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On the index of nilpotency of some nil algebras

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Summary. - *According to G. Higman's proof* [1] *of M. Nagata's conjecture* [2], *if $x^2 = 0$, for every element x of an (linear associative) algebra $\mathcal{A}$ over a field of characteristic p, then the index of nilpotency of $\mathcal{A}$ is ≤ 3, provided $p > 2$ (including $p = \infty$).*

Below, we prove that if $x^2 = 0$, for every element x of an algebra $\mathcal{A}$ over a field of characteristic 2, then the index of nilpotency N of $\mathcal{A}$ is $\leq m$, provided dim $\mathcal{A}$ (dimension of $\mathcal{A}$) is $< 2^m - 1$. Moreover, we show that the upper bound m of N is attained for every integer m (of course $m \geq 2$). Furthermore, we show that under the same hypothesis, there are non-nilpotent infinite dimensional algebras.

LEMMA 1. - *Let $\mathcal{A}$ be an algebra over a field $\mathcal{F}$ such that $x^2 = 0$, for every $x \in \mathcal{A}$. Let N denote the index of nilpotency of $\mathcal{A}$. Then*

$$\dim \mathcal{A} < 2^m - 1 \quad \textit{implies} \quad N \leq m$$

PROOF. - To prove the lemma, it is enough to show that if there exist m elements $x_1, x_2, \ldots, x_m$ of $\mathcal{A}$ such that

$$x_1 x_2 \ldots x_m \neq 0 \tag{1}$$

then

$$\dim \mathcal{A} \geq 2^m - 1$$

Thus, in what follows we assume (1).

Clearly, the hypothesis of the lemma implies that $\mathcal{A}$ is anticommutative, i.e., $xy = -yx$, for every two elements x and y of $\mathcal{A}$.

We order a subset S_i, $(i = 1, 2, \ldots, m)$, of $\{x_1, x_2, \ldots, x_m\}$ which has i elements according to the natural order of the

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subscripts of the elements of S_i. Moreover, for a given i, we order the set of all such subsets S_i according to the principle of first differences. Let S_{ij} $(j = 1, 2, \dots, \alpha_i)$ represent the $j-th$ subset S_i, where $\alpha_i = \binom{m}{i}$.

Let x_{ij} represent the product of all the elements of S_{ij} in the natural order of their subscripts. Obviously, there are $2^m - 1$ such products x_{ij}. We shall show that these products are linearly independent over $\mathfrak{F}$. To this end, it is enough to prove that if

$$(2) \qquad \sum_{i=1}^{m} \sum_{j=1}^{i} a_{ij} x_{ij} = 0, \quad a_{ij} \varepsilon \mathfrak{F}$$

then $a_{ij} = 0$, for every $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, \alpha_i$.

In view of the hypothesis of the lemma and the anticommutativity of $\mathfrak{A}$, multiplication of both sides of equality (2) by $x_1 \dots x_{s-1} x_{s+1} \dots x_m$ yields $a_{1s} x_{1s} = 0$, which in view of (1) implies $a_{1s} = 0$, for $s = 1, 2. \dots, \alpha_1$. Hence, (2) reduces to

$$(3) \qquad \sum_{i=2}^{m} \sum_{j=1}^{i} a_{ij} x_{ij} = 0,$$

Multiplication of both sides of equality (3) by

$$x_1 \dots x_{u-1} x_{u+1} \dots x_{v-1} x_{v+1} \dots x_m, \text{ yields } a_{2r} x_{2r} = 0$$

for $r = 1, 2, \dots, \alpha_2$. Continuing in this way, we derive

$$a_{m1} x_{m1} = 0$$

implying $a_{m1} = 0$. Thus, indeed in (2), $a_{ij} = 0$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, \alpha_i$, as desired.

In view of the above, the $2^m - 1$ products x_{ij} are linearly independent and hence $\dim \mathfrak{A} \geq 2^m - 1$.

Thus, Lemma 1 is proved.

In view of Lemma 1, we have

COROLLARY. - *If* $\dim \mathfrak{A} < 3$, *where* $x^2 = 0$, *for every* $x \varepsilon \mathfrak{A}$ *then the index of nilpotency of* $\mathfrak{A}$ *is equal to* 2.

Clearly, the result in the above Corollary also could not have been obtained from the abovementioned result of HIGMAN.

LEMMA 2. - *For every integer* $m \geq 2$, *there exists an algebra*

$\mathfrak{A}$ over a field of characteristic 2 such that

$$X^2 = 0, \text{ for every } X \varepsilon \mathfrak{A},$$

with

$$\dim \mathfrak{A} = 2^{m-1} - 1 \text{ and } N = m$$

where N *is the index of nilpotency of $\mathfrak{A}$.*

PROOF. - Consider the $m-1$ indeterminates $x_1, x_2, \ldots, x_{m-1}$ and let $\mathfrak{R}$ be the ring over $GF(2)$ of all polynomials $P(x_1, \ldots, x_{m-1})$ with zero constant term. Let $\mathfrak{Q}$ be the ideal of $\mathfrak{R}$ consisting of all polynomials $P(x_1, \ldots, x_{m-1})$ whose non-zero terms are at least of degree 2 in some x_i.

Take the quotient algebra $\mathfrak{R}/\mathfrak{Q}$ for $\mathfrak{A}$.

Now, if $P(x_1, \ldots, x_{m-1}) \varepsilon \mathfrak{R}$ then in view of the definition of $\mathfrak{Q}$ and the fact that $\mathfrak{R}$ is over $GF(2)$, we see at once that $P^2(x_1, \ldots, x_{m-1}) \varepsilon \mathfrak{Q}$. From this it follows that

$$X^2 = 0. \text{ for every } X \varepsilon \mathfrak{R}/\mathfrak{Q} \tag{4}$$

Let us denote the element $x_i + \mathfrak{Q}$ of $\mathfrak{R}/\mathfrak{Q}$ by X_i. From the definition of $\mathfrak{Q}$ and from (4) it follows that $\mathfrak{R}/\mathfrak{Q}$ is the algebra of all polynomials $Q(X_1, \ldots, X_{m-1})$ whose constant terms are zero and whose non-zero terms are of degree less than 2 in every X_i.

We claim that the $2^{m-1} - 1$ non-zero elements

$$X_1, X_2, \ldots, X_1X_2, X_1X_3, \ldots, X_1X_2 \ldots X_{m-1} \tag{5}$$

of $\mathfrak{R}/\mathfrak{Q}$ form a basis for $\mathfrak{R}/\mathfrak{Q}$. Clearly, every abovementioned polynomial $Q(X_1, \ldots, X_{m-1})$ is a linear combination over $GF(2)$ of the elements listed in (5). Moreover, no non-trivial linear combination of the elements listed in (5) can be equal to *0*. Thus indeed

$$\dim \mathfrak{R}/\mathfrak{A} = 2^{m-1} - 1.$$

Furthermore, every term of any product of m elements of $\mathfrak{R}/\mathfrak{Q}$ must contain X_i^2 for some i and hence by (4), every such product is equal to *0*. Consequently, the index of nilpotency N of $\mathfrak{A}$ is less than or equal to m. Finally, since $X_1X_2 \ldots X_{m-1} \neq 0$, we see that $N = m$.

Thus, Lemma 2 is proved.

LEMMA 3. – *There exists a non-nilpotent infinite dimensional algebra $\mathfrak{A}$ over a field of characteristic 2 such that*

$$X^2 = 0, \quad \textit{for every} \quad X \varepsilon \mathfrak{A}$$

PROOF. – Consider the infinitely many indeterminates $x_1, x_2, \ldots$ and let $\mathfrak{R}$ be the ring over $GF(2)$ of all polynomials $P(x_1, x_2, \ldots)$ with zero constant term. As in the case of the proof of Lemma 2, we construct the corresponding ideal $\mathfrak{Q}$ and we take the quotient algebra $\mathfrak{R}/\mathfrak{Q}$ for $\mathfrak{A}$. Clearly, again

$$X^2 = 0, \quad \text{for every} \quad X \varepsilon \mathfrak{R}/\mathfrak{Q}.$$

Here again we denote the element $x_i + \mathfrak{Q}$ of $\mathfrak{R}/\mathfrak{Q}$ by X_i and here again for every integer $m = 1$, the $2^m - 1$ elements

$$(6) \qquad X_1, X_2, \ldots, X_1X_2, X_1X_3, \ldots, X_1X_2 \ldots X_m$$

are linearly independent over $GF(2)$ and every element $Q(X_1, X_2, \ldots)$ of $\mathfrak{R}/\mathfrak{A}$ is a linear combination over $GF(2)$ of the elements listed in (6), for a suitable m. Consequently, $\mathfrak{R}/\mathfrak{Q}$ is an infinite dimensional algebra. However, in this case $\mathfrak{R}/\mathfrak{Q}$ cannot be nilpotent since $X_1X_2 \ldots X_m \neq 0$ for every integer $m \geq 1$.

Thus, Lemma 3 is proved

In view of Lemmas 1, 2 and 3, we have:

THEOREM. – *Let $\mathfrak{A}$ be an algebra over a field of characteristic 2 such that $x^2 = 0$, for every $x \varepsilon \mathfrak{A}$. Let N be the index of nilpotency of $\mathfrak{A}$. Then*

$$\dim \mathfrak{A} < 2^m - 1 \quad \textit{implies} \quad N \leq m$$

Moreover, the upper bound m of N is attained for every integer m. Furthermore, there exist infinite dimensional algebras $\mathfrak{A}$ which are not nilpotentent.

The above Theorem together with the abovementioned Higman's result give an upper bound of the index of nilpotency (when it exists) of any (linear associative) algebra $\mathfrak{A}$ in which $x^2 = 0$, for every $x \varepsilon \mathfrak{A}$.

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