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On a Generalized Hermite Polynomial and a problem of Carlitz

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Summary. - *The classical formula relating Hermite and Laguerre polynomials is generalized.*

Carlitz [1] has proposed the question: If $\{f_n(x)\}$ is a set of polynomials and we define

$$\Phi_{2n}(x) = f_n(x^2),$$

then what conditions on the set $\{f_n(x)\}$ guarantee that there exists a set of orthogonal polynomials $\{\Phi_n(x)\}$? One well-hnown answer (stated by Carlitz)is the following: If the $f_n(x)$ are certain Laguerre polynomials,

$$(1) \qquad f^n(x) = (-1)^n n!\ L_n^{(-1/2)}(x),$$

then the $\Phi_n(x)$ are monic Hermite polynomials,

$$(2) \qquad \Phi_n(x) = 2^{-n} H_n(x),$$

where the odd-indexed $\Phi_n(x)$ are given by

$$(3) \qquad \Phi_{2n+1}(x) = (-1)^n n!\ x\ L_n^{(1/2)}(x^2).$$

We will show here (with a further example) that if $\{f_n(x)\}$ is a sequence of polynomials orthogonal over a non-negative domain, then $\{\Phi_{2n}(x) = f_n(x^2)\}$ is the set of even polynomials from a set $\{\Phi_n(x)\}$ of orthogonal polynomials.

Let us assume that $\{f_n(x)\}$ is a set of monic orthogonal polynomials satisfying

$$(4) \qquad f_n(x) - (x + a_n) f_{n-1}(x) + b_n f_{n-2}(x) = 0, \ (n \geq 2),$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 5 gingno 1963.

where

$$(5) \qquad b_n > 0$$

and furthermore that

$$(6) \qquad (-1)^n f_n(0) > 0, \qquad (n \geq 0).$$

Condition (6) can be replaced by the assumption that the domain of orthogonality of the $f_n(x)$ is non-negative. For if the domain is non-negative, the zeros of all the $f_n(x)$ are positive and $(-1)^n f_n(0)$, the product of the zeros of $f_n(x)$, is positive.

The polynomials $\{\Phi_{2n}(x) = f_n(x^2)\}$ have, from (4), a recurrence relation

$$(7) \qquad \Phi_{2n}(x) - (x^2 + a_n)\Phi_{2n-2}(x) + b_n \Phi_{2n-4}(x) = 0, \quad (n \geq 2).$$

Consider now the polynomial

$$\Phi_{2n}(x) - \frac{\Phi_{2n}(0)}{\Phi_{2n-2}(0)} \Phi_{2n-2}(x). \qquad (n \geq 1).$$

Because of (6) the denominator $\Phi_{2n-2}(0)$ is never zero. This polynomial is of degree $2n$ and has no constant term. Hence we may define the $\{\Phi_m(x)\}$ for odd m by

$$x\Phi_{2n-1}(x) = \Phi_{2n}(x) - \frac{\Phi_{2n}(0)}{\Phi_{2n-2}(0)} \Phi_{2n-2}(x), \qquad (n \geq 1),$$

and thus we have another recurrence relation for the $\{\Phi_n(x)\}$,

$$(8) \qquad \Phi_{2n}(x) \quad x\Phi_{2n-1}(x) - \frac{\Phi_{2n}(0)}{\Phi_{2n-2}(0)} \Phi_{2n-2}(x) = 0 \qquad (n \geq 1).$$

Between (7) and (8) we may eliminate certain even indexed polynomials and thus obtain

$$(9) \qquad \Phi_{2n-1}(x) - x\Phi_{2n-2}(x) - \frac{\Phi_{2n-4}(0)}{\Phi_{2n-2}(0)} b_n \Phi_{2n-3}(x) = 0, \quad (n \geq 2).$$

Notice that the coefficients

$$-\Phi_{2n}(0)/\Phi_{2n-2}(0) \text{ and } -\Phi_{2n-4}(0) b_n/\Phi_{2n-2}(0)$$

of (8) and (9) are, from (5) and (6), both positive. Thus the relations (8) and (9) together form a sufficient condition [3] that the $\{\Phi_n(x)\}$ be orthogonal.

Consider formula (1). If we reintroduce the parameter α of the generalized Laguerre polynomial, the polynomials are still orthogenal over a non-negative domain and we may thus use our result to form a generalization of the Hermite polynomials.

The monic polynomials

$$f_n^a(x) = (-1)^n n!\ L_n^{(a)}(x) \qquad (a > -1),$$

where

$$f_n^a(0) = (-1)^n \Gamma(n+1+a)/\Gamma(1+a)$$

have, from the Laguerre polynomial recurrence relation, the recurrence relation

$$f_n^a(x) - (x - 2n + 1 - a) f_{n-1}^a(x) +$$
$$+ (n - 1 + a)(n - 1) f_{n-2}^a(x) = 0, \qquad (n \geq 2).$$

From the facts above it follows that the polynomials

$$\{\Phi_{2n}^a(x) = f_n^a(x^2) = (-1)^n n!\ L_n^{(a)}(x^2)\} \tag{10}$$

are from an orthogonal set with the recurrence relation

$$\begin{cases} \Phi_n^a(x) - x\Phi_{n-1}^a(x) + [(n/2) + a]\Phi_{n-2}^a(x) = 0, & (n \text{ even}) \\ \Phi_n^a(x) - x\Phi_{n-1}^a(x + [(n/2) - (1/2)]\Phi_{n-2}^a(x) = 0, & (n \text{ odd}), \end{cases} \tag{11}$$

and therefore the set $\{\Phi_n^a(x)\}$, with $a > -1$, is a set of orthogonal polynomials.

For the odd polynomials of this set, we have

$$\Phi_{2n-1}^a(x) = \frac{1}{x}\left[\Phi_{2n}^a(x) - \frac{\Phi_{2n}^a(0)}{\Phi_{2n-2}^a(0)}\Phi_{2n-2}^a(x)\right]$$

$$= (-1)^{n-1}(n-1)!\ x^{-1}[-nL_n^{(a)}(x^2) + (n+a)L_{n-1}^{(a)}(x^2)].$$

But, from the contiguous relations for the Laguerre polynomials, page 203 of [2], it readily follows that

$$\Phi^{\alpha}_{2n-1}(x) = (-1)^{n-1}(n-1)!\; x\; L^{(\alpha+1)}(x^2), \quad (\alpha > -1),\ (n \geq 1). \tag{12}$$

The polynomials $\{\Phi_n(x)\}$ given by (2) are the same as the polynomials $\{\Phi_n^{-1/2}(x)\}$ formed by setting $\alpha = -1/2$ in (10) and (12). Thus the polynomials $\{\Phi^{\alpha}_n(x)\}$ are generalizations of the Hermite polynomials. The orthogonality relatioa is, for $\alpha > -1$,

$$\int_{-\infty}^{+\infty} \Phi^{\alpha}_n(x)\Phi^{\alpha}_m(x)\,|x|^{2\alpha+1}\, e^{-x^2}dx = \tag{13}$$

$$= \begin{cases} 0 \text{ for } m \neq n, \\ \Gamma\left(\frac{n}{2}+\frac{3}{2}+\alpha\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right) \text{ for } n \text{ odd}, \\ \Gamma\left(\frac{n}{2}+1+\alpha\right)\Gamma\left(\frac{n}{2}+1\right) \text{ for } n \text{ even}. \end{cases}$$

Ther hypergeometric representation is

$$\begin{aligned} \Phi^{\alpha}_n(x) &= x^n {}_2F_0\left(-\frac{n}{2},\ -\frac{n}{2}-\alpha;\ -;\ -1/x^2\right), & (n \text{ even}), \\ \Phi^{\alpha}_n(x) &= x^n {}_2F_0\left(-\frac{n}{2}+\frac{1}{2},\ -\frac{n}{1}+\frac{1}{2}-\alpha;\ -;\ -1/x^2\right), & (n \text{ odd}). \end{aligned} \tag{14}$$

Each $\Phi^{\alpha}_n(x)$ satisffies a differential equation:

$$\begin{aligned} &[x^2D^2 + (1+2\alpha-2x^2)xD + (\qquad 2nx^2)]\,\Phi^{\alpha}_n(x) = 0, & (n \text{ even}) \\ &[x^2D^2 + (1+2\alpha-2x^2)xD + (-1-2\alpha+2nx^2)]\,\Phi^{\alpha}_n(x) = 0, & (n \text{ odd}). \end{aligned} \tag{15}$$

The formulas (13), (14), and (15) and follow from the analogous formulas for the Laguerre polynomials by simple changes of variable.

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