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SEZIONE STORICO-DIDATTICA

A simple procedure for the determination of the axes of symmetry and metrical elements of the conics

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Summary. - *See the Note on the end of this paper.*

1. Central Conics

The general form of the equation of the second degree curve is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0. \tag{1.1}$$

For the second degree curves with a center, we have

$$\delta \equiv \begin{vmatrix} A & B \\ B & C \end{vmatrix} \neq 0. \tag{1.2}$$

The equations

$$\begin{aligned} Ax_0 + By_0 + D &= 0, \\ Bx_0 + Cy_0 + E &= 0, \end{aligned} \tag{1.3}$$

under hypothesis (1.2), have an unique solution for x_0 and y_0. The point $O(x_0, y_0)$ is the center of the curve (1.1). If a transformation of coordinates is carried out such that the new origin coincides with the point O, and if we denote the new system coordinates again with x and y, the equation (1.1) becomes

$$Ax^2 + 2Bxy + Cy^2 = F', \tag{1.4}$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 25 novembre 1963.

where

$$(1.5) \qquad F' = -\frac{\Delta}{\delta} \quad \left(\Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} \right).$$

If $\Delta = 0 \Rightarrow F' = 0$, the equation (1.1) represents either two straight-lines or a point. Therefore, we shall assume that $\Delta \neq 0$.

If $B = 0$, all the necessary elements can be found directly from (1.4). Therefore, we shall assume that $B \neq 0$.

Let us prove that the equation (1.4) can be given in the following form

$$(1.6) \qquad \lambda\, (y - \alpha x)^2 + \mu\, (y - \beta x)^2 = 1,$$

where

$$(1.7) \qquad \alpha\beta = -1.$$

Comparing the equation (1.4) with (1.6), we shall obtain the following set of equations

$$(1.8) \qquad \lambda\alpha^2 + \mu\beta^2 = A/F',$$

$$(1.9) \qquad \lambda\alpha + \mu\beta = -B/F',$$

$$(1.10) \qquad \lambda + \mu = C/F'.$$

By eliminating λ and μ from the equations (1.8) (1.9), and (1.10), we shall obtain

$$\begin{vmatrix} \alpha^2 & \beta^2 & A \\ \alpha & \beta & -B \\ 1 & 1 & C \end{vmatrix} = 0.$$

By making use of the equality (1.7), then it follows that $\alpha \neq \beta$, we find, in turn, that

(1.11)
$$\begin{vmatrix} \alpha^2 & \alpha+\beta & A \\ \alpha & 1 & -B \\ 1 & 0 & C \end{vmatrix} = 0,$$

(1.12)
$$\begin{vmatrix} 1 & \alpha+\beta & A \\ 0 & 1 & -B \\ 1 & 0 & C \end{vmatrix} = 0,$$

(1.13)
$$\alpha + \beta = -\frac{A - C}{B} = -\varkappa.$$

According to the (1.7) and (1.13), we can conclude that α and β are the roots of the following quadratic equation

(1.14)
$$z^2 + \varkappa z - 1 = 0.$$

The roots of this equation are real having different signs. Let us assume that

(1.15)
$$\alpha > 0, \qquad \beta < 0.$$

The equations (1.9) and (1.10) have a unique solution for λ and μ:

(1.16)
$$\lambda = -\frac{B + C\beta}{F'(\alpha - \beta)}, \qquad \mu = \frac{B + C\alpha}{F'(\alpha - \beta)}.$$

Hence

(1.17)
$$\lambda\mu = \frac{\delta}{(4 + \varkappa^2)\, F'^2}.$$

Thus we have proved that the equation (1.4) may, indeed, be given in the form (1.6) if $\Delta \neq 0$ and $B \neq 0$. The constants α and β are determined from (1.14), whereas λ and μ are obtained from (1.16).

If the axes are turned through the angle $\theta = \arctan \alpha$ $(0 < \theta < \pi/2)$, the equation (1.6) becomes

$$\lambda (\alpha^2 + 1) \xi^2 + \mu (\beta^2 + 1) \eta^2 = 1 \tag{1.18}$$

where ξ and η are new coordinates.

Thus, the straight-lines

$$y = \alpha x \text{ and } y = \beta x \tag{1.19}$$

are the axes of the curve (1.4).

According to (1.18), (1 17), (1.10), (1.5) we can assert that the equation (1.1) determines

1° - An ellipse, provided $\delta > 0$ and $C\Delta < 0$;

2° - A hyperbola, provided $\delta < 0$ and $\Delta \neq 0$:

3° - A set of two straight-lines intersecting each other, provided $\delta < 0$ and $\Delta = 0$;

4° - A point, provided $\delta > 0$ and $\Delta = 0$.

The equation (1.1) has no geometrical interpretation in the real domain if $\delta > 0$ and $C\Delta > 0$.

Although the preceding discussion has been made under assumption that $B \neq 0$, it is easy to see that it holds good also when $B = 0$.

The squares of the lengths of semi-axes of the curve (1.1) are determined by means of the following formula

$$\pm a^2 = \frac{1}{\lambda (\alpha^2 + 1)}, \qquad \pm b^2 = \frac{1}{\mu (\beta^2 + 1)}. \tag{1.20}$$

2. Non-Central Conics

For these curves, we have

$$\delta \equiv \begin{vmatrix} A & B \\ B & C \end{vmatrix} = 0 \tag{2.1}$$

Provided that A or C are equal to zero, then $B = 0$, and it is seen directly that the equation (1.1) represents a parabola, and its axis, vertex, and principal parameter of latus rectum can be determined directly. Therefore, we shall assume that $AC \neq 0$.

If we write $C = B^2/A$, the equation (1.1) ca be given in the following form

$$(Ax + By)^2 + A(2Dx + 2Ey + F) = 0. \tag{2.2}$$

Then, instead of (2.2), we can write

$$(Ax + By + \lambda)^2 = 2A(\lambda - D)x + 2(B\lambda - AE)y + \lambda^2 - AF, \tag{2.3}$$

where λ is a constant which we shall determine in such a way that the straight-lines

$$Ax + By + \lambda = 0, \tag{2.4}$$

$$2A(\lambda - D)x + 2(B\lambda - AE)y + \lambda^2 - AF = 0 \tag{2.5}$$

are perpendicular to each other. The condition of perpendicularity of these straight-lines is as follows

$$A^2(\lambda - D) + B(\lambda B - AE) = 0,$$

whence it follows that

$$\lambda = \frac{A(AD + BE)}{A^2 + B^2}. \tag{2.6}$$

Since for this value of λ, the lines (2.4) and (2.5) are perpendicular to each other, and since the equation (2.3) expresses the fact that the distance of the point (x, y) from the line (2.5) is proportional to the square of the distance of the same point from the line (2.4), we can conclude that (2.3) is the equation of a parabola.

The axis of this parabola is the straight-line (2.4), the straight-line (2.5) being its tangent at the principal vertex. If $\lambda^2 - AF > 0$, the parabola lies on the side of the line (2.5) where the origin is located. But if $\lambda^2 - AF < 0$, the parabola is on the other side of the said straight-line.

There is one exception, namely, when the coefficients beside x and y in (2.5) are equal to zero. Then. if $\lambda = D$, we obtain

$$A^2D + ABE - (A^2 + B^2)\,D \Rightarrow B\,(BD - AE) = 0,$$

and the equation (2.3) becomes

$$(2.7) \qquad (Ax + By + D)^2 = D^2 - AF.$$

Therefore, if $\delta = 0$ and $B \neq 0$, the equation (1.1) represents

1° - A parabola, provided $BD - AE \neq 0$;

2° - Two parallel straight-lines, provided $BD - AE = 0$ and $D^2 - AF > 0$;

3° - A double straight-line, provided $BD - AE = 0$ and $D^2 - AF = 0$.

The equation (1.1) has no sense in the real domain when

$$\delta = 0,\ B \neq 0,\ BD - AE = 0 \text{ and } D^2 - AF < 0.$$

If $B = 0$, the preceding discussion does not hold good.

In the case 1° the parabola's parameter is

$$(2.8) \qquad p = \frac{\sqrt{A^2(\lambda - D)^2 + (B\lambda - AE)^2}}{A^2 + B^2} = \frac{|\,A\,(AE - BD)\,|}{(A^2 + B^2)^{3/2}}\,.$$

3. Examples

EXAMPLE 1. - For the curve

$$3x^2 + 2xy + 3y^2 + 6x - 2y - 5 = 0 \tag{3.1}$$

we have

$$\delta = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8, \quad \Delta = \begin{vmatrix} 3 & 1 & 3 \\ 1 & 3 & -1 \\ 3 & -1 & -5 \end{vmatrix} = -76.$$

Since $\delta \neq 0$, the curve has a center, the coordinates of which are $x_0 = -\frac{5}{4}$, $y_0 = \frac{3}{4}$. In this case, $\varkappa = 0$, $F' = \frac{19}{2}$ and the equation (1.4) is as follows

$$3x^2 + 2xy + 3y^2 = \frac{19}{2}.$$

The constants α and β are determined from the equation

$$z^2 - 1 = 0 \Rightarrow \alpha = 1, \ \beta = -1.$$

Thus, the axes of the curve (3.1) are the straight-lines

$$y - \frac{3}{4} = x + \frac{5}{4}, \quad y - \frac{3}{4} = -\left(x + \frac{5}{4}\right),$$

i. e.,

$$y - x = 2, \quad y + x = -\frac{1}{2}.$$

Then, according to (1.16), $\lambda = \frac{2}{19}$, $\mu = \frac{4}{19}$. Therefore, the given curve is an ellipse whose semi-axes are determined by the relations

$$a^2 = \frac{19}{4}, \quad b^2 = \frac{19}{8}.$$

EXAMPLE 2. - Let us consider the curve whose equation is

$$9x^2 - 24xy + 16y^2 - 16x - 12y - 4 = 0. \tag{3.2}$$

Here, we have $\delta = 0$, $BD - AE = 150$. Therefore, the given curve is a parabola. According to (2.6), we find that $\lambda = 0$. By using the equations (2.4) and (2.5), we can conclude that the straight-line

$$9x - 12y = 0, \quad \text{i. e.} \quad 3x = 4y$$

is the axis of the parabola (2.3), and the line

$$4x + 3y + 1 = 0$$

being its tangent at the vertex. From (2.8), we find that the parameter of this parabola is $p = \frac{2}{5}$. Since $\lambda^2 - AF = 36 > 0$ the parabola (3.2) and the origin are located on the same side of the straight-line $4x + 3y + 1 = 0$.

4. Note

In this paper, we have discussed a classic chapter of Analytical Geometry. In literature there exist a number of procedures referring to the problems discussed above. We have been unable to establish whether the procedure outlined in our paper for Central Conics is a new procedure, but it is certainly a brief and simple procedure, and can be used with some advantage in teaching Analytical Geometry.

For Non-Central Conics, the procedure presented in this paper is not a new one, but still there are some new details in it. See A. GEARY - H. V. LOWRY - H. A. HAYDEN:

Advanced Mathematics for Technical Students, part I, London 1948, p. 213.