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An integral formula for the Jacobi polynomial.

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Zanichelli

<http://www.bdim.eu/item?id=BUMI_1962_3_17_3_273_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1962.

An integral formula for the Jacobi polynomial

by LEONARD CARLITZ (a Durham, U. S. A.) (*) (**)

Summary. - *It is shown that*

$$P_n^{(\alpha,\beta)}(x) = \frac{2^{\alpha+\beta-1}}{\pi} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+n+1}$$

$$\cdot \int_{-\pi}^{\pi} e^{\frac{1}{2}(\alpha-\beta)\theta i} \cos^{\alpha+\beta}\frac{1}{2}\theta \cdot$$

$$\{ x(1+\cos\theta) - i\sin\theta \}^n d\theta.$$

We have

$$P_n^{(\alpha,\beta)} = \sum_{r=0}^{n} \binom{\alpha+n}{n-r}\binom{\beta+n}{r}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r}$$

$$= \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)}$$

$$\cdot \sum_{r=0}^{n} \binom{n}{r}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r} \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+r+1)\Gamma(\beta+n-r+1)}.$$

Now it is known that [2, p. 463]

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-\nu)\theta i} \cos^{\mu+\nu}\theta\, d\theta = \frac{\pi\Gamma(\mu+\nu+1)}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)},$$

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so that

$$\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x-1}{2}\right)^{r}\left(\frac{x+1}{2}\right)^{n-r}\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+r+1)\Gamma(\beta+n-r+1)}=$$

$$=\frac{2^{\alpha+\beta+n}}{\pi}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x-1}{2}\right)^{r}\left(\frac{x+1}{2}\right)^{n-r}\cdot$$

$$\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta+2r-n)\theta i}\cos^{\alpha+\beta+n}\theta\, d\theta =$$

$$=\frac{2^{\alpha+\beta+n}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta-n)\theta i}\cos^{\alpha+\beta+n}\theta\left(\frac{x-1}{2}e^{2\theta i}+\frac{x+1}{2}\right)^{n}d\theta =$$

$$=\frac{2^{\alpha+\beta+n}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)\theta i}\cos^{\alpha+\beta+n}\theta\cdot\left(\frac{x-1}{2}e^{\theta i}+\frac{x+1}{2}e^{-\theta i}\right)^{n}d\theta =$$

$$=\frac{2^{\alpha+\beta+n}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)\theta i}\cos^{\alpha+\beta+n}\theta\cdot(x\cos\theta-i\sin\theta)^{n}d\theta.$$

We have therefore

$$P_n^{(\alpha,\beta)}(x)=\frac{2^{\alpha+\beta+n}}{\pi}\,\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)}\cdot \tag{1}$$

$$\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)\theta i}\cos^{\alpha+\beta+n}\theta(x\cos\theta-i\sin\theta)^{n}d\theta.$$

Since

$$2^{n}\cos^{n}\theta(x\cos\theta-i\sin\theta)^{n}=(2x\cos^{2}\theta-2i\sin\theta\cos\theta)^{n}=$$

$$=\{x(1+\cos 2\theta)-i\sin 2\theta\}^{n},$$

we have also

$$(2) \qquad P_n^{(\alpha,\beta)}(x) = \frac{2^{\alpha+\beta-1}}{\pi} \frac{\Gamma(\alpha + n\ 1)\Gamma(\beta + n + 1)}{n\,!\,\Gamma(\alpha + \beta + n + 1)} \cdot$$

$$\cdot \int_{-\pi}^{\pi} e^{\frac{1}{2}(\alpha-\beta)\theta i} \cos^{\alpha+\beta}\frac{1}{2}\theta \cdot \{ x(1 + \cos\theta) - i \sin\theta \}^n d\theta.$$

In particular, for the ultraspherical polynomial, (2) reduces to

$$(3) \qquad P_n^{(\alpha,\alpha)}(x) = \frac{2^{2\alpha-1}}{\pi} \frac{\Gamma(\alpha + n + 1)\Gamma(\alpha - n - 1)}{n\,!\,\Gamma(2\alpha + n + 1)}$$

$$\cdot \int_{-\pi}^{\pi} \cos^{2\alpha}\frac{1}{2}\theta \{ x(1 + \cos\theta) + i \sin\theta \}^n d\theta.$$

Therefore for the LEGENDRE polynomial

$$(4) \qquad P_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{ x(1 + \cos\theta) + i \sin\theta \}^n d\theta =$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \{ x(1 + \cos\theta) + i \sin\theta \}^n d\theta,$$

a formula due to CATALAN [1].

REFERENCES

[1] E. CATALAN, *Nouvelles proprietés des fonctions X*, « Mémoires de l'Academie Royale des Sciences », des lettres, et des beaux-arts de Belgique, vol. 47 (1889) pp. 3-24.

[2] E. T. WHITTAKER and G. N. WATSON, *A course of modern analysis*, fourth edition, Cambridge, 1927.