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# A note on complex elliptic integrals.

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# A note on complete elliptic integrals.

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**Summary.** - *In an earlier paper [1] we have derived certain new formulae relating to the complete elliptic integrals of the first and second kind, and then deduced four well-known properties of these integrals. Subsequently the question regarding the range of validity of some resultes has been raised. Here an attempt has been made to discuss this question satisfactorily.*

In [1] we have found the formulae:

$$(1)\qquad \frac{2}{\sqrt{\pi\alpha}}\, K\left(\sqrt{\frac{\lambda}{\alpha}}\right) = M\left[e^{-\alpha x} L_{-\frac{1}{2}}^{(0)}(\lambda x);\ \frac{1}{2}\right]$$

$$= L\left[x^{-\frac{1}{2}} L_{-\frac{1}{2}}^{(0)}(\lambda x);\ \alpha\right]$$

$$\frac{2}{\sqrt{\pi\alpha}}\, E\left(\sqrt{\frac{\lambda}{\alpha}}\right) = M\left[e^{-\alpha x} L_{\frac{1}{2}}^{(0)}(\lambda x);\ \frac{1}{2}\right]$$

$$= L\left[x^{-\frac{1}{2}} L_{\frac{1}{2}}^{(0)}(\lambda x);\ \alpha\right]$$

where

$$M[f(x);\ s] = \int_0^\infty f(x) x^{s-1} dx$$

and

$$L[f(x);\ s] = \int_0^\infty e^{-sx} f(x) dx.$$

We ask now: Have we found the images of the complete elliptic integrals of the first and second kind for MELLIN and LAPLACE transforms? In answer to this question we say that we have found

(*) Pervenuta alla Segreteria dell' U. M. I. il 30 agosto 1962.

the complete elliptic integrals of the first and second kind as the images or transforms of suitable MELLIN and LAPLACE transformations.

Next in proving (1) and (2) we have started from the integral formula [2]:

$$\int_0^\infty e^{-\alpha x}\Phi(\beta,\ \varphi;\ \lambda x)x^{s-1}dx = \alpha^{-s}\Gamma(s)F(\beta,\ s;\ \varphi;\ \lambda\alpha^{-1}); \tag{3}$$

which is valid if $\mathrm{Re}\, s > 0$, $\mathrm{Re}\,\alpha > \mathrm{Max}\,(0,\ \mathrm{Re}\,\lambda)$, and $|\alpha| > |\lambda|$.

Thus for real values of $s$, $\alpha$ and $\lambda$, we observe that (3) holds $s > 0$, $\alpha > |\lambda|$.

Using (3) we have derived the formulae:

$$K\left(\sqrt{\frac{\lambda}{\alpha}}\right) = \frac{1}{2}\sqrt{\pi\alpha}\int_0^\infty e^{-\alpha x}L_{-\frac{1}{2}}^{(0)}(\lambda x)x^{-\frac{1}{2}}\,dx \tag{4}$$

$$E\left(\sqrt{\frac{\lambda}{\alpha}}\right) = \frac{1}{2}\sqrt{\pi\alpha}\int_0^\infty e^{-\alpha x}L_{\frac{1}{2}}^{(0)}(\lambda x)x^{-\frac{1}{2}}\,dx \tag{5}$$

We now observe that (4) and (5) are valid when $\alpha > |\lambda|$.

From (4) and (5) we have also found the following formulae:

$$K(k) = \frac{1}{2}\sqrt{\pi}\int_0^\infty e^{-x}L_{-\frac{1}{2}}^{(0)}(k^2x)x^{-\frac{1}{2}}\,dx \tag{6}$$

$$E(k) = \frac{1}{2}\sqrt{\pi}\int_0^\infty e^{-x}L_{\frac{1}{2}}^{(0)}(k^2x)x^{-\frac{1}{2}}\,dx \tag{7}$$

each of (6) and (7) holds when $|k| < 1$.

$$\frac{2}{\alpha\sqrt{\pi}}\,K\left(i\frac{\lambda}{\alpha}\right) = \int_0^\infty e^{-\alpha^2 x}\,L_{-\frac{1}{2}}^{(0)}(-\lambda^2 x)x^{-\frac{1}{2}}\,dx \tag{8}$$

$$\frac{2}{\alpha\sqrt{\pi}}\,E\left(i\frac{\lambda}{\alpha}\right)=\int_0^\infty e^{-\alpha^2 x}\,L_{\frac{1}{2}}^{(0)}(-\lambda^2 x)x^{-\frac{1}{2}}\,dx \tag{9}$$

each of (8) and (9) holds when $\alpha > |\lambda|$.

Further we have found that

$$K\left(i\frac{\lambda}{\alpha}\right)=\alpha K(\lambda), \quad \text{where} \quad \alpha^2+\lambda^2=1. \tag{10}$$

$$E\left(i\frac{\lambda}{\alpha}\right)=\frac{1}{\alpha}E(\lambda), \quad \text{where} \quad \alpha^2+\lambda^2=1. \tag{11}$$

We remark here that each of (10) and (11) holds when $\alpha>|\lambda|$.

Lastly we have deduced the known formulae:

$$\frac{d}{dk}E(k)=\frac{1}{kk'^2}E(k)-\frac{1}{k}\cdot K(k) \quad \text{where} \quad k^2+k'^2=1 \tag{12}$$

$$\frac{d}{dk}K(k)=\frac{1}{k}[E(k)-K(k)]. \tag{13}$$

In this connection we mention that the notations $F(a+1)$ and $F(b+1)$, which we have employed in [1], are well-known notations of [2, p. 103]. Again the notation $\Phi(a, c; x)$ and $F(a, b; c; x)$, which are used in [1], for ${}_1F_1(a; c; x)$ and ${}_2F_1(a, b; c; x)$ respectively, are well-known notations of [2, pp 248, 182, 269]. $\Phi(a, c; x)$ is Humbert's symbol. Thus uses of these notations or symbols are proper.

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## REFERENCES

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[3] Math. Reviews, Vol. 23, 1962. pag. 341, # A 1849.