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A note on asymptotic behavior of differential equations

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Summary. - *Using a representation of the solution of the Riccati equation in terms of a minimization operator, we derive some known asymptotic results for the solutions of* $u'' = (1 + f(t))u = 0$.

1. Introduction.

The asymptotic behavior of solutions of the equation

$$u'' - (1 + f(t))u = 0 \tag{1.1}$$

has been extensively investigated; see [1]. It is of interest, however, to present a new method which is quite easy to apply and which seems capable of yielding a number of useful results.

2. The associated Riccati equation.

Setting $u'/u = v$, we obtain the RICCATI equation

$$v' + v^2 - 1 - f(t) = 0, \qquad v(0) = v_0. \tag{2.1}$$

We wish to show that $v \longrightarrow 1$ as $t \longrightarrow \infty$, if $f(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and $v(0) > 0$. Let us suppose that $1 + f(t) \geq \delta > 0$ for $t \geq 0$. It follows that v cannot be negative.

3. Quasilinearization.

Using the representation

$$-v^2 = \min_w (w^2 - 2wv), \tag{3.1}$$

we obtain from (2.1) the relation

$$v' = \min_w [w^2 - 2wv + (1 + f(t))]. \tag{3.2}$$

As demonstrated in [2], we can solve the equation

$$v' = w^2 - 2wv + (1 + f(t)), \qquad v(0) = v_0, \tag{3.3}$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 16 dicembre 1962.

for w an arbitrary function of t, and then minimize over w. Thus we have the representation

$$(3.4)\quad v = \min_{w} \left[v_0 e^{-2\int_0^t w ds} + e^{-2\int_0^t w ds} \cdot \left[\int_0^t e^{2\int_0^{t_1} w ds} [w^2 + (1 + f(t_1))] dt_1 \right] \right].$$

Hence for all $w(t)$,

$$(3.5)\quad v \geq \left[v_0 e^{-2\int_0^t w ds} + e^{-2\int_0^t w ds} \left[\int_0^t e^{-2\int_0^t w ds} [w^2 + (1 + f(t_1))] dt_1 \right] \right].$$

4. First inequality.

Setting $w = 1$, for $t \geq 0$, we obtain the inequality

$$(4.1)\quad v \leq \left[v_0 e^{-2t} + e^{-2t} \int_0^t [e^{2t_1}(2 + f(t_1))] dt_1 \right]$$

$$\leq v_0 e^{-2t} + (1 - e^{-2t}) + e^{-2t} \int_0^t e^{2t_1} f(t_1) dt_1 \leq 1 + o(1)$$

as $t \to \infty$.

5. Second inequality.

To obtain an inequality in the other direction, we return to (2.1) and set $v = 1/z$, [3]. The equation for z is

$$z' + (1 + f(t))z^2 - 1 = 0, \qquad z(0) = \frac{1}{v_0}.$$

Following the foregoing procedure, we have

$$(5.2)\quad v \leq \frac{e^{-2\int_0^t (1+f(t_1)) w(t_1)}}{v_0} + e^{-2\int_0^t (1+f(t_1)) w(t_1)}.$$

$$\cdot \left[\int_0^t [(1 + f(t_1))w^2 + 1] e^{2\int_0 (1+f(s))w(s)ds} dt_1 \right]$$

for all w. Setting $w = 1$ for $t \geq 0$, we have

$$(5.3) \qquad z \leq \frac{e^{-2\int_0^t (1+f(t_1))dt_1}}{v_0} + \left[1 - e^{-2\int_0 (1+f(t_1))dt_1} \right] -$$

$$- e^{-2\int_0 (1+f(t_1))dt_1} \int_0^t f(t_1) e^{2\int_0 (1+f(t_2))dt_2} dt_1 .$$

Hence, if $f(t) \longrightarrow 0$,

$$(5.4) \qquad z \leq 1 + o(1).$$

6. Discussion.

It is clear that we can obtain more precise estimates under fnrther conditions on $f(t)$ by using better estimates for w. For example, we might use $w = 1 + f(t)/2$. Furthermore, the method is applicable to the study of more general equation such as

$$(6.1) \qquad v' = g(v) + h(t)$$

and to the study of matrix RICCATI equations.

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