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BREVI NOTE

Arithmetical Functions of a Greatest Common Divisor, II. An Alternative Approach

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Summary. - *This paper is concerned with the average order of arithmetical functions* f((m, n)) *where* f *is of divisor or totient type. The methods are elementary, and some of the results obtained refine previously proved estimates.*

1. INTRODUCTION. – Let $g(n)$ be a (complex-valued) arithmetical function, α a real number, and let the greatest common divisor of positive integers m, n be denoted by (m, n). In this paper, as in I[3], we consider the average order of the function $f_\alpha((m, n))$, where

$$f_\alpha(n) = \sum_{d\delta=n} g(d)\delta^\alpha, \tag{1.1}$$

$\alpha \geqq 1$, and $g(n)$ is bounded. Note, in the cases $g(n) = 1$ and $g(n) = \mu(n)$ where $\mu(n)$ is the Möbius function, that $f_\alpha(n)$ reduces respectively to the basic divisor and totient functions, $\sigma_\alpha(n)$ and $\varphi_\alpha(n)$. For simplicity we write $\sigma_1(n) = \sigma(n)$, $\varphi_1(n)$, $\sigma_0(n) = \tau(n)$.

The method of I was based on a lemma which reduced the problem under discussion to an analogous problem for functions in a single variable. The key lemma of the present paper is a result (Lemma 4.1) which makes it possible to determine the average order of $f_\alpha((m, n))$ on the basis of that of $\sigma_\alpha((m, n))$. The main result is contained in Theorem 4.1. The result obtained for $\sigma_\alpha((m, n))$ is given by Theorem 3.1, while that for $\varphi_\alpha((m, n)$ appears in Corollary 4.1. The treatement of $\sigma_\alpha((m, n))$ is along the lines of Césaro's discussion of the special cases, $\alpha = 1$ and $\alpha = 2$ [2]. Our results are, however, more precise than those obtained by

(*) Pervenuta alla Segreteria dell'U. M. I. il 20 giugno 1962.

Césaro, who failed to consider the remainder term. In this connection we note that the order of the remainder obtained in I in the case $\alpha = 1$, $O(x^{3/2}\log x)$, is improved in the present paper to $O(x^{3/2})$. While the method of the present paper is quite different from the treatment employed in *I*, both papers are entirely elemen- in nature.

2. Preliminary lemmas. - If $x \geqq 1$, we define the summatory function $H(x)$ of a function $h(n)$ and the summatory function $H^*(x)$ of $h((m, n))$ by

$$H(x) = \sum_{n \leqq x} h(n), \qquad H^*(x) = \sum_{m,n \leqq x} h((m, n)).$$

Similarly let $G(x)$ and $G^*(x)$ represent the summatory functions of $g(n)$ and $g((m, n))$. Also place

$$f(n) = \sum_{d\delta = n} g(d)h(\delta), \qquad k(n) = \sum_{d \mid n} h(d). \tag{2.1}$$

We recall the classical identities [6, (19), (27)],

$$\sum_{n \leqq x} k(n) = \sum_{n \leqq x} h(n)\left[\frac{x}{n}\right], \tag{2.2}$$

$$\sum_{n \leqq x} k(n) = \sum_{n \leqq \delta_1} H\left(\frac{x}{n}\right) + \sum_{n \leqq \delta_2} h(n)\left[\frac{x}{n}\right] - [\delta_1]H(\delta_2), \tag{2.3}$$

where $[x]$ denotes the largest integer $\leqq x$, and

$$\delta_1\delta_2 = x, \qquad 0 < \delta_1,\ \delta_2 \leqq x. \tag{2.4}$$

These identities are special cases $(g(n) = 1)$ of the relations [3, (2.2) and (2.3)],

$$\sum_{n \leqq x} f(n) = \sum_{n \leqq x} g(n)H\left(\frac{x}{n}\right) = \sum_{n \leqq x} h(n)G\left(\frac{x}{n}\right), \tag{2.5}$$

$$\sum_{n \leqq x} f(n) = \sum_{n \leqq \delta_1} g(n)H\left(\frac{x}{n}\right) = \sum_{n \leqq \delta_2} h(n)G\left(\frac{x}{n}\right) - G(\delta_1)H(\delta_2). \tag{2.6}$$

We first prove the following two-dimensional analogue of (2.5).

Lemma 2.1. - *If $f(n)$ is defined by* (2.1), *then*

$$\sum_{a, b \leqq x} f((a, b)) = \sum_{n \leqq x} g(n)H^*\left(\frac{x}{n}\right) = \sum_{n \leqq x} h(n)G^*\left(\frac{x}{n}\right). \tag{2.7}$$

PROOF. - It is sufficient to prove the equality of the first and third members of (2.7). We have

$$\sum_{a,b\leq x} f((a,b)) = \sum_{a,b\leq x}\ \sum_{d\delta=(a,b)} g(d)h(\delta) = \sum_{\substack{a,b\leq x\\ d\delta=(a,b)\\ (\delta d_1=a,\ \delta d_2=b)}} g(d)h(\delta)$$

$$= \sum_{\substack{\delta d_1,\ \delta d_2\leq x\\ d=(d_1,d_2)}} g(d)h(\delta) = \sum_{\delta\leq x} h(\delta) \sum_{d_1,d_2\leq \frac{x}{\delta}} g((d_1,\ d_2)) = \sum_{\delta\leq x} h(\delta)G^*\left(\frac{x}{\delta}\right).$$

In case $g(n)=1$, Lemma 2.1 yelds

LEMMA. 2.2. - *If* k(n) *is defined as in* (2.1), *then*

$$\sum_{a,b\leq x} k((a,\ b)) = \sum_{n\leq x} h(n)\left[\frac{x}{n}\right]^2 \tag{2.8}$$

From this result we can deduce another relation for $k((m,\ n))$ due to Césaro [2].

LEMMA 2.3 - *If* k(n) *is defined as in* (2.1), *then*

$$\sum_{a,b\leq x} k((a,b)) = \sum_{n\leq x} (2n-1)H\left(\frac{x}{n}\right). \tag{2.9}$$

PROOF. - By (2.8)

$$\sum_{a,b\leq x} h((a,\ b)) = \sum_{n\leq x} h(n) \sum_{a,b\leq \frac{x}{n}} 1 = \sum_{a,b\leq x}\ \sum_{n\leq \min\left(\frac{x}{a},\frac{x}{b}\right)} h(n)$$

$$= \sum_{\substack{a\leq x\\ b\leq a}} H\left(\frac{x}{a}\right) + \sum_{\substack{b\leq x\\ a\leq b}} H\left(\frac{x}{b}\right) - \sum_{\substack{n\leq x\\ (a=b=n)}} H\left(\frac{x}{n}\right) = 2\sum_{n\leq x} nH\left(\frac{x}{n}\right) - \sum_{n\leq x} H\left(\frac{x}{n}\right),$$

from which (2.9) results.

Finally, we prove the following analogue of (2.5) for functions of two variables.

LEMMA 2.4. - (cf. Césaro [2, (7)]). *If* δ_1, δ_2 *satisfy* (2.4), *then*

$$\sum_{a,b\leq x} k((a,\ b)) = \sum_{n\leq x} (2n-1)H\left(\frac{x}{-}\right) + \sum h(n)\left[\frac{x}{-}\right]^2 - [\text{illegible}]. \tag{2.10}$$

PROOF. – We apply (2.5) and (2.6) to the right member of (2.9) obtaining, with $g(n) = 2n - 1$,

$$\sum_{a,b \leq x} k((a, b)) = \sum_{n \leq \delta_1} (2n-1) H\left(\frac{x}{n}\right) + \sum_{n \leq \delta_2} h(n) \sum_{m \leq \frac{x}{n}} (2m-1) -$$

$$- H(\delta_2) \sum_{n \leq \delta_1} (2n - 1)$$

and (2.10) follows on summing arithmetical progressions.

3. THE AVERAGE ORDER OF $\sigma_\alpha((m, n))$. – We first list several simple estimates that will be needed in this and the following section,

$$\sum_{n>x} \frac{1}{n^\alpha} = O\left(\frac{1}{x^{\alpha-1}}\right) \qquad \text{if } \alpha > 1; \tag{3.1}$$

$$\sum_{n>x} \frac{\log n}{n^\alpha} = O\left(\frac{\log x}{x^{\alpha-1}}\right) \qquad \text{if } \alpha > 1; \tag{3.2}$$

$$\sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + \begin{cases} O(x^\alpha) & \text{if } \alpha > 0 \\ O(1) & \text{if } -1 < \alpha < 0; \end{cases} \tag{3.3}$$

$$\sum_{n \leq x} \frac{1}{n} = \log x + \nu + O\left(\frac{1}{x}\right), \tag{3.4}$$

where ν denotes Euler's constant. We use $\zeta(s)$ to denote the Riemann zeta-function.

THEOREM 3.1. – (Cf. [3, Corollary 3.1]). *If* $\alpha > 1$, *then*

$$\sum_{a,\, b \leq x} \sigma_\alpha((a, b)) = \frac{x^{\alpha+1}}{\alpha+1}\left(2\zeta(\alpha) - \zeta(\alpha+1)\right) + O(E_\alpha(x)), \tag{3.5}$$

where

$$E_\alpha(x) = \begin{cases} x^\alpha & (\alpha > 2) \\ x^2 \log x & (\alpha = 2) \\ x^2 & (1 < \alpha < 2); \end{cases} \tag{3.6}$$

$$\sum_{a,b \leq x} \sigma((a, b)) = x^2\left(\log x + 2\nu - \frac{1}{2} - \frac{\zeta(2)}{2}\right) + O\left(x^{3/2}\right). \tag{3.7}$$

PROOF. CASE 1 ($\alpha > 1$). - In this case we have, by Lemma 2.3 (with $h(n) = n^{\alpha}$) and (3.3),

$$(3.8) \quad \sum_{a,b \leq x} \sigma_{\alpha}((a, b)) = \sum_{n \leq x} (2n - 1) \sum_{d \leq \frac{x}{n}} d^{\alpha}$$

$$= \frac{x^{\alpha+1}}{\alpha + 1}\left(2 \sum_{n \leq x} \frac{1}{n^{\alpha}} - \sum_{n \leq x} \frac{1}{n^{\alpha+1}}\right) + O\left(x^{\alpha} \sum_{n \leq x} \frac{2n-1}{n^{\alpha}}\right)$$

$$= \frac{x^{\alpha+1}}{\alpha + 1}\left(2\zeta(\alpha) - \zeta(\alpha + 1)\right) + O\left(x^{\alpha+1} \sum_{n>x} \frac{1}{n^{\alpha}}\right)$$

$$+ O\left(x^{\alpha} \sum_{n \leq x} \frac{1}{n^{\alpha-1}}\right).$$

so that, by (3.1), (3.3), and (3.4),

$$\sum_{a,b \leq x} \sigma_{\alpha}((a, b)) = \frac{x^{\alpha+1}}{\alpha + 1}\left(2\zeta(\alpha) - \zeta(\alpha + 1)\right) + O(x^2) + O(E_{\alpha}(x)),$$

which proves (3.5).

CASE 2 ($\alpha = 1$). - In (2.10) we place $h(n) = n$, $z = \delta_1 = \delta_2 = \sqrt{x}$ to obtain, with a slight computation omitted

$$(3.9) \quad \sum_{a,b \leq x} \sigma((a, b)) = \sum_{n \leq z} \left(2n - \frac{1}{2}\right)\left[\frac{x}{n}\right]^2 + \sum_{n \leq z} \left(n - \frac{1}{2}\right)\left[\frac{x}{n}\right] - \frac{[z]^4}{2} - \frac{[z]^3}{2};$$

hence

$$\sum_{a,b \leq x} \sigma((a, b)) = \sum_{n \leq x^{1/2}} \left(2n - \frac{1}{2}\right)\left(\frac{x}{n} + O(1)\right)^2 - \frac{1}{2}\left(x^{1/2} + O(1)\right)^4 + O(x^{3/2})$$

$$= x^2 \sum_{n \leq x^{1/2}} \left(2n - \frac{1}{2}\right)\frac{1}{n^2} - \frac{x^2}{2} + O(x^{3/2})$$

$$= 2x^2 \sum_{n \leq x^{1/2}} \frac{1}{n} - \frac{x^2}{2}\zeta(2) - \frac{x^2}{2} + O\left(x^2 \sum_{n > x^{1/2}} \frac{1}{n^2}\right) + O(x^{3/2}),$$

and (3.7) results, by virtue of (3.4) and (3.1). This completes the proof of Theorem 3.1

4. The general case. – In this section we define $q(n)$ to be the « arithmetical derivative » of $g(n)$,

$$q(n) = \sum_{d\delta=n} g(d)\mu(\delta). \tag{4.1}$$

Correspondingly, $k(n)$ as defined in (2.1) in the « arithmetical integral » of $h(n)$. We now prove

Lemma 4.1. – *If* f(n) *and* k(n) *are defined as in* (2.1), *then*

$$\sum_{a, b \leq x} f((a, b)) = \sum_{n \leq x} q(n) K^*\left(\frac{x}{n}\right), \tag{4.2}$$

where $K^*(x)$ *is the summatory function of* $k((m, n))$.

Proof. – We recall the classical relation between arithmetical derivatives and integrals ([4, Chapter X, (13)], [1]),

$$f(n) = \sum_{d\delta=n} g(d)h(\delta) = \sum_{d\delta=n} q(d)k(\delta);$$

the lemma results on applying Lemma 2.1.

We shall also need the estimate,

Lemma 4.2. – *If* g(n) *is bounded, then* $q(n) = O(n^\varepsilon)$ *for all* $\varepsilon > 0$.

Proof. – By (4.1) and the boundedness of $\mu(n)$ and $g(n)$, it follows that $q(n) = O(\tau(n))$; hence the lemma is a consequence of the familiar estimate, $\tau(n) = O(n^\varepsilon)$, [5, Theorem 315, p. 260].

We place, as in *I*,

$$L(s, g) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \qquad s > 1,$$

and denote the derivative of $L(s, g)$ by $L'(s, g)$. It is recalled that

$$\zeta^{-1}(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s} \qquad (s > 1),$$

so that

$$L(s, q) = \frac{L(s, g)}{\zeta(s)}, \qquad L'(s, q) = \frac{L'(s, g)}{\zeta(s)} - \frac{L(s, g)\zeta'(s)}{\zeta^2(s)}, \qquad s > 1. \tag{4.3}$$

Our principal result is contained in

THEOREM 4.1 - (cf. [3, Theorem, § 3]). *Suppose* g(n) *to be bounded. then if* $\alpha > 1$,

$$\sum_{a,\, b \leq x} f_\alpha((a, b)) = \frac{L(\alpha+1, g)}{(\alpha+1)\zeta(\alpha+1)} (2\zeta(\alpha) - \zeta(\alpha+1)) x^{\alpha+1} + O(E_\alpha(x)), \tag{4.4}$$

where $E_\alpha(x)$ is defined by (3.6); in case $\alpha = 1$,

$$\sum_{a,\, b \leq x} f_1((a, b)) = \frac{x^2}{\zeta(2)} \left\{ L(2, g) \left(\log x + 2\nu - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + L'(2, g) \right\} + O(x^{3/2}). \tag{4.5}$$

PROOF. - Denote by $S_\alpha^*(x)$ the summatory function of $\sigma_\alpha((m, n))$. Then by Lemma 4.1, with $h(n) = n^\alpha$,

$$\sum_{a,\, b \leq x} f_\alpha((a, b)) = \sum_{n \leq x} q(n) S_\alpha^* \left(\frac{x}{n}\right). \tag{4.6}$$

We now apply the result obtained in Theorem 3.1 for $S_\alpha^*(x)$, considering separately the two cases $\alpha > 1$ and $\alpha = 1$.

CASE 1 ($\alpha > 1$). - In this case let ϵ represent any real number satisfying $0 < \epsilon < 1$. Also place $c_1 = (\alpha+1)^{-1}(2\zeta(\alpha) - \zeta(\alpha+1))$. We have then by (4.6), (3.5), and Lemma (4.2),

$$\sum_{a,\, b \leq x} f_\alpha((a, b)) = c_1 x^{\alpha+1} \sum_{n \leq x} \frac{q(n)}{n^{\alpha+1}} + O\left(\sum_{n \leq x} |\, q(n) \,|\, E_\alpha\left(\frac{x}{n}\right) \right) =$$

$$= c_1 x^{\alpha+1} L(\alpha+1, q) + 0\left(x^{\alpha+1} \sum_{n \leq x} \frac{1}{n^{\alpha+1-\epsilon}} \right) + O\left(\sum_{n \leq x/2} n^\epsilon E_\alpha\left(\frac{x}{n}\right) \right) +$$

$$+ O\left(\sum_{n \leq x} n^\epsilon \right).$$

It follows readily, using (3.1) and (3.6), that

$$\sum_{a,\, b \leq x} f_\alpha((a, b)) = c_1 x^{\alpha+1} L(\alpha+1, q) + O(x^{1+\epsilon}) + O(E_\alpha(x)),$$

and (4.4) results by (4.3).

CASE 2 ($\alpha = 1$). - In this case let ϵ denote a real number, $0 < \epsilon < \frac{1}{2}$; also place $c_2 = 2\nu = (\zeta(2) + 1)/2$. On the basis of (4.6) with $\alpha = 1$, (3.7), and Lemma 4.2, it follows that

$$\sum_{a,\, b \leqq x} f_1((a, b) = (x^2 \log x + c_2) \sum_{n \leqq x} \frac{q(n)}{n^2} - x^2 \sum_{n \leqq x} \frac{q(n) \log n}{n^2} +$$

$$+ O\left(x^{3/2} \sum_{n \leqq x} \frac{|\, q(n) \,|}{n^{3/2}}\right) = (x^2 \log x + c_2) L(2, q) + x^2 L'(2, q) +$$

$$+ O\left(x^2 \log x \sum_{n > x} \frac{1}{n^{2-\epsilon}}\right) + O\left(x^2 \sum_{n \leqq x} \frac{\log n}{n^{2-\epsilon}}\right) + O\left(x^{3/2} \sum_{n \leqq x} \frac{1}{n^{3/2-\epsilon}}\right).$$

Application of (3.1) and (3.2) yields then

$$\sum_{a,\, b \leqq x} f_1((a, b)) = (x^2 \log x + c_2) L(2, q) + x^2 L'(2, q) +$$

$$+ 0(x^{1+\epsilon} \log x) + O(x^{3/2}),$$

and (4.5) follows on the basis of (4.3). The theorem is proved.

The special case $q(n) = \mu(n)$ yields the following result for $\varphi_\alpha((m, n))$.

COROLLARY 4.1 - (cf. [3, Corollary 3.2]). *If* $\alpha > 1$, *then*

$$\text{(4.7)} \qquad \sum_{a,\, b \leqq x} \varphi_\alpha((a, b) = \frac{x^{\alpha+1}}{(\alpha + 1)\zeta^2(\alpha + 1)} (2\zeta(\alpha) - \zeta(\alpha + 1)) + O(E_\alpha(x)),$$

where $E_\alpha(x)$ *is defined by* (3.6);

$$\text{(4.8)} \qquad \sum_{a,\, b \leqq x} \varphi((a, b)) = \frac{x^2}{\zeta^2(2)} \left(\log x + 2\nu - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right) + O(x^{3/2}).$$

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