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On a characterization of a certain set of orthogonal polynomials

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Summary. - *In this paper we characterize all orthogonal polynomials* $\{f_n(x)\}$ *with a generating function of the form*

$$\Phi(2xt+t^2) = \Sigma f_n(x) \frac{t^n}{n!}.$$

We assume weaker conditions than have previously been done.

Let $\Phi(u)$ be a function which has a TAYLOR series expansions with $\Phi(o) \neq 0$; and put

$$\Phi(2xt + t^2) = \sum_{n=0}^{\infty} f_n(y) \frac{t^n}{n!}, \tag{1}$$

where $f_n(x)$ is a polynomial of degree n in x. Some years ago, K. P. WILLIAMS [5] proved that if $\{f_n(x)\}$ satisfy a recurrence relation of the form

$$h(n) f_{n+1}(x) = l(n) x f_n(x) + k(n) f_{n-1}(x) \tag{2}$$

where $h(n)$, $l(n)$ and $k(n)$ are polynomials then $f_n(x)$ is either the HERMITE or the GEGENBAUER polynomial (for definition see [3, pp. 191-201]).

It is well known [1,4] that a necessary and sufficient condition for a sequence of polynomials $\{f_n\}$, where $f_n(x)$ is of exact degree n, to be orthogonal is that they satisfy a recurrence relation of the form

$$f_{n+1}(x) = (A_n x + B_n) f_n(x) + C_n f_{n-1}(x) \tag{3}$$

$$f_0(x) = 1, \quad f_{-1}(x) = o. \quad A_n C_n \neq 0.$$

Thus it might be of interest to replace (2) in WILLIAMS'

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result by the more general condition (3). We prove the following:

THEOREM: *If* $\{f_n(x)\}$ *is a set of orthogonal polynomials which has a generating function of the form* (1) *then* $f_n(x)$ *is either the Hermite or the Gegenbauer polynomial.*

PROOF. - We have from (1)

$$(-1)^n f_n(-x) = f_n(x), \tag{4}$$

which leads to $B_n = o$ in (3). Formula (1) also implies (see [3, p. 131])

$$nf_n(x) = xf'_n(x) + nf'_{n-1}(x). \tag{5}$$

Put

$$f_n(x) = \sum_{k=o}^{n} b(n, k)x^{n-2k}. \tag{6}$$

Thus (5) and (6) imply

$$\begin{aligned} b(n, k) &= \frac{n(n-2k+1)}{2k} b(n-1, k-1) \\ &= \frac{2^k k!(n-2k)!}{n!} b(n-k, o). \end{aligned}$$

On the other hand if we equate coefficients of x^{n-2k} in (3) (with $B_n = o$) we obtain

$$\begin{aligned} n(n+1)b(n+1-k, o) = A_n n(n+1-2k)b(n-k, o) + \\ + 2k\, C_n b(n-k, o). \end{aligned} \tag{7}$$

Since $A_n = \dfrac{b(n+1, o)}{b(n, o)}$ then (7) can be written as

$$n(n+1)A_{n-k} = n(n+1-2k)A_n + 2k\, C_n \quad (o \leq 2k \leq n). \tag{8}$$

Putting $k = 1$ we obtain

$$2C_n = n(n+1)A_{n-1} - n(n-1)A_n.$$

If we now substituite this value in (8) and put $k = 2$, we get, after some reduction,

$$A_n - 2A_{n-1} + A_{n-2} = o.$$

Hence

$$A_n = a + bn$$

where a, b are arbitrary constants.

If $b = o$ we have $b(n, o) = a^n$. Thus

$$f_n(x) = \left(-\frac{a}{2}\right)^{n/2} H_n\left(x\sqrt{-\frac{a}{2}}\right).$$

On the other hand if $b \neq o$ we get

$$b(n, o) = b^n (a/b)_n \, b(1, o)$$

$$f_n(x) = (-1)^n i^n 2^{\frac{n}{2}} n!\, b^n C_n^{\left(\frac{a}{b}\right)}\left(\frac{ix}{\sqrt{2}}\right).$$

We remark that the condition of WILLIAMS' Theorem can be relaxed even further. For instance we may assume that $\{f_n(x)\}$ satisfy the recurrence

$$(9) \qquad f_{n+1}(x) = (A_n x + B_n) f_n(x) + C_n f_{n-1}(x) + D_n \sum_{i=0}^{n-2} T_i f_i(x)$$

where $A_n C_n \neq o$. Quasi-orthogonal polynomials are of this type [2]. However this is somewhat trivial since it follows from (4) and (9) that $B_n = D_n = o$.

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