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An integral formula for the Jacobi polynomial

by LEONARD CARLITZ (a Durham, U. S. A.) (*) (**)

Summary. - *It is shown that*

$$P_n^{(\alpha, \beta)}(x) = \frac{2^{\alpha+\beta-1}}{\pi} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n! \Gamma(\alpha+\beta+n+1)} \\ \cdot \int_{-\pi}^{\pi} e^{\frac{i}{2}(\alpha-\beta)\theta} \cos^{\alpha+\beta} \frac{1}{2}\theta \cdot \\ |x(1+\cos\theta) - i \sin\theta|^n d\theta.$$

We have

$$P_n^{(\alpha, \beta)} = \sum_{r=0}^n \binom{\alpha+n}{n-r} \binom{\beta+n}{r} \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \\ = \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n! \Gamma(\alpha+\beta+n+1)} \\ \cdot \sum_{r=0}^n \binom{n}{r} \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+r+1)\Gamma(\beta+n-r+1)}.$$

Now it is known that [2, p. 463]

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-\nu)\theta} \cos^{\mu+\nu}\theta d\theta = \frac{\pi \Gamma(\mu+\nu+1)}{2^{\mu+\nu} \Gamma(\mu+1)\Gamma(\nu+1)},$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 5 maggio 1962.

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so that

$$\begin{aligned}
 & \sum_{r=0}^n \binom{n}{r} \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \frac{\Gamma(x+\beta+n+1)}{\Gamma(x+r+1)\Gamma(\beta+n-r+1)} = \\
 & = \frac{2^{\alpha+\beta+n}}{\pi} \sum_{r=0}^n \binom{n}{r} \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \\
 & \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta+2r-n)\theta i} \cos^{\alpha+\beta+n}\theta d\theta = \\
 & = \frac{2^{\alpha+\beta+n}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta-n)\theta i} \cos^{\alpha+\beta+n}\theta \left(\frac{x-1}{2} e^{2\theta i} + \frac{x+1}{2}\right)^n d\theta = \\
 & = \frac{2^{\alpha+\beta+n}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)\theta i} \cos^{\alpha+\beta+n}\theta \cdot \left(\frac{x-1}{2} e^{\theta i} + \frac{x+1}{2} e^{-\theta i}\right)^n d\theta = \\
 & = \frac{2^{\alpha+\beta+n}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)\theta i} \cos^{\alpha+\beta+n}\theta \cdot (x \cos \theta - i \sin \theta)^n d\theta.
 \end{aligned}$$

We have therefore

$$\begin{aligned}
 (1) \quad P_n^{(\alpha, \beta)}(x) & = \frac{2^{\alpha+\beta+n}}{\pi} \frac{\Gamma(x+n+1)\Gamma(\beta+n+1)}{n! \Gamma(x+\beta+n+1)} \\
 & \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)\theta i} \cos^{\alpha+\beta-n}\theta (x \cos \theta - i \sin \theta)^n d\theta.
 \end{aligned}$$

Since

$$\begin{aligned}
 2^n \cos^n \theta (x \cos \theta - i \sin \theta)^n & = (2x \cos^2 \theta - 2i \sin \theta \cos \theta)^n = \\
 & = |x(1 + \cos 2\theta) - i \sin 2\theta|^n,
 \end{aligned}$$

we have also

$$(2) \quad P_n^{(\alpha, \beta)}(x) = \frac{2^{\alpha+\beta-1}}{\pi} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n! \Gamma(\alpha+\beta+n+1)} \cdot \int_{-\pi}^{\pi} e^{\frac{1}{2}(\alpha-\beta)\theta i} \cos^{\alpha+\beta} \frac{1}{2} \theta \cdot |x(1+\cos \theta) - i \sin \theta|^n d\theta.$$

In particular, for the ultraspherical polynomial, (2) reduces to

$$(3) \quad P_n^{(\alpha, \alpha)}(x) = \frac{2^{2\alpha-1}}{\pi} \frac{\Gamma(\alpha+n+1)\Gamma(\alpha-n-1)}{n! \Gamma(2\alpha+n+1)} \cdot \int_{-\pi}^{\pi} \cos^{2\alpha} \frac{1}{2} \theta |x(1+\cos \theta) + i \sin \theta|^n d\theta.$$

Therefore for the LEGENDRE polynomial

$$(4) \quad P_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x(1+\cos \theta) + i \sin \theta|^n d\theta = \frac{1}{2\pi} \int_0^{2\pi} |x(1+\cos \theta) + i \sin \theta|^n d\theta,$$

a formula due to CATALAN [1].

REFERENCES

- [1] E. CATALAN, *Nouvelles propriétés des fonctions X*, « Mémoires de l'Académie Royale des Sciences », des lettres, et des beaux-arts de Belgique, vol. 47 (1889) pp. 3-24.
- [2] E. T. WHITTAKER and G. N. WATSON, *A course of modern analysis*, fourth edition, Cambridge, 1927.