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T. Pati, M. S. Ramanujan

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On iteration products preserving absolute convergence

by T. Pati (Allahabad, India) and M. S. Ramanujan
(Madras, India) (*)

Summary. - *Given two summation processes A and B, we shall write $|A| \subset |AB|$ if every sequence (or function), belonging to a suitable class, and absolutely A-summable has the property that its B-transform is also absolutely A-summable. In this paper we discuss certain pairs of methods A, B with the above property.*

1. Introduction.

The question as to when A-summability of a sequence would imply its AB-summability, where AB denotes the usual interation product, has been discussed for various pairs A, B by several authors (see [2] thru [13]). The analogous question for absolute summability is discussed here. Each section in the sequel contains at its start the relevant definitions.

2. The product of $[J, f]$ and Hausdorff transformations:

Definitions.

The $[J, f]$ transformations. Let $f(x)$ be a real or complex valued function defined for $x > x_0 \geq 0$ and possessing derivatives of all orders. The transform defined by

$$t(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^m}{m!} f^{(m)}(x) s_m$$

(for all sufficiently large x) is called the *$[J, f]$-transform* of the sequence $\{s_n\}$. $\{s_n\}$ is defined to be *$[J, f]$-summable* to s if $\lim_{x \to \infty} t(x) = s$. Analogously we define a sequence $\{s_n\}$ to be absolutely summable $[J, f]$ or simply *$|J, f|$-summable* if $t(x) \in BV\,(h, \infty)$, where h is some finite positive constant.

As shown by Jakimovski [2], the well-known Abel and Borel methods are special cases of the $[J, f]$ method. Also, it has been established by Jakimovski that the $[J, f]$ transformation is convergence-preserving if and only if there exists a function $\alpha(t)$ of

(*) Pervenuta alla Segreteria dell'U.M.I. il 12-11-61.

bounded variation in [0, 1] such that

$$f(x)=\int_0^1 t^x d\alpha(t).$$

The familiar Hausdorff or (H, μ_n) – transform of a sequence is defined by the matrix $\lambda=(\lambda_{n\nu})$, where

$$\lambda_{n\nu}=\binom{n}{\nu}\Delta^{n-\nu}\mu_\nu,\ (n\geq\nu),\ \text{and}=0,\ (n<\nu).$$

This method is known to be regular if and only if μ_n is a regular moment constant, i.e. it has the representation

$$\mu_n=\int_0^1 t^n d\chi(t),\quad n=0,\ 1,\ 2,\dots,$$

where $\chi(t)$ is a function of bounded variation over [0,1] and $\chi(0)=\chi(+0)=0$ and $\chi(1)=1$. (For further details see HARDY [1]).

We shall call the transform $\{\sigma_n\}$ of the sequence $\{s_n\}$, defined by

$$\sigma_n=\sum_\nu \lambda_{n\nu}s_\nu,$$

the Hausdorff or (H,μ_n) — transform of the sequence $\{s_n\}$.

We are now in a position to prove the following theorem.

THEOREM 1. *Let $[J, f]$ be a convergence-preserving transformation. If $\{s_n\}$ is a sequence such that*

$$i)\ \sum_{n=0}^{\infty}\frac{x^n}{n!}\ |\,f^{(n)}(x)\,|\ s_n^*,\ (s_n^*=\max_{m\leq n}|\,s_m\,|),\ \text{is}$$

convergent for $x>0$

and ii) the $[J, f]$ — transform of the sequence $\{s_n\}$, denoted by

$$A(x,\ s),\quad\in BV_x[0,\ \infty],$$

then any regular (H, μ_n) — transform of $\{s_n\}$ is also summable $|J, f|$.

PROOF. Under the condition $i)$ it follows after a brief calculation (which we suppress here for brevity and borrow from

JAKIMOVSKI [2], that

$$A(x, t) = \int_0^1 A(xu, s) d\chi(u),$$

or equivalently,

$$F(x) = \int_0^1 g(xu) d\chi(u),$$

where $A(x, t)$ denotes the $[J, f]$ — transform of $\{t_n\}$ which is the (H, μ_n) — transform of $\{s_n\}$ and $A(x, s)$ denotes the $[J, f]$ — transform of $\{s_n\}$. Thus, by hypothesis, $g(x) \in BV[0, \infty]$ and we need to show that $F(x) \in BV[B, \infty]$.

For $x \in [0, X]$, where X is arbitrary, and any arbitrary partition $0 = x_0 < x_1 < \ldots < x_n = X$ of $[0, X]$, we have

$$| F(x_i) - F(x_{i-1}) | = | \int_0^1 [g(x_i u) - g(x_{i-1} u)] \, d\chi(u) |$$

$$\leq \int_0^1 | g(x_i u) - g(x_{i-1} u) | \, | d\chi(u)|,$$

so that

$$\sum_{i=1}^{n} | F(x_i) - F(x_{i-1}) | \leq \int_0^1 \left\{ \sum_{i=1}^{n} | g(x_i u) - g(x_{i-1} u) | \right\} | d\chi(u) |.$$

Since, by ii), $g(x) \in BV[0, \infty]$ and also since $xu \leq x$ for $0 \leq u \leq 1$, we get $Var_x [g(xu)] \leq Var_x [g(x)]$, uniformly, for each u in $0 \leq u \leq 1$, and both the variations being taken over the same range $[0, X]$. Now the conclusion follows from the bounded variation of $\chi(u)$ over $[0, 1]$, ensured by the regularity of the (H, μ) — method.

This completes the proof of the theorem.

A specialised form of Theorem 1 will be the following analogue of a known theorem of RAJAGOPAL'S ([8], Theorem 1). We omit the proof of the theorem.

THEOREM 2. *Let T be a regular sequence-to-sequence transfor-*

mation, defined by

$$T\{s_n,\ x\} = \frac{\sum\limits_{n=0}^{\infty} p_n x^n s_n}{\sum\limits_{n=0}^{\infty} p_n x^n}\ ,\ p_n \geq 0,\ 0 < x < \rho,$$

where $\rho(\leq \infty)$ *is the radius of convergence of the power series* $f(x) = \sum\limits_{n=0}^{\infty} p_n x^n$, *it being assumed that* $\Sigma p_n \rho^n = +\infty$, *and that for every* u *in* $0 < u < 1$,

$$\text{(i)} \qquad T_u\{s_n,\ x\} = \frac{1}{f(x)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x - xu)}{n!} x^n u^n s_n\,,\ (0 < x < \rho)$$

is absolutely convergent,

$$\text{(ii)} \qquad T_u\{\,|\, s_n\,|,\ x\} \leq F(x),\quad F(x) \in BV(0,\ \rho)$$

and

$$\text{(iii)} \qquad T_u\{s_n\,, x\} \in BV_r(0,\ \rho)$$

uniformly in $0 < u < 1$

Then for every sequence $\{s_n\}$ *which is* $|\,T\,|$ *— summable its transform by a regular* $(H,\ \mu_n)$ *method is also* $|\,T\,|$ *— summable.*

3. In this section we consider the iteration product of the $[K,\ c_n]$ — summability method of Jakimovski [4] with the quasi-Hausdorff transformations.

Definitions. Given a sequence $\{c_n\}$, the $[K,\ c_n]$ *transform* $t(x)$ of a sequence $\{s_n\}$ is defined by

$$t(x) = x^{-1} \sum_{n=0}^{\infty} s_n \sum_{m=0}^{n} (-1)^m \binom{n}{m} c_m x^{-m}.\ \ x > x_0 > 0.$$

The sequence $\{s_n\}$ is said to be *summable* $[K,\ c_n]$ to s if $\lim_{x \to \infty} t(x) = s$, and by analogy we say that $\{s_n\}$ is *summable* $|K,\ c_n|$ if $t(x) \in BV(k,\ \infty)$, where k is some fixed positive number.

JAKIMOVSKI [3] has shown that $[K, c_n]$ is conservative if

$$c_n = \int_0^1 t^{n+1} d\alpha(t), \qquad n \geq 0,$$

where $\alpha(t) \in BV[0, 1]$.

The quasi-HAUSDORFF or (H^*, μ_n)-transform of a sequence is defined by

$$\sigma_n^* = \sum_{m=n}^{\infty} \binom{m}{n} \Delta^{m-n} \mu_n \, s_m$$

and this is known to be a conservative transformation if and only if

$$\mu_n = \int_0^1 u^{n+1} d\chi(u),$$

where $\chi(u)$ is of of bounded variation in $[0, 1]$. This result is due to RAMANUJAN [9]

We now prove

THEOREM 3. *Let $[K, c_n]$ be a conservative transformation. If the sequence $\{s_n\}$ is bounded and summable $|K, c_n|$, then its transform σ_n^* by any conservative quasi-Hausdorff method (H^*, μ_n) is also summable $|K, c_n|$.*

PROOF. Let σ_n^* denote the (H^*, μ_n) — transform of the sequence $\{s_n\}$ and $G(x)$ the $[K, c_n]$ — transform of $\{\sigma_n^*\}$. Following the analysis of JAKIMOVKI ([3], Theorem 4. 1) we obtain

$$G(x) = \int_0^1 g(x/u) d\chi(u).$$

Now, given that $g(x) \in BV[A, \infty]$, we have to show that $G(x) \in$ $\in BV[B, \infty]$. Since $g(x) \in BV[A, X]$ for each X. given an arbitrarily small positive ε, there exists a number x', such that $\operatorname{var} g(x) \leq \varepsilon$ for $x \geq x'$; that is, in each interval $[x', X]$, $\operatorname{var} g(x) < \varepsilon$.

Let us fix $\delta > 0$ such that $x'\delta = A$: evidently $0 < \delta < 1$.

Let $x \in [A, X]$, $X > A$ arbitrary. Then for u in $[0, \delta]$ we have

$x/u \geq x/\delta \geq A/\delta = x'$ and therefore

$$\operatorname{var}_x \int_0^\delta g(x/u)d\chi(u) \leq N,$$

where N is finite, independent of δ and in fact as small as we like. Also, for u in $[\delta, 1]$ and x in $[A, X]$, we have $A \leq x/u \leq x/\delta = X$, say, and therefore

$$\operatorname{var}_x \int_\delta^1 g(x/u)d\chi(u) \leq M$$

since $g(x/u) \in BV[A, X]$ for each X.

This completes the proof of the theorem.

Remark. The condition of boundedness on the sequence $\{s_n\}$ in the hypothesis of the theorem above may be relaxed to an apparently weaker restriction, as indicated by Jakimovski ([3], Theorem 4 2).

4. In this last section we give results analogous to those in Theorems 1 and 3, for the products of a Wienerian transformation with a continuous function-to-function Hausdorff transformation and with the $[M, \alpha(u)]$ — transformation, defined by Jakimovski [4], as a function-to-function analogue of the quasi-Hausdorff methods. We start with the relevant definitions.

The function-to-function *Wienerian transformation* of any function $s(x)$ is given by

$$T_s(t) = t^{-1} \int_0^\infty \Psi(x/t)s(x)dx,$$

where $\Psi(u)$ is a positive, monotonic decreasing function of u for $u \geq 0$ and the *T-summability* of $s(x)$ to s is defined by:

$$T_s(t) \rightarrow s \text{ as } t \rightarrow +\infty,$$

with the implication that, for every $t < \infty$, $T_s(t)$ is an absolutely convergent integral. The *summability* $|T|$ of $s(x)$ may be defined by: $T_s(t) \in BV[A, \infty]$, where A is some positive constant. Without any loss of generality it may be taken to be zero.

The function-to-function Hausdorff transformation of $s(x)$ is

$$\sigma_s(x) = \int_0^1 s(xu)d\chi(u),$$

where, with $\chi(u)$ as defined in the sequence-to-sequence method, the method becomes conservative or regular. (for further details, see HARDY [1]).

THE $[M, \alpha(u)]$ TRANSFORMATION. Let $\alpha(u)$ be defined in $[0, 1]$. The $[M,\alpha(u)]$ — *transform* $\sigma_s^*(x)$ of a given function $s(x)$ is given by

$$\sigma_s^*(x) = \int_0^1 s(x/u)d\alpha(u).$$

This method has been defined by JAKIMOVSKI [4] and he shows that $\alpha(u) \in BV[0, 1]$ is a sufficient condition for the method to be conservative. In what follows we assume that this condition of BV of $\alpha(u)$ is satisfied.

THEOREM 4. *For a given function $s(x)$ let its Wienerian transform $T_s(t)$ be of $BV[0, \infty]$. Then any regular function-to-function Hausdorff transformation of $s(x)$ is also summable $|T|$.*

PROOF. In this case

$$T_\sigma(t) = t^{-1} \int_0^\infty \Psi(x/t)\sigma_s(x)dx,$$

where

$$\sigma_s(x) = \int_0^1 s(xu)d\chi(u).$$

By a reasoning given in PATI [6], we have

$$T_\sigma(t) = \int^1 T_s(tu)d\chi(u).$$

Hence, proceeding as in the proof of Theorem 1, we get the result.
We conclude the paper with the demonstration of

THEOREM 5. *Let $[M, \alpha(u)]$ be a conservative transformation corresponding to the function $\alpha(u) \in BV[0, 1]$. Then for every bounded function $s(x)$ its $|T|$-summability implies the $|T|$-summability of its $[M, \alpha(u)]$-transform.*

PROOF. Using the notation already indicated, we have

$$\sigma^*(x) = \sigma_s^*(x) = \int_0^1 s(x/u)d\alpha(u),$$

so that

$$T_{\sigma^*}(t) = t^{-1}\int_0^\infty \Psi(x/t)\sigma_s^*(x)dx$$

$$= t^{-1}\int_0^\infty \Psi(x/t)\left[\int_1^\infty s(x/u)\, d\alpha(u)\right] dx$$

$$= t^{-1}\int_0^1 \left[\int_0^\infty \Psi(x/t)\, s(x/u)\, dx\right] d\alpha(u),$$

(the inversion in the order of integration being validated by the absolute convergence of the double integral, guaranteed by the bounded nature of $s(x)$ and also by our choice of $[M, \alpha(u)]$ as one corresponding to the function $\alpha(u)$ assumed to be of bounded variation in $[0, 1]$). Thus, putting $x = uv$

$$T_{\sigma^*}(t) = \int_0^1 \left[\int_0^\infty \Psi(uv/t)\, s(v)(u/t)\, dv\right] d\alpha(u)$$

$$= \int_0^1 \left[\int_0^\infty (t/u)^{-1}\, \Psi(v/(t/u))s(v)dv\right] d\alpha(u)$$

$$= \int_0^1 T_s(t/u)\, d\alpha(u).$$

Now, proceeding with arguments similar to those employed in the proof of Theorem 3, we obtain the complete proof of the theorem.

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