Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **20** (1965), n.1, p. 125–127.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1965_3_20_1_125_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1965.

On some problems posed by Karlin and Szegö concerning orthogonal polynomials

by RICHARD ASKEY (a Madison, Wisconsin)

Summary. - *Two results are obtained for Turàn determinants of the classical polynomials.*

In a very interesting paper [3] KARLIN and SZEGÖ have given a number of generalizations of an inequality of TURÁN. They also pose a number of questions and formulate some conjectures. We have two comments to make on these questions.

Turán's inequality is

$$(1) \qquad \Delta(x) = \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} < 0$$

for $-1 < x < 1$, where $P_n(x)$ is the LEGENDRE polynomial of degree n with the usual normalization, $P_n(1) = 1$. SZEGÖ [5] has shown that (1) holds for

(*a*) ultraspherical polynomials, $P_n^\lambda(x)/P_n^\lambda(1)$, $\lambda > -1/2$,

(*b*) LAGUERRE polynomials, $L_n^\alpha(x)/L_n^\alpha(0)$, $\alpha > -1$, $x > 0$,

(*c*) HERMITE polynomials, $H_n(x)$, $-\infty < x < \infty$.

We use the same notation as in SZEGÖ [4].

For ultraspherical polynomials $P_n^\lambda(x)$ with the usual normalization we have (1) for $-1 < x < 1$ only for $\lambda \geq \lambda/2$, see [3, p. 131]. KARLIN and SZEGÖ ask the question as to what normalizations of the classical polynomials give rise to an inequality of the form (1) for all x in the interior of the interval of support of the measure for which they are orthogonal. They give the following condition as a sufficient condition and we notice that it is also necessary.

THEOREM 1. - Let $Q_n(x)$ be one of the polynomials given in (*a*), (*a*), or (*c*). Then $R_n(x) = c_n Q_n(x)$ satisfies (1) for at least as large a set of x, if and only if

(i) $c_n \cdot c_{n+2} > 0, \ n = 0, 1, \ldots$

(ii) $c_n \cdot c_{n+2} - c^2_{n+1} \leq 0$.

The sufficiency of these conditions follows from

$$(2) \qquad R_n \cdot R_{n+2} - R^2_{n+1} = c_n \cdot c_{n+2}[Q_n \cdot Q_{n+2} - Q^2_{n+1}] + \\ + [c_n \cdot c_{n+2} - c^2_{n+1}] Q^2_{n+1}.$$

The necessity of (i) follows from (2) if we choose x as a zero of Q_{n+1}. For $Q_n(x) = P_n^\lambda(x)/P_n^\lambda(1)$ we have $\Delta(1) = 0$ and $Q_n(1) = 1$. But $\Delta(x)$ is continuous and so to have (1) for x close to 1 we must have (ii). The same argument works for polynomials (*b*). For $H_n(x)$ we notice that H^2_{n+1} is a polynomial of degree $2n+2$ and $H_n(x) \cdot H_{n+2}(x) - [H_{n-1}(x)]^2$ is a polynomial of degree $2n$. Thus for large x we must have (ii).

One of the conjectures of KARLIN and SZEGÖ is that the determinants

$$D_n(h, k, x) = \begin{vmatrix} P_n(x) & P_{n+h}(x) \\ P_{n+k}(x) & P_{n+h+k}(x) \end{vmatrix}$$

have $h - 1 + k - 1$ zeros in the interior of the interval of support of the measures for which the polynomials $P_n(x)$ are orthogonal.

Turan's inequality and the generalizations in [5] are just the case $h = k = 1$. For $h = 1$ and k arbitrary this result was shown by KARLIN and SZEGÖ [3]. They also have a few other special cases of this conjecture. For ultraspherical polynomials with n odd. $h = k = 2$, this result is due to FORSYTHE [2] for $\lambda = 1/2$, DANESE [1] for $1/2 < \lambda \leq 1$ and SZEGÖ [6] for $0 < \lambda < 1/2$. However it turns out that this conjecture is false for n odd. $h = k = 2$, $\lambda > 1$ and also for HERMITE polynomials $H_n(x)$. $D_{2n+1}(2, 2, x)$ has a double zero at $x = 0$. For x a zero of $P_n^\lambda(x)$ or of $H_n(x)$ we have $D_{2n+1}(2, 2, x) < 0$. If we show that $D''_{2n+1}(2, 2, 0) > 0$, then $D_{2n+1}(2, 2, x)$ is positive for small x and so $D_{2n+1}(2, 2, x)$ has at least four zeros instead of two. A simple computation shows that

$$D''_{2n+1}(2, 2, 0) = 16 \left[\frac{(2n)!}{(n)!}\right]^2 (2n+1)$$

for $H_n(x)$ and

$$D''_{2n+1}(2, 2, 0) = \frac{2^5\lambda^2(\lambda-1)}{(2n+2\lambda+3)(2n+3)} \cdot$$
$$\cdot\left[\frac{\Gamma(n+\lambda+2)\Gamma(2\lambda)(2n+3)!}{\Gamma(2n+2\lambda+3)\Gamma(\lambda+1)(n+1)!}\right]^2$$

for $P_n^\lambda(x)/P_n^\lambda(1)$.

So many interesting and deep results are true for determinants of Turán type that it is hard to believe that some nice results are not true for the determinants $D_n(h, k, x)$. However we are lacking enough special cases to formulate a reasonable conjecture.

REFERENCES

[1] Arthur E. Danese, *Some identities and inequalities involving ultraspherical polynomials*, «Duke Mathematical Journal», vol. 26, pp. 349-360.

[2] George E. Forsythe, *Second order determinants of Legendre polynomials*, «Duke Mathematical Journal», vol. 18 (1951), pp. 361-371.

[3] S. Karlin and G. Szegö, *On certain determinants whose elements are orthogonal polynomials*, «Journal d'Analyse Mathematique», vol. 8 (1961), pp. 1-157.

[4] G. Szegö, *Orthogonal Polynomials*, «American Mathematical Society Colloquium Publications», vol. 23, New York, 1959.

[5] ——, *On an inequality of P. Turán concerning Legndre polynomials*, «Bulletin of the American Mathematical Society», vol. 54 (1948), pp. 401-405.

[6] ——, *An inequality for Jacobi polynomials*, in *Studies in Mathematical Analysis and Related Topics*, edited by G. Szegö and others, Stanford, California, 1962, pp. 392-398.

Supported in part by N.S.F. grant GP-4985.
Pervenuta alla Segreteria dell'U.M.I.
il 7 ottobre 1964.