\frac{\delta(\beta-\mu)}{\beta} + \frac{r\mu}{\beta}}\right), \\ \\ s_n^{k} = on^{(k-r-\beta)}. \end{cases} \qquad \mu = 1, 2, \ldots, k-1,$$

Let $S^0(x) = s(x) = \sum_{\nu=1}^{n} a_\nu$ when $n \leq x < n+1$,

$$S^q(x) = \frac{1}{\sqrt{q}} \int_0^x S^0(x)(x-t)^{q-1} dt, \; q > 0,$$

and

$$D_n^{\mu} \equiv \left[\frac{d^{\mu}}{dx^{\mu}}\right]_{x=n}.$$

On integration by parts k times and by virtue of (3.5) we have

$$\Phi_1 = -t \int_0^n S^0(x)\Phi(xt)dx$$

$$= t \sum_{\nu=1}^{k-1} (-1)^{\mu} S^{\mu}(n) D_n^{\mu-1} \Phi(xt)$$

$$+ (-1)^k t S^k(n) D_n^{k-1} \Phi(xt)$$

$$+ (-1)^{k+1} t \int_0^n S^k(x) \left(\frac{d^k}{dx^k}\right) \Phi(xt) dx$$

$$(3.7) \qquad \begin{cases} = \Phi_3 + \Phi_4 + \Phi_5, \text{ say for } \beta < k, \\ \\ = \Phi_3 + \Phi_4, \text{ for } \beta = k. \end{cases}$$

Now, since

$$tS^{\mu}(n) D_n^{\mu-1} \Phi(xt) = S^{\mu}(n) t^{\mu} \left[\frac{d^{\mu-1}}{d(xt)^{\mu-1}} \Phi(xt)\right]_{x=n}$$

and hence

$$\Phi_3 = t \sum_{\mu=1}^{k-1} (-1)^{\mu} S^{\mu}(n) D_n^{\mu-1} \Phi(xt)$$

$$= o\left(n^{\frac{\delta(\beta-\mu)}{\beta} + \frac{r\mu}{\beta}} \cdot t^{\mu} \cdot n^{-p} \cdot t^{-p}\right), \text{ by (3.6).}$$

$$= o\left(t^{\mu-p} \cdot (\varepsilon t)^{-\rho\left(-p + \frac{\delta(\beta-\mu)}{\beta} + \frac{r\mu}{\beta}\right)}\right)$$

$$= o\left(\varepsilon^{\rho\left(-p - \frac{\delta(\beta-\mu)}{\beta} - \frac{r\mu}{\beta}\right)} \cdot t^{\mu-p+\rho\left(p - \frac{\delta(\beta-\mu)}{\beta} - \frac{r\mu}{\beta}\right)}\right), \text{ by (3 3).}$$

In this expression the index of the power of t is

$$\mu - p + \rho\left(p - \frac{\delta(\beta-\mu)}{\beta} - \frac{r\mu}{\beta}\right)$$

$$= \left[(\mu - p)\beta - \frac{p}{p-\delta} \{ \beta(\delta - p) + \mu(r - \delta) \} \right] / \beta$$

$$= \frac{1}{\beta}\left[\beta\mu - \frac{p}{p-\delta} \cdot \mu(r-\delta)\right], \text{ by (3.3)}$$

$$= \frac{\mu}{\beta}\left[\beta - \frac{\beta+1-p}{r+1-p}(r-\delta)\right], \text{ by (3.3)}$$

$$= \frac{\mu(\beta - r)}{\beta(r+1-p)}, \text{ since } \delta = \frac{p(\beta - r)}{r+1-p}$$

which is positive for all $\mu = 1, 2, \dots, k-1$ and $0 < \delta < 1$, $\beta \geq 0$, $p \geq 1$.

Therefore

(3.8) $$\Phi_3(t) = o(1), \text{ as } t \to 0.$$

Evaluating $\Phi_4(t)$, we obtain

$$\Phi_4(t) = (-1)^k t S^k(n) D_n^{k-1} \Phi(xt)$$

$$= o(t \cdot n^{k+r-\beta} \cdot t^{k-1} \cdot t^{-p} \cdot n^{-p}), \text{ by (3.6) and Lemma 2,}$$

$$= o(t^{k-p} \cdot (\varepsilon t)^{-\rho(k+r-\beta-p)}), \text{ by (3.3).}$$

In this expression the index of the power of t is

$$k-p-\rho(k+r-\beta-p)=\frac{1}{p-\delta}\{(p-\delta)(k-p)-p(k+r-\beta-p)\},\ \text{by (3.3)}$$

$$=\frac{1}{p-\delta}\{-\delta k+\delta p-pr+p\beta\}$$

$$=\frac{1}{p-\delta}\{-k\delta+(\beta+1)\delta\},\ \text{since } \delta=\frac{p(\beta-r)}{\beta+1-p}$$

$$=\frac{\delta}{p-\delta}\{\beta+1-k\}$$

which is positive, since $r+1>p$ and $\beta>r$. Hence, we have

(3.9) $$\Phi_4(t)=o(1),\ \text{as } t\rightarrow 0.$$

Estimating $\Phi_5(t)$, we have

(3.10) $$\Phi_5(t)=(-1)^{k+1}t\int_0^n S^k(x)\frac{d^k}{dx^k}\Phi(xt)dx$$

$$=(-1)^{k+1}t\int_0^n \frac{d^k}{dx^k}\Phi(xt)dx\int_0^x(x-u)^{k-\beta-1}S^\beta(u)du$$

$$=(-1)^{k+1}t\int_0^n S^\beta(u)du\int_u^n(x-u)^{k-\beta-1}\frac{d^k}{dx^k}\Phi(xt)dx,$$

by change of order of integration

$$=(-1)^{k+1}\left\{\int_0^{t^{-1}}du\int_u^{u+t^{-1}}dx+\int_{t^{-1}}^{(\varepsilon t)^{-\rho}}du\int_u^{u+t^{-1}}dx+\int_0^{(\varepsilon t)^{-\rho}-t^{-1}}du\int_{u+t^{-1}}^{(\varepsilon t)^{-\rho}}dx\right.$$

$$\left.-\int_{(\varepsilon t)^{-\rho}-t^{-1}}^{(\varepsilon t)^{-\rho}}du\int_{(\varepsilon t)^{-\rho}}^{u+t^{-1}}dx\right\}S^\beta(u)(x-u)^{k-\beta-1}\frac{d^k}{dx^k}\Phi(xt)$$

$$=\psi_1(t)+\psi_2(t)+\psi_3(t)+\psi_4(t),\ \text{say.}$$

Considering $\psi_1(t)$, we have

$$\psi_1(t) = (-1)^{k+1} t \int_0^{t^{-1}} S^{\beta}(u) du \int_u^{u+t^{-1}} (x-u)^{k-\beta-1} \frac{d^k}{dx^k} \Phi(xt) dx \tag{3.11}$$

$$= O(t) \Big\{ \int_0^{t^{-1}} | S^{\beta}(u) | \, du \int_u^{u+t^{-1}} t^k \cdot (x-u)^{k-\beta-1} dx \Big\}, \text{ by Lemma 1}$$

$$= O(t)^{k+1} \int_0^{t^{-1}} o(u^r) du [(u-x)^{k-\beta}]_u^{u+t^{-1}}$$

$$= o(t^{\beta+1}) [u^{r+1}]_0^{t^{-1}}$$

$$= o(t^{\beta-r})$$

$$= o(1), \text{ as } t \to o, \text{ since } \beta > r > 0$$

Next

$$\psi_2(t) = (-1)^{k+1} t \int_{t^{-1}}^{(\varepsilon t)^{-\rho}} S^{\beta}(u) du \int_u^{u+t^{-1}} (x-u)^{k-\beta-1} \frac{d^k}{dx^k} \Phi(xt) dx \tag{3.12}$$

$$= O(t^{k+1}) \int_{t^{-1}}^{(\varepsilon t)^{-\rho}} | S^{\beta}(u) | \, du \int_u^{u+t^{-1}} (x-u)^{k-\beta-1} \cdot O(xt)^{-p} dx$$

$$= o(t^{k+1-p}) \int_{t^{-1}}^{(\varepsilon t)^{-\rho}} u^r \cdot u^{-p} [(x-u)^{k-\beta}]_u^{u+t^{-1}} du$$

$$= o(t^{k+1-p}) \int_{t^{-1}}^{(\varepsilon t)^{-\rho}} u^{r-p} \, . \, du t^{\beta-k}$$

$$= o(t^{\beta+1-p}) [u^{r+1-p}]_{t^{-1}}^{(\epsilon t)^{-\rho}}$$

$$= o(t^{-(r-\beta)}) + o(1), \text{ by } (3.3)$$

$$= o(1), \text{ as } t \to 0.$$

Considering $\psi_3(t)$ and on integrating, by parts its inner integral, we get

$$(3.13) \qquad \psi_3(t) = (-1)^{k+1} t \int_0^{(\varepsilon t)^{-\rho} - t^{-1}} S^{\beta}(u) \int_{u+t^{-1}}^{(\varepsilon t)^{-\rho}} (x-u)^{k-\beta-1} \frac{d^k}{dx^k} \Phi(xt) dx$$

$$= (-1)^{k+1} \int_0^{(\varepsilon t)^{-\rho} - t^{-1}} S^{\beta}(u) du \left\{ \left[(x-u)^{k-\beta-1} \frac{d^{k-1}}{dx^{k-1}} \Phi(xt) \right]_{u+t^{-1}}^{(\varepsilon t)^{-\rho}} \right.$$

$$\left. - (k-\beta-1) \int_{u+t^{-1}}^{(\varepsilon t)^{-\rho}} (x-u)^{k-\beta-2} \frac{d^{k-1}}{dx^{k-1}} \Phi(xt) dx \right\}$$

$$= X_1(t) - (k-\beta-1) X_2(t), \text{ say},$$

where

$$(3.14) \qquad X_1(t) = O(t^k) \int_0^{(\varepsilon t)^{-\rho} - t^{-1}} S^{\beta}(u) du \, \{ t^{-p} [(\varepsilon t)^{\rho p} \cdot ((\varepsilon t)^{-\rho} - u)^{k-\beta-1}$$

$$- (u + t^{-1})^{-p} t^{-k+\beta+1}] \}$$

$$= X_3(t) + X_4(t), \text{ say}.$$

Determining the value of $X_3(t)$, we obtain

$$X_3(t) = o(\varepsilon^{p\rho}) (t^{k-p+\rho p}) \int_0^{(\varepsilon t)^{-\rho}} u^r ((\varepsilon t)^{-\rho} \quad u)^{h-\beta-1} du$$

$$= o(t^{k-p+\rho(p-k-r+\beta)}).$$

Here the exponent of t is

$$k-p+\rho(p-k-r+\beta)=\frac{1}{(r+1-p)}\{(k-p)(r+1-p)+(\beta+1-p)(p-k-r+\beta)\}$$

$$=\frac{(\beta-r)(\beta+1-k)}{(r+1-p)}, \text{ by (3.3)}$$

which is positive, since $k<\beta+1$.

Hence

(3.15) $$X_3(t)=o(1), \text{ as } t\to o.$$

Also

(3.16) $$X_4(t)=o(t^{\beta+1-p})\int_0^{(\varepsilon t)^{-\rho}-t^{-1}} u^r(u+t^{-1})^{-p}dt$$

$$=o(t^{\beta+1-p})[u^{r+1-p}]_0^{(\varepsilon t)^{-\rho}}$$

$$=o(t^{(\beta+1-p)-\rho(r+1-p)})$$

$=o(1)$, as $t\to 0$, by virtue of (3.3).

Thus from (3.14), (3.15) and (3.16) we get

(3.17) $$X_1(t)=o(1) \text{ as } t\to 0.$$

Estimating $X_2(t)$, we have

(3.18) $$X_2(t)=O(t^k)\int_0^{(\varepsilon t)^{-\rho}-t^{-1}} |\, S^{\beta}(u)\,|\, du\int_{u+t^{-1}}^{(\varepsilon t)^{-\rho}} O(xt)^{-p}(x-u)^{k-\beta-2}du$$

$$=o(t^{k-p})\int_0^{(\varepsilon t)^{-\rho}-t^{-1}} u^{r-p}[(x-u)^{k-\beta-1}]_{u+t^{-1}}^{(\varepsilon t)^{-\rho}}du$$

$$=o(t^{\beta+1-p})\int_0^{(\varepsilon t)^{-\rho}-t^{-1}} u^{(r-p)}du+o(t^{k-p-\rho(k-\beta-1)})\int_0^{(\varepsilon t)^{-\rho}-t^{-1}} u^{r-p}du$$

$$=o(t^{\beta+1-p}+t^{k-p-\rho(k-\beta-1)})(u^{r-p+1})_0^{(\varepsilon t)^{-\rho}-t^{-1}}$$

$$=o(1) \text{ as } t\to 0, \text{ by viture of (3.3) and (3.9).}$$

On collecting (3.13), (3.17) and (3.18), we find that

(3.19) $$\psi_3(t) = o(1), \text{ as } t \to 0.$$

Lastly we observe that

(3.20) $$\psi_4(t) = (-1)^{k+1} t \int_{(\varepsilon t)^{-\rho}-t^{-1}}^{(\varepsilon t)^{-\rho}} S^{\beta}(u) du \int_{(\varepsilon t)^{-\rho}}^{u+t^{-1}} (x-u)^{k-\beta-1} \frac{d^k}{dx^k} \Phi(xt) dx$$

$$= o(t^{k+1-p}) \int_{(\varepsilon t)^{-\rho}-t^{-1}}^{(\varepsilon t)^{-\rho}} u^r . t^{\rho p} du [(x-u)^{k-\beta}]_{(\varepsilon t)^{-\rho}}^{u+t^{-1}}$$

$$= o(t^{\beta+1-p+\rho p}) [u^{r+1}]_{(\varepsilon t)^{-\rho}-t^{-1}}^{(\varepsilon t)^{-\rho}}$$

$$= o(t^{(\beta+1-p)-\rho(r+1-p)})$$

$$= o(1), \text{ as } t \to 0, \text{ by } (3.3).$$

Now on collecting (3.10), (3.11), (3.12) and (3.20) we get

(3.21) $$\Phi_5(t) = o(1) \text{ as } t \to 0.$$

Combining (3.1), (3.4) and (3.20) we find that for an arbitrarily chosen ε,

$$| t \sum_{\nu=1}^{\infty} s_\nu \Phi(\nu t) | < A \varepsilon^p, \text{ as } t \to 0.$$

Since ε is arbitrarily small, hence we have

(3 22) $$t \sum_{\nu=1}^{\infty} s_\nu \Phi(\nu t) \to 0 \text{ as } t \to 0.$$

This completes the proof of Theorem *A*.

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