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A note on Dougall's theorem

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Summary. - *The writer shows that Dougall's theorem on the sum of an $_7F_6$ is equivalent to the series transformation* (9) *below. Some special cases are discussed also.*

It is familiar that Saalschütz's theorem

$$(1)\qquad {}_3F_2\left[\begin{matrix}-n, & a, & b; \\ & c & d\end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},$$

where

$$(2)\qquad c + d = a + b - n + 1,$$

is equivalent to Euler's formula

$$(3)\qquad F(c-a,\ c-b;\ c;\ z) = (1-z)^{a+b-c}F(a,\ b;\ c;\ z).$$

Thus it is of some interest to find a series transformation that is equivalent to Dougall's theorem [1, p. 26]

$$(4)\qquad {}_7F_6\left[\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & -m: \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+m\end{matrix}\right]$$

$$= \frac{(1+a)_m(1+a-c-d)_m(1+a-b-d)_m(1+a-b-c)_m}{(1+a-b)_m(1+a-c)_m(1+a-d)_m(1+a-b-c-d)_m},$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 25 marzo 1964.

where

(5) $$1 + 2a = b + c + d + e - m.$$

To begin with, we replace e by $e + m$. Then (4) becomes

(6) $${}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e+m & -m; \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e-m & 1+a+m \end{matrix}\right]$$

$$= \frac{(1+a)_m(1+a-c-d)_m(1+a-b-d)_m(1+a-b-c)_m}{(1+a-b)_m(1+a-c)_m(1+a-d)_m(1+a-b-c-d)_m},$$

where now

(7) $$1 + 2a = b + c + d + e.$$

Next since

$$(1 + a - e - m)_r = (-1)^r \frac{(e-a)_m}{(e-a)_{m-r}}$$

and by (7)

$$e - a = 1 + a - b - c - d,$$

(6) becomes

(8) $$\sum_{r=0}^{m} \frac{(a)_r\left(1+\frac{1}{2}a\right)_r(b)_r(c)_r(d)_r(e)_{m+r}(e-a)_{m-r}}{r!\,(m-r)!\left(\frac{1}{2}a\right)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r(1+a)_{m+r}}$$

$$= \frac{(e)_m(1+a-c-d)_m(1+a-b-d)_m(1+a-h-c)_m}{m!\,(1+a-b)_m(1+a-c)_m(1+a-d)_m}.$$

If we multiply both sides of (8) by x^m and sum over m we get

(9) $${}_4F_3\left[\begin{matrix} e, & 1+a-c-d, & 1+a-b-d, & 1+a-b-c; & x \\ & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c)_r(d)_r(e)_{2r}}{r!\,(1+a-b)_r(1+a-c)_r(1+a-d)_r(a)_{2r}} x^r$$

$$\cdot F(e + 2r,\ e - a;\ 1 + a + 2r;\ x),$$

where of course (7) it assumed to hold.

If in (9) we replace d, e by $1+a-d$, $1+a-e$ and let $a \to \infty$, the left member becomes

$$F(d-b,\ d-c;\ d;\ x),$$

while the right member becomes

$$(1-x)^{-1+e}F(b,\ c;\ d;\ x).$$

Since (7) is now

$$1-e=b+c-d,$$

it is clear that we have (3). On the other hand, if in (9) we replace c, d by $1+a-c$, $1+a-d$ and let $a \to \infty$. we get

$$\text{(10)} \qquad {}_3F_2\begin{bmatrix} e,\ d-b,\ c-b;\ x \\ c,\ d \end{bmatrix} = (1-x)^{-e}\sum_{r=0}^{\infty}(-1)^r \frac{(b)_r(e)_{2r}}{r!\,(c)_r(d)_r}\frac{x^r}{(1-x)^{2r}},$$

where now

$$\text{(11)} \qquad 1+b+e=c+d.$$

We may rewrite (10) in the form

$$\text{(12)} \qquad {}_3F_2\begin{bmatrix} e,\ 1+e-c,\ 1+e-d;\ x \\ c,\ d \end{bmatrix} =$$

$$= (1-x)^{-e}{}_3F_2\begin{bmatrix} \frac{1}{2}e,\ \frac{1}{2}+\frac{1}{2}e,\ c+d-1-e;\ -4x/(1-x)^2 \\ e,\ d \end{bmatrix}$$

a result due to Whipple [3, (7.1)]. We remark that when $c=d=1$, $e=-a$ and $x \to -1$, (10) reduces to

$$\text{(13)} \qquad \sum_{r=0}^{\infty}\binom{a}{r}^2 = 2^a\sum_{r=0}^{\infty}(-1)^r\binom{-1-a}{r}\binom{\frac{1}{2}a}{r}\binom{-\frac{1}{2}+\frac{1}{2}a}{r}.$$

As observed by Bailey [2, p. 497] (12) is a consequence of Saalschütz's theorem.

If in (9) we replace c by $c-d$ and let $d \to \infty$ we get

$$(14) \qquad F(e,\ e-a+b;\ 1+a-b;\ x)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(e)_{2r}}{r!\,(1+a-b)_r(a)_{2r}} x^r F(e+2r, e-a;\ 1+a+2r;\ x).$$

On the other hand, if we replace d by $d-e$ and x by x/e^2 and then let $e \to 0$, the left member of (9) reduces to

$${}_1F_2(1+a-b-c;\ 1+a-b,\ 1+a-c;\ x),$$

while the right member reduces to

$$\sum_{r=0}^{\infty} (-1)^r \frac{(a)_r(b)_r(c)_r}{r!\,(1+-b)_r(1+a-c)_r(a)_{2r}} x^r {}_0F_1(1+a+2r;\ x).$$

The condition (7) now reduces to

$$(15) \qquad 1+2a = b+c+d.$$

However, since the parameter d no longer appears, (15) may be disregarded and we get

$$(16) \qquad {}_1F_2(1+a-b-c;\ 1+a-b,\ 1+a-c;\ x)$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r(b)_r(c)_r}{r!\,(1+a-b)_r(1+a-c)_r(a)_{2r}} x^r {}_0F_1(1+a+2r;\ x).$$

Incidentally (16) is equivalent to [1, p. 25]

$$(17) \qquad {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & -m; \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a+m \end{matrix}\right] =$$

$$= \frac{(1+a)_m(1+a-b-c)_m}{(1+a-b)_m(1+a-c)_m};$$

indeed it is not difficult to verify that (14) is also equivalent to (17).

REFERENCES

[1] W. N. Bailey, *Generalized hypergeometric series*, Cambridge, 1935.

[2] W. N. Bailey, *Transformation of generalized hypergeometric series* Proceedings of the London Mathematical Society (2), vol. 29 (1929), pp. 495-502.

[3] F. J. W. Whipple, *Some transformations of generalized hypergeometri c series*, Proceedings of the London Mathematical Journal (2), vol. 26 (1927), pp. 257-272.