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SEZIONE SCIENTIFICA

BREVI NOTE

Approximation To The Generating Function By The Cesàro Means of Its Ultraspherical Series.

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Summary. - *The author has extended one of the resultes of Obrechkoff « On the approximation to the generating function by the Cesaro means of its ultraspherical series ».*

1. Let $f(\theta, \varphi)$ be a function defined for the range $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$; the ultraspherical series corresonding to it on the sphere S is

$$(1.1) \qquad f(\theta, \varphi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\lambda) \int\int_S \frac{f(\theta', \varphi') P_n^{(\lambda)}(\cos \omega) \sin \theta' d\theta' d\varphi'}{[\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{\frac{1}{2}-\lambda}},$$

where

$$\cos \omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi').$$

The LAPLACE series is a particular case of (1.1) for $\lambda = \frac{1}{2}$.

We write [See also; 3]

$$(1.2) \qquad f(\omega) = \frac{1}{2\pi(\sin \omega)^{2\lambda}} \int_{c_\omega} \frac{f(\theta', \varphi') ds'}{[\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{\frac{1}{2}-\lambda}}$$

and

$$(1.3) \qquad \varphi(\omega) = \left[f(\omega) - \frac{A\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\lambda\right)} \right] (\sin \omega)^{2\lambda};$$

$$\Phi_p(x) = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} \varphi(t) dt;$$

$$\Phi_0(x) = \varphi(x)$$

$$\varphi_p(x) = \Gamma(p+1) x^{-p} \Phi_p(x), \qquad p \geq 0;$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 19 dicembre 1961.

and

$$\Phi_p(x) = \frac{d}{dx} \Phi_{p+1}(x), \qquad -1 < p < 0.$$

The absolute integrability of the integrand in (1.2) is assumed throughout.

OBRECHKOFF [2] has proved the following result on the order of the CESÀRO means σ_n^k of the series (1.1):

THEOREM 1. - If $p \geq 0$, $0 \leq \alpha < 1$, such that

$$\int_0^t | \varphi_p(t) | \, dt = O(t^{1+2\lambda+\alpha})$$

$$\varphi_{p+1}(t) = o(t^{2\lambda+\alpha}), \qquad t \longrightarrow 0$$

then, for each k,

$$p + \lambda + 1 \geq k > p + \lambda + \alpha,$$

we have

$$\sigma_n^k - A = O\left(\frac{1}{n^\alpha}\right).$$

We shall prove the following theorem:

THEOREM 2. - If

$$\Phi^*(t) = \int_0^t \frac{| \varphi_p(t) |}{t^{1+2\lambda}} \, dt = O\left[\left(\log \frac{1}{t}\right)^{r+1}\right]$$

and

$$\varphi_{p+1}(t) = O\left[t^{2\lambda}\left(\log \frac{1}{t}\right)^{r+1}\right], \qquad t \rightarrow 0.$$

for $-1 < r < \sim$ and any $p \geq 0$ then

$$\sigma_n^k - A = O\,[(\log n)^{r+1}]$$

where

$$\lambda + [P[+ 1 \geq k > p + \lambda.$$

2. For the proof of the theorem we shall require the following Lemmas:

Lemma 1. – If $S_n^k(\omega)$ denotes the n^{th} Cesàro means of order $k \geq 0$ of the series

$$\Sigma\,(n+\lambda)P_n^{(\lambda)}(\cos\omega) \tag{2.1}$$

Then we have, for $\lambda > 0$ and $p \geq 0$

$$S_n^{(p)}(w) = \frac{d^p\{S_n^k(\omega)\}}{d\omega^p} = \begin{cases} O\left(n^{2\lambda+p+1}\right) \text{ for } 0 \leq \omega \leq \pi, \quad k > 0; \\ O\left(\dfrac{n^{\lambda+p-k}}{w^{k+\lambda+1}}\right) + O\left(\dfrac{1}{nw^{2\lambda+p+2}}\right) \\ \qquad \text{for } 0 < \omega \leq a < \pi; \\ O\left(\dfrac{n^{\lambda+p-k}}{w^{k+\lambda+1}}\right) \text{ for } 0 < \omega \leq a < \pi \\ \qquad \text{and } \lambda + 1 + [p] \geq k. \end{cases} \tag{2.2}$$

Lemma 2. – If η be any fixed positive constant less than π, then we have

$$\sigma_n^k - A = \int_0^\eta \varphi(\omega) S_n^k(\omega)\, d\omega + o(1) \tag{2.3}$$

for $k > \lambda$.

Lemma 3. – For a non-integral

$$p = m + \sigma \qquad (0 < \sigma < 1), \quad \text{we have}$$

$$\int_0^\eta \Phi_p(\omega) S_n^{(p)}(\omega)\, d\omega = \Phi_{m+1}(\eta) S_n^{(m)}(\eta) - \int_0^\eta \Phi_m(t) S_n^{(m)}(t)\, dt. \tag{2.4}$$

Lemmas 1 and 3 are due to Obrechkoff and for 2 see Kogbetliantz [1] and Obrechkoff [2].

3. Proof of the Theorem. – In view of Lemma 2 it will be sufficient to prove that

$$\int_0^{\eta} \varphi(\omega) S_n^k(\omega) d\omega = O[(\log n)^{r+1}].$$

Integrating by parts m times we ha e

$$\int_0^{\eta} \varphi(\omega) S_n^k(\omega) d\omega$$

$$= \left[\sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(\omega) S_n^{(\rho-1)}(\omega) \right]_0^{\eta} + (-1)^m \int_0^{\eta} \Phi_m(\omega) S_n^{(m)}(\omega) d\omega .$$

Also, for a non-integral

$$p = m + \sigma \ (0 < \sigma < 1),$$

we have

$$J = \int_0^{\eta} \varphi(\omega) S_n^k(\omega) d\omega$$

$$= \left[\sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(\omega) S_n^{(\rho-1)}(\omega) \right]_0^{\eta} + (-1)^m \left\{ \Phi_{m+1}(\eta) S_n^{(m)}(\eta) \right.$$

$$\left. - \int_0^{\eta} \Phi_p(\omega) S_n^{(p)}(\omega) d\omega \right\}$$

$$= J_1 + J_2 + J_3 .$$

(3.1) $$J_2 = o(1) \text{ as } n \to \infty$$

and

(3.2) $$J_2 = o(1) \text{ as } n \to \infty$$

since $k > p + \lambda$ [see 4].

Hence

$$J = o(1) + (-1)^{m+1}\int_0^{\eta} \Phi_p(u) S_n^{(p)}(u)\,du$$

$$= o(1) + \frac{(-1)^{m+1}}{\Gamma(p+1)}\int_0^{\eta} u^p \varphi_p(u) S_n^{(p)}(u)\,du$$

$$= o(1) + \frac{(-1)^{m+1}}{\Gamma(p+1)} L, \text{ say.}$$

Write

$$L = \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\eta}\right) u^p \varphi_p(u) S_n^{(p)}(u)\,du$$

$$= L_1 + L_2, \text{ say.}$$

Now

(3.3) $$L_1 = \Big[\Phi_{p+1}(u) S_n^{(p)}(u)\Big]_0^{\frac{1}{n}} - \int_0^{\frac{1}{n}} \Phi_{p+1}(u) S_n^{(p)}(u)\,du$$

$$= J_1' - J_2'$$

$$J_2' = O\left[\int_0^{\frac{1}{n}} u^{p+1}\varphi_{p+1}(u) S_n^{(p+1)}(u)\,du\right]$$

$$= O\left[\int_0^{\frac{1}{n}} u^{p+1}\cdot u^{2\lambda}\left(\log\frac{1}{u}\right)^{r+1}\cdot O(n^{2\lambda+p+2})\,du\right]$$

$$= O[(\log n)^{r+1}].$$

(3.4) $$J_2' = O\left[\left(\frac{1}{n}\right)^{p+1} \frac{n^{\lambda+p-k}}{\left(\frac{1}{n}\right)^{\lambda+k+1}} \varphi_{p+1}\left(\frac{1}{n}\right)\right] \text{ as } \lambda+[p]+1 \geq k.$$

$$= O[(\log n)^{r+1}].$$

Thus

(3.5) $$L_1 = O[\log n]^{r+1}].$$

Now

$$L_2 = \int_{\frac{1}{n}}^{\eta} u^p \varphi_p(u) S_n^{(p)}(u) du$$

$$= \int_{\frac{1}{n}}^{\eta} |\varphi_p(u)| \cdot O\left(\frac{n^{\lambda+p-k}}{u^{k+\overline{\lambda+1}}}\right) u^p \cdot du$$

$$= O(n^{\lambda+p-k}) \cdot \int_{\frac{1}{n}}^{\eta} \frac{|\varphi(\omega)|}{u^{1+2\lambda}} \cdot u^{p+\lambda-k} du$$

$$= O(n^{\lambda+p-k}) \cdot [\Phi^*(u) \cdot u^{p+\lambda-k}]_{\frac{1}{n}}^{\eta}$$

$$- O(n^{\lambda+p-k}) \int_{\frac{1}{n}}^{\eta} \Phi^*(u) \cdot u^{p+\lambda-k-1} \cdot du$$

$$= O(n^{\lambda+p-k}) \cdot \Phi^*(\eta)\eta^{\lambda+p-k}$$

$$+ O(n^{\lambda+p-k} \cdot \Phi^*\left(\frac{1}{n}\right) \cdot \frac{1}{n^{\lambda+p-k}}$$

$$+ O(n^{\lambda+p-k}) \cdot \int_{\frac{1}{n}}^{\eta} \left(\log \frac{1}{u}\right)^{r+1} \cdot u^{p+\lambda-k-1} du$$

$$= k_1 + k_2 + k_3, \text{ say}$$

(3.6) $$k_1 = o(1), \text{ since } k > \lambda + p$$

(3.7) $$k_2 = O[(\log n)^{r+1}]$$

and

(3.8) $$k_3 = O(n^{\lambda+p-k}) \cdot \int_{\frac{1}{n}}^{\eta} \frac{1}{u^{-p-\lambda+k+1}} du$$

$$= O(n^{\lambda+p-k}) \cdot \left[\frac{1}{u^{k-p-\lambda}}\right]_{\frac{1}{n}}^{\eta}$$

$$= o(1) + O(1)$$

$$= O(1).$$

Thus

(3.9) $$L_2 = o(1) + O[(\log n)^{r+1}] + O(1)$$

$$= O[(\log n)^{r+2}]$$

Combining (3.1), (3.2) ... and (3.9) the result is proved.

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