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The sum of the first n terms of an ${}_5F_4$

by LEONARD CARLITZ (a Durham U.S.A.) (*)

Summary. - *The sum of the first* n *terms of a special* ${}_5F_4$ *is obtained. Some related series are also discussed.*

1. Put

$$u_m = \frac{(1+b+c+d)_m(1+b)_m(1+c)_m(1+d)_m}{m\,!\,(1+c+d)_m(1+b+d)_m(1+b+c)_m}. \tag{1}$$

Then

$$u_m - u_{m-1} = \frac{(1+b+c+d)_{m-1}(1+b)_{m-1}(1+c)_{m-1}(1+d)_{m-1}}{m\,!\,(1+c+d)_m(1+b+d)_m(1+b+c)_m}$$
$$\cdot \{(m+b+c+d)(m+b)(m+c)(m+d) -$$
$$- m(m+c+d)(m+b+d)(m+b+c)\}.$$

The quantity within braces reduces to

$$(b+c+d)bcd + 2bcdm = bcd(b+c+d+2m),$$

so that

$$u_m - u_{m-1} = \frac{b+c+d+2m}{b+c+d}\,\frac{(b+c+d)_m(b)_m(c)_m(d)}{m!\,(1+c+d)_m(1+b+d)_m(1+b+c)_m}.$$

Now

$$\frac{\left(1+\frac{1}{2}a\right)_m}{\left(\frac{1}{2}a\right)_m} = \frac{a+2m}{a}.$$

Thus if we put

$$a = b + c + d, \tag{2}$$

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we have

$$(3)\qquad u_m - u_{m-1} = \frac{(a)_m\left(1+\frac{1}{2}a\right)_m (b)_m(c)_m(d)_m}{m!\left(\frac{1}{2}a\right)_m (1+a+b)_m(1+a+c)_m(1+a+d)_m}$$

Let

$$_pF_q\begin{bmatrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{bmatrix}_m = \sum_{r=0}^{m} \frac{(a_1)_r \cdots (a_p)_r}{r!\,(b_1)_r \cdots (b_q)_r}.$$

Then it follows from (3) that

$$(4)\qquad {}_5F_4\begin{bmatrix} a, & 1+\frac{1}{2}a, & b, & c, & d; \\ & \frac{1}{2}a, & 1+a-b & 1-a-c, & 1+a-d \end{bmatrix}_m$$

$$= \frac{(1+a)_m(1+b)_m(1+c)_m(1+d)_m}{m!\,(1+a-b)_m(1+a-c)_m(1+a-d)_m}$$

provided (2) is satisfied.

If we let $m \longrightarrow \infty$, (4) becomes

$$(5)\qquad {}_5F_4\begin{bmatrix} a, & 1+\frac{1}{2}a, & b, & c, & d; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-a & \end{bmatrix}$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(1+d)}$$

This is of course a special case of the theorem [1, p. 27] in which (2) is not assumed:

$$(6)\qquad {}_5F_4\begin{bmatrix} a, & 1+\frac{1}{2}a, & b, & c, & d; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d & \end{bmatrix}$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-b-d)\Gamma(1+a-c-b)}.$$

2. It is of some interest to consider what we get using the method above when (2) is not assumed. We now put

$$(7) \qquad u_m = \frac{(1+a)_m(1+s+b)_m(1+s+c)_m(1+s+d)}{(1+s)_m(1+a-b)_m(1+a-c)_m(1+a-d)},$$

where

$$(8) \qquad s = a - b - c - d.$$

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$$(9) \qquad u_m - u_{m-1} = \frac{(1+s)_{m-1}(1+s+b)_{m-1}(1+s+c)_{m-1}(1+s+d)_{m-1}}{(1+s)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}$$

$$\cdot \{ (m+a)(m+s+b)(m+s+c)(m+s+d)$$

$$-(m+s)(m+a-b)(m+a-c)(m+a-d) \}.$$

The quantity in braces reduces after a little manipulation to

$$(10) \qquad bcd(s+a+2m)$$

for $m \geq 1$. It therefore follows that

$$(11) \quad 1+bcd\sum_{r=1}^{m}(s+a+2r)\frac{(1+a)_r(1+s+b)_{r-1}(1+s+c)_{r-1}(1+s+d)_{r-1}}{(1+s)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r}$$

$$= \frac{(1+s)_m(1+s+b)_m(1+s+c)_m(1+s+d)_m}{(1+s)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m},$$

where s is defined by means of (8).

If we prefer we may replace the left member of (11) by

$$(12) \quad 1+\frac{bcd}{a(s+b)(s+c)(s+d)}\sum_{r=1}^{m}(s+a+2r)\frac{(a)_r(s+b)_r(s+c)_r(s+d)_r}{(1+s)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r}$$

$$=1+\frac{bcd}{a(s+b)(s+c)(s+d)}\sum_{r=1}^{m}\frac{\left(1+\frac{1}{2}(s+a)\right)_r(a)_r(s+b)_r(s+c)_r(s+d)_r}{\left(\frac{1}{2}(s+a)\right)_r(1+s)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r}.$$

Thus when $s=0$ it is evident that (11) reduces to (4).

If we let $m \to \infty$, (11) becomes

$$(13)\quad 1+\frac{bcd}{a(s+b)(s+c)(s+d)}\sum_{r=1}^{\infty}(s+a+2r)\frac{(a)_r(s+b)_r(s+c)_r(s+d)_r}{(1+s)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r}$$
$$=\frac{\Gamma(1+s)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+s+b)\Gamma(1+s+c)\Gamma(1+s+d)}.$$

The left member of (13) can be expressed in terms of an ${}_6F_5$.

3. Returning to (4), one can obtain a more general result in the following way. If we put

$$(14)\qquad u_m=\frac{(1+a)_m(1+b)_m(1+c)_m(1+d)_m}{m\,!\,(1+b')_m(1+c')_m(1+d')_m},$$

then we have

$$u_m-u_{m-1}=\frac{(1+a)_{m-1}(1+b)_{m-1}(1+c)_{m-1}(1+d)_{m-1}}{m\,!\,(1+b')_m(1+c')_m(1+d')_m}K_m,$$

where

$$K_m=(m+a)(m+b)(m+c)(m+d)-m(m+b')(m+c')(m+d').$$

In general K_m is a polynomial in m of degree 3. If the parameters satisfy

$$(15)\qquad a+b+c+d=b'+c'+d'$$

and

$$(16)\qquad a(b+c+d)+cd+db+bc=c'd'+d'b'+b'c'$$

then K_m is of at most the first degree. When (15) and (16) are satisfied we may put

$$(17)\qquad K_m=abcd+\lambda m,$$

vhere

$$(18)\qquad \lambda=a(cd+db+bc)+bcd-b'c'd'.$$

It follows that

$$u_m - u_{m-1} = \frac{(a)_m(b)_m(c)_m(d)_m}{m!(1+b')_m(1+c')_m(1+d')_m}\left(1+\frac{\lambda m}{abcd}\right),$$

so that

$$\sum_{r=0}^{m}\left(1+\frac{\lambda m}{abcd}\right)\frac{(a)_m(b)_m(c)_m(d)_m}{m!(1+b')_m(1+c')_m(1+d')_m} \tag{19}$$

$$= \frac{(1+a)_m(1+b)_m(1+c)_m(1+d)_m}{m!(1+b')_m(1+c')_m(1+d')}.$$

In particular, when $m \to \infty$, we get

$$\sum_{r=0}^{\infty}\left(1+\frac{\lambda m}{abcd}\right)\frac{(a)_m(b)_m(c)_m(d)_m}{m!(1+b')_m(1+c')_m(1+d')_m} \tag{20}$$

$$= \frac{\Gamma(1+b')\Gamma(1+c')\Gamma(1+d')}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(1+d)}.$$

It is assumed in both (19) and (20) that both (15) and (16) are satisfied.

Note that if b', c', d' are the roots of

$$x^3 - (a+b+c+d)x^2 + (ab+ac+ad+cd+db+bc)x$$
$$- a(cd+db+bc) - bcd = 0,$$

then (20) reduces to

$$ {}_4F_3\left[\begin{matrix} a, & b, & c, & d; \\ 1+b', & 1+c', & 1+d' \end{matrix}\right] = \frac{\Gamma(1+b')\Gamma(1+c')\Gamma(1+d')}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(1+d)}. \tag{20}$$

The last result can be generalized to series ${}_{p+1}F_p$ in an obvious way.

REFERENCES

[1] W. N. Bailey, *Generalized hypergeometric series*, Cambridge 1935.