W. A. Al-Salam

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Zanichelli

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Operational Derivation of Some Formulas for the Hermite and Laguerre Polynomials

by W. A. AL-SALAM (*)

Summary. - *Various generating functions and formulas are derived by means of the operators* e^{-D^2} *and* D^n.

1. The Hermite polynomials $\{H_n(x)\}$ may be defined by means of the Rodrigue's formula

$$D^n\{e^{-x^2}\} = (-1)^n\ e^{-x^2}\ H_n(x), \qquad D = D_x = d/dx. \tag{1 1}$$

Another operational representation, mentioned by GOULD and HOPPER [4], is

$$e^{-D^2}\{x^n\} = H_n(x/2). \tag{1.2}$$

We shall employ these operators to obtain, in a simple and rapid manner, various properties of the Hermite polynomials. One of these, namely formula (1.7), is believed to be new.

To start with let us operate on the identity

$$e^{xt} = \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}$$

by means of e^{-D^2}. The familiar shift rule gives for the left hand side

$$\begin{aligned} e^{-D^2}\ e^{xt} &= e^{xt}\ e^{-(D+t)^2}\{1\} \\ &= e^{xt}\ e^{-(D^2+2tD+t^2)}\{1\} \\ &= e^{xt}\ e^{-t^2} = e^{xt-t^2} \end{aligned}$$

(*) Pervenuta alla Segreteria dell'U. M I. il 24 maggio 1963.

On the other hand the right hand side yields

$$\sum_{n=0}^{\infty} H_n(x/2) \frac{t^n}{n!}$$

We thus arrive at the familiar generating function

$$e^{2xt-t^2} = \Sigma\, H_n(x)\, t^n/n! \tag{1.3}$$

which is often used as a definition of the Hermite polynomials. This derivation may be interepreted as a simple proof of (1 2).

Another way of deriving (1.3) is the following: We have formally

$$e^{-tD} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} D^n$$

Now multiply from the right by e^{-x^2} we get for the left hand side by TAYLOR's theorem

$$e^{-tD}\, e^{-x^2} = e^{-(x+t)^2}$$

The right hand side gives

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)\, e^{-x^2}$$

Combining these two expressions we get (1.3).

We shall now derive the Mehler formula

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)\, H_n(y) = (1-t^2)^{-1/2} \exp\left[\frac{2xyt - t^2(x^2+y^2)}{2(1-t^2)}\right] \tag{1.4}$$

by means of this method.

We need the formula [3].

$$e^{aD^2}\, e^{-kx^2} = (1+4ak)^{-1/2} \exp\left[\frac{kx^2}{1+4ak}\right] \tag{1.5}$$

which may be proved by expanding e^{aD^2} and e^{-kx^2} in their power series, performing the differentiation operation and then summing the resulting series

Now replace t by tD_y in (1.3) and operate on both sides e^{-y^2}. We have from the left hand side

$$\exp\{2xtD_y - t^2D^2_y\}[e^{-y^2}] = e^{2xtD_y} \frac{1}{\sqrt{1-4t^2}} \exp\left[\frac{y^2}{1-4t^2}\right]$$

$$= \frac{1}{\sqrt{1-4t^2}} \exp\left[\frac{(y+2xt)^2}{1-t^2}\right]$$

The right hand side gives

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n D^n_y\, e^{-y^2} = e^{-y^2} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} H_n(x)\; H_n(y)$$

Combining these two sides we get (1.4).

We also mention that (1.4) can be derived by operating on $e^{-x^2-y^2}$ by means of the operator $e^{-tD_xD_y}$.

A third variation of this method is to replace t by ty in (1.3) and operate on both sides with $e^{-D^2_y}$.

Other generating functions can be obtained in this manner. For example if we operate on e^{-x^2} by means of $D^k\, e^{-tD}$ we get [5, p. 197],

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x)}{n!} t^n = e^{2xt-t^2}\; H_k(x-t). \tag{1.6}$$

On the other hand if we employ the operator $D^k\, e^{-tD^2}$ we get

$$\sum_{n=0}^{\infty} \frac{H_{2n+k}(x)}{n!} t^n = (1+4t)^{-(k+1)/2} \exp\left[\frac{4tx^2}{1+4t}\right] H_k\left(\frac{x}{1+4t}\right). \tag{1.7}$$

This formula may also be derived by operating on e^{tx} by means of $D^k\, e^{-D^2}$. To the best knowledge of the writer formula (1.7) is new. The cases $k=0$, $k=1$ are given by Rooney [6].

We now derive the formula [2]

$$(1.8)\quad H_m\left(\frac{x+y}{\sqrt{2}}\right)H_m\left(\frac{x-y}{\sqrt{2}}\right) = \sum_{k=0}^{m}(-1)^{m+k}\binom{m}{k}H_{2k}(x/2)H_{2m-2k}(y/2).$$

Operate on both sides of the identity

$$(x+y)^m(x-y)^m = (x^2-y^2)^m = \sum_{k=0}^{m}(-1)^{m+k}\binom{m}{k}x^{2k}\,y^{2m-2k}$$

with the operator $e^{-D_x^2-D_y^2} = e^{-D_x^2}\,e^{-D_y^2}$.

The right hand side gives the right hand side of (1.8) To evaluate the left hand side we first rewrite

$$D_x^2 + D_y^2 = \frac{1}{2}(D_x+D_y)^2 + \frac{1}{2}(D_x-D_y)^2.$$

We then make the change of variables $x+y=u$, $x-y=v$. Thus the left hand side becomes

$$\left\{\exp\left(-\frac{1}{2}D_v^2\right)v^m\right\}\left\{\exp\left(-\frac{1}{2}D_u^2\right)u^m\right\} = 2^m H_m\left(\frac{v}{\sqrt{2}}\right)H_m\left(\frac{v}{\sqrt{2}}\right)$$

and thus (1.8) follows.

In a similiar manner the identity

$$(x+y)^{2m} + (x-y)^{2m} = 2\sum_{k=0}^{m}\binom{2m}{2k}x^{2k}\,y^{2m-2k}$$

yields the formula [2]

$$2^{m-1}\{H_{2m}(x+y) + H_{2m}(x-y)\} = \sum_{k=0}^{m}\binom{2m}{2k}H_{2k}(\sqrt{2}\,x)\,H_{2m-2k}(\sqrt{2}\,y)$$

2. The special Laguerre polynomials $\{L_n^{(a-n)}(x)\}$ possess the Rodrique's formula

$$(2.1)\qquad D^n\left[x^a e^{-x}\right] = n!\,x^{a-n}e^{-x}L_n^{(a-n)}(x)\qquad n=0,1,2,3,\ldots$$

We have by TAYLOR's theorem

$$e^{tD}[x^{a}e^{-xt}] = (x + t)^{a}\ e^{-x-t}$$

Thus we get the generating, due to Erdelyi,

$$(1 + t)^{a}\ e^{-xt} = \sum_{n=0}^{\infty} t^{n} L_{n}^{(a-n)}(x). \tag{2.2}$$

If now we replace t by tD_y in (2.2) and operate on $y^{\beta}\ e^{-y}$ we get from the right hand side

$$y^{\beta}\ e^{-y} \sum_{n=0}^{\infty} n!\ (t/y)^{n}\ L_{n}^{(\alpha-n)}(x)\ L_{n}^{(\beta-n)}(y).$$

The left hand side yields

$$(1 + tD_y)^{a}\ e^{-xtD_y}[y^{\beta}\ e^{-y}] = (1 + tD_y)^{a}(y - xt)^{\beta}\ e^{-y+xt} \tag{2.3}$$

Now replace $y - xt$ by u then $D_y = D_u$ and (2.3) becomes

$$\begin{aligned}(1 + tD_y)^{a}\, e^{-xtD_y}\ [y^{\beta}e^{-y}] &= (1 + tD_u)^{a}[u^{\beta}\ e^{-u}] \\ &= e^{-u}(1 + tD_u - t)^{a}[u^{\beta}] \\ &= e^{-u}(1 - t)^{a}(1 + \frac{t}{1-t} D_u)^{a}[u^{\beta}] \\ &= e^{-u}(1 - t)^{a}u^{\beta}\ {}_2F_0[- a, - \beta; —; \\ &\qquad \frac{t}{(1-t)u}\Big].\end{aligned}$$

Here we need to add the assumption that either a or β is a positive integer. In this case the ${}_2F_0$ is a finite series and may aè writter as a ${}_1F_1$ by summing backward. We get, after some reduction, the formula [1, p. 151]

$$\sum_{=0}^{\infty} n!\, t^{n}L_{n}^{(a-n)}(x)L_{n}^{(\beta-n)}(y) = e^{xyt} \left\{ \begin{array}{l} (1\text{-}yt)^{a-\beta}t^{\beta}\beta!L_{\beta}^{(a-\beta)}\Big(- \frac{(1\text{-}xt)(1\text{-}yt)}{t}\Big) \\ (1\text{-}xt)^{\beta-a}t^{a}a!L_{a}^{(\beta-a)}\Big(- \frac{(1\text{-}xt)(1\text{-}yt)}{t}\Big) \end{array} \right.$$

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