Bollettino Unione Matematica Italiana

W. W. Comfort

Functions linearly continuous on a product of Baire spaces.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **20** (1965), n.1, p. 128–134.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1965_3_20_1_128_0>

Functions Linearly Continuous on a Product of Baire Spaces

by W. W. COMFORT [1]

Summary. - *The author uses the topology defined by the family of linearly (i.e., separately) continuous functions on a finite product of Baire spaces to obtain a generalization of a much-studied theorem of Sierpiński.*

Introduction. - A theorem of SIERPIŃSKI asserts that a real-valued function on R^n, if it is continuous separately in each of its variables, is determined by its values on any dense subset of R^n (see [5]). New proofs appear in [6] and [3]; more recently GOFFMAN and NEUGEBAUER, by carefully analyzing their own (new) argument, have obtained a substantial generalization of SIERPIŃSKI's result (see [2]).

The present generalization, though stated not to include the result of [2], is applicable to a broad class of spaces not handled by any of the authors cited above (see 2.5 below). Our approach is elementary: We consider the topology defined by the family of all functions linearly continuous on the given product space, and we show that each nonempty set which is open in this topology contains a nonempty set which is open relative to the usual product topology.

Definitions and Notational Conventions. - Though the expression «separately continuous» seems preferable, we adopt in our first definition the terminology of [2].

DEFINITION. - A function f on a topological product space $X = \prod_{k=1}^{n} X_k$ into a topological space Y is said to be linearly continuous if for each $z = (z_1, z_2, \ldots, z_n)$ in X and each k the map

$$x_k \to f(z_1, \ldots, z_{k-1}, x_k, z_{k+1}, \ldots, z_n)$$

is continuous from X_k into Y.

DEFINITION. - The topological space X is a BAIRE space if each nonempty open subset of X is of second category — i.e., if

(1) The author was supported in part by the National Science Foundation, U.S.A., under contract NSF-GP 2200.

the countable intersection of dense open subsets of X is dense in X.

DEFINITION. - (OXTOBY. See [4]). A subset $\mathfrak{B}$ of $\mathfrak{U}$ is a pseudobase for the topological space $(X, \mathfrak{U})$ provided

(1) $\emptyset \notin \mathfrak{B}$; and (2) if $\emptyset \neq U \in \mathfrak{U}$, then $B \subset U$ for some $B \in \mathfrak{B}$.

In what follows we will be concerned with a finite family $\{(X_k, \mathfrak{U}_k)\}_{1 \leq k \leq n}$ of topological spaces, and the symbols X, $\mathfrak{U}$, $\mathfrak{F}$ and $\mathfrak{T}$ will be used in connection with this family. The symbol X will always denote the cartesian product $\prod_{k=1}^{n} X_k$, and $\mathfrak{U}$ the product topology on X; $\mathfrak{F}$ is the set of linearly continuous real-valued functions on X, and $\mathfrak{T}$ is the smallest topology on X relative to which each element of $\mathfrak{F}$ is continuous.

1. - The results. For each x in X, f in $\mathfrak{F}$ and $\varepsilon > 0$, the set $N(x, f, \varepsilon)$ defined by the identity

$$N(x, f, \varepsilon) = \{ y \in X : | f(x) - f(y) | < \varepsilon \}$$

is clearly $\mathfrak{T}$-open in X, and the family $\mathfrak{B}$ of all such subsets of X is a subbase for $\mathfrak{T}$. We first strengthen this result.

1.1. LEMMA. - $\mathfrak{B}$ is a base for $\mathfrak{T}$.

PROOF. - The result follows easily from 3 G of [1]. In detail, let $z \in N(x_1, f_1, \varepsilon_1) \cap (x_2, f_2, \varepsilon_2)$ and for $1 \leq k \leq 2$ set $\eta_k = \varepsilon_k - | f_k(x_k) - f_k(z) |$, so that

$$z \in \bigcap_{k=1}^{2} N(z, f_k, \eta_k) \subset \bigcap_{k=1}^{2} N(x_k, f_k, \varepsilon_k).$$

With $g_k = | f_k - f_k(z) |$ for $1 \leq k \leq 2$ and $g = g_1 \vee g_2$ and $\varepsilon = \min(\eta_1, \eta_2)$, we have $g \in \mathfrak{F}$ and hence $N(z, g, \varepsilon) \in \mathfrak{B}$. The relation

$$z \in N(z, g, \varepsilon) \subset \bigcap_{k=1}^{2} N(z, f_k, \eta_k)$$

completes the proof.

1.2 THEOREM. - Let X_1 and X_2 be BAIRE spaces and let X_2 admit a countable pseudobase. Then for each point (p_1, p_2) in X, each f in $\mathfrak{F}$, and each $\varepsilon > 0$, we have

$$\operatorname{int}_{\mathfrak{U}} N((p_1, p_2), f, \varepsilon) \neq \emptyset.$$

PROOF. - There is a neighborhood U_1 of p_1 such that $|f(x_1, p_2) - f(p_1, p_2)| < \varepsilon/2$ whenever $x_1 \varepsilon U_1$. If $\{C_k\}_{1 \leq k < \infty}$ is a pseudobase for X_2, then there is for each x_1 in X_1 an integer k such that

$$(*) \qquad |f(x_1, x_2) - f(x_1, p_2)| < \varepsilon/2 \text{ for each } x_2 \text{ in } C_k.$$

Let $V_k = \{x_1 \varepsilon U_1 : (*) \text{ holds}\}$. Then $\bigcup_{k=1}^{\infty} V_k = U_1$, so $\operatorname{int} \operatorname{cl} V_n \neq \emptyset$ for some integer n To complete the proof, it will suffice to show

$$(**) \qquad (\operatorname{cl} V_n) \times C_n \subset N((p_1, p_2), f, \varepsilon).$$

To this end, let $x_1 \varepsilon \operatorname{cl} V_n$, say $x_1 = \lim_\alpha x^\alpha$ where each $x^\alpha \varepsilon V_n$. For each $x_2 \varepsilon C_n$ we have $f(x_1, x_2) = \lim_\alpha f(x^\alpha, x_2)$ and $f(x_1, p_2) = \lim_\alpha (f(x^\alpha, p_2)$. Since

$$|f(x^\alpha, x_2) - f(x^\alpha, p_2)| < \varepsilon/2 \text{ for each } \alpha,$$

we have $|f(x_1, x_2) - f(x_1, p_2)| \leq \varepsilon/2$ whenever $x_1 \varepsilon \operatorname{cl} V_n$ and $x_2 \varepsilon C_n$. For such (x_1, x_2), then, we have

$$|f(x_1, x_2) - f(p_1, p_2)| \leq \varepsilon/2 + |f(x_1, p_2) - f(p_1, p_2)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so that (**) holds.

1.3. - Our aim is to show that if X is a finite product of BAIRE spaces all but one of which admit a countable pseudobase, then each nonempty $\mathfrak{T}$-open subset of X has nonempty $\mathfrak{U}$-interior. The proof depends on the following observation (in which, of course, the symbols need not have the highly restricted meaning assigned to them above), and on a result of OXTOBY which we state in 1.4 below.

LEMMA. - Let $\mathfrak{T}$ and $\mathfrak{U}$ be topologies for a set X, and let $\mathfrak{T} \supset \mathfrak{U}$. Suppose that each nonempty $\mathfrak{T}$-open subset of X has nonempty $\mathfrak{U}$-interior. If $(X, \mathfrak{U})$ is a BAIRE space, then $(X, \mathfrak{T})$ is a BAIRE space.

PROOF. - We show first that if D is a $\mathcal{T}$-dense. $\mathcal{T}$-open subset of X, then the $\mathfrak{A}$-open set $\mathrm{int}_{\mathfrak{A}} D$ is $\mathfrak{A}$-dense in X. Indeed. if $\emptyset \neq U \in \mathfrak{A}$, then $U \in \mathcal{T}$ and $D \cap U$ is nonempty and $\mathcal{T}$-open. so that $\mathrm{int}_{\mathfrak{A}}(D \cap U) \neq \emptyset$; that is, $U \cap \mathrm{int}_{\mathfrak{A}} D \neq \emptyset$.

Now let $\{D_n\}_{1 \leq n < \infty}$ be a sequence of $\mathcal{T}$-dense, $\mathcal{T}$-open subsets of X, and let $\emptyset \neq T \in \mathcal{T}$. Each set $\mathrm{int}_{\mathfrak{A}} D_n$ is $\mathfrak{A}$-dense and $\mathfrak{A}$-open, and $\mathrm{int}_{\mathfrak{A}} T \neq \emptyset$. Thus

$$(\mathrm{int}_{\mathfrak{A}} T) \cap \bigcap_n \mathrm{int}_{\mathfrak{A}} D_n \neq \emptyset,$$

so surely $T \cap \bigcap_n D_n \neq \emptyset$.

1.4. - The form of OXTOBY'S result best suited to our needs is a direct consequence of his Lemma 2.5 and Theorem 3. We shall refer to it later as «OXTOBY's theorem».

THEOREM. - (OXTOBY. See [4]). - The cartesian product of a countable family of BAIRE spaces. each of which has a countable pseudobase, is a BAIRE space with a countable pseudobase.

1.5. THEOREM. - Let $(X_1, \mathfrak{A}_1)$ be a BAIRE space, and for $2 \leq k \leq n$ let $(X_k, \mathfrak{A}_k)$ be a BAIRE space with a countable pseudobase. Then each nonempty $\mathcal{T}$-open subset of X has nonempty $\mathfrak{A}$-interior.

PROOF. - We proceed by induction on n. The assertion is true for $n = 2$ by 1.1 and 1.2 above, and we shall suppose the theorem true for $n = m \geq 2$. We consider the case $n = m + 1$, so that $X = \prod_{k=1}^{m+1} X_k$ and $\mathfrak{F}$ denotes the family of linearly continuous real-valued functions on X.

Let $X' = \prod_{k=1}^{m} X_k$, let $\mathfrak{A}'$ be the product topology on X', and let $\mathcal{T}'$ be the topology induced on X' by the family of linearly continuous real-valued functions on X'. Then $(X', \mathfrak{A}')$ is a BAIRE space by OXTOBY's theorem, so $(X', \mathcal{T}')$ is a BAIRE space by the inductive hypothesis and lemma 1.3.

To complete the proof, it is (according to 1.1) sufficient to show that for each p in X, each f in $\mathfrak{F}$, and each $\varepsilon > 0$. we have $\mathrm{int}_{\mathfrak{A}} N(p, f, \varepsilon) \neq \emptyset$. We consider that product topology on the space $X' \times X_{m+1}$ which this space receives when X' carries the topology $\mathcal{T}'$ and X_{m+1} carries the topology $\mathcal{U}_{m+1}$. The function f, considered for the moment as a function of two variables, is linearly continuous on $X' \times X_{m+1}$, so that by the case $n=2$ of the present theorem there exist nonempty sets $T \in \mathcal{T}'$ and $U_{m+1} \in \mathcal{U}_{m+1}$ such that

$$T \times U_{m+1} \subset N(p, f, \varepsilon).$$

But T contains a nonempty set of the form $\prod_{k=1}^{m} U_k$ with each $U_k \in \mathcal{U}_k$, so that

$$\prod_{k=1}^{m+1} U_k \subset N(p, f, \varepsilon).$$

2. Corollaries and Remarks. - Our first corollary, a generalization of the SIERPIŃSKI result quoted earlier, is similar to theorem 1′ of [2].

2.1 COROLLARY. - Let X_1 be a BAIRE space, and for $2 \leq k \leq n$ let X_k be a BAIRE space with a countable pseudobase. If two linearly continuous real-valued functions agree on a $\mathcal{U}$-dense subset of $\prod_{k=1}^{n} X_k$, then they agree throughout $\prod_{k=1}^{n} X_k$.

PROOF. - The subset of X on which the functions differ is $\mathcal{T}$-open, hence must be empty by 1.5.

2.2. COROLLARY. - Let X_1 be a BAIRE space, and for $2 \leq k \leq n$ let X_k be a BAIRE space with a countable pseudobase. The topology $\mathcal{T}_Y$ induced on X by the family $\mathcal{F}_Y$ of linearly continuous functions from X into the completely regular space Y has the following property: each nonempty $\mathcal{T}_Y$-open set has nonempty $\mathcal{U}$-interior.

PROOF. - The collection

$$\{f^{-1}(V) : f \in \mathcal{F}_Y \text{ and } V \text{ is open in } Y\}$$

is a subbase for the topology $\mathcal{T}_Y$ on X. If z belongs to the basic set $\bigcap_{k=1}^{n} f^{-1}(V_k)$, then for each k there is a continuous function h_k mapping Y to $[0, 1]$ such that $h_k(f_k(z)) = 0$ and $h_k \equiv 1$ off V_k. Then $h_k \circ f_k$ is a linearly continuous real-valued function on X, and we have

$$z \in \bigcap_{k=1}^{n} N(z, h_k \circ f_k, 1/2) \subset \bigcap_{k=1}^{n} f_k^{-1}(V_k).$$

Thus each $\mathcal{T}_Y$-open set is $\mathcal{T}_R$-open, and the result follows from 1.5.

2.3. Remark. - The analogues of our theorem 1.5 and its corollary 2.1 for an infinite product are false. If $X = \prod_{k=1}^{\infty} [0, 1]$ and $f(x) = \inf_{1 \leq k < \infty} x_k$ for each x in X, then f is linearly continuous on the compact metric space X. The function f does not vanish identically on X, but f is everywhere zero on the $\mathfrak{A}$-dense set

$$D = \{x \in X : x_k = 0 \text{ for all but finitely many integers } k\};$$

thus 2.1 fails.

The set $f^{-1}]1/2, 1]$, while non-empty and $\mathcal{T}$-open, misses D, so that 1.5 also fails.

In consonance with Oxtoby's [4], we say that a topological space is quasi-regular if each of its nonempty open subsets contains the closure of some nonempty open set. Any regular space, and hence any completely regular space, is quasi-regular. The following easy lemmas make it possible to construct many product spaces which satisfy the hypotheses of theorem 2.1. We content ourselves with a single example in 2.5.

2.4. Lemma. - Let X be dense in Y.

(a) If X is a Baire space, so is Y;

(b) If X has a countable pseudobase and Y is quasi-regular, then Y has a countable pseudobase.

Proof. (a) If $\{D_n\}_{1 \leq n < \infty}$ is a sequence of dense open subsets of Y, then for any nonempty (relatively) open subset U of X we have $D_n \cap U \neq \emptyset$ for each n, so that each set $D_n \cap X$ is (relatively) dense and open X. If If V is an arbitrary nonempty open subset of Y, then, we set $U = V \cap X$ and we have

$$\begin{aligned} V \cap \bigcap_n D_n &\supset V \cap (\bigcap D_n) \cap X \\ &= (V \cap X) \cap \bigcap_n (D_n \cap X) \\ &\neq \emptyset. \end{aligned}$$

(b) Let $\{U_n\}_{1\leq n<\infty}$ be a countable pseudobase for X, and for each n set $V_n = Y\setminus \mathrm{cl}_Y(X\setminus U_n)$. To see that $\{V_n\}_{1\leq n<\infty}$ is a pseudobase for Y, let V be an arbitrary nonempty open subset of Y and let W be a nonempty open subset of Y for which $\mathrm{cl}_Y W\subset V$. Choosing n so that $U_n\subset W\cap X$, we find that for each p in V_n we have

$$p\in \mathrm{cl}_Y U_n \subset \mathrm{cl}_Y W \subset V.$$

Thus $V_n\subset V$.

The class of BAIRE spaces which admit a countable pseudobase is extensive. According to a well-known theorem of BAIRE the class contains any separable metric space which is locally compact or completely metrizable; it contains any BAIRE space which admits a dense subset that is countable and first-countable; and according to OXTOBY's theorem the countable product of elements of this class is an element of this class. The following example can be easily generalized, then, but it gives an indication of the scope of our theorem.

2.5. EXAMPLE. - For $1\leq k\leq n$ let M_k be separable metric space which admits a complete metric, and let $M_k\subset X_k\subset \beta M_k$. If f is a linearly continuous function on the space $X=\prod_{k=1}^{n} X_k$ to a space Y whose points are distinguished by real-valued continuous functions, and if $f^{-1}(p)$ is dense in X for some p in Y, then $f(x)=p$ for each x in X,

REFERENCES

[1] LEONARD GILLMAN and MEYER JERISON, *Rings of Continuous Functions*, «D. Van Nostrand Company», Princeton, 1960.

[2] CASPER GOFFMAN and C. J. NEUGEBAUER, *Linearly continuous functions*, «Proc. Amer. Math. Soc.», vol. 12 (1961) pp. 997-998.

[3] S. MARCUS, *On functions continuous in each variable*, «Doklady Akad. Nauk SSSR», vol. 112 (1957) pp. 812-814.

[4] J. C. OXTOBY, *Cartesian products of Baire spaces*, «Fundamenta Math.», vol. 49 (1961) pp. 157-166.

[5] WACLAW SIERPINSKI, *Sur une propriété de fonctions de deux variables réelles. continues par rapport à chacune de variables*, «Publ. Math. Univ. Belgrade» vol. 1 (1932) pp. 125-128.

[6] G. Tolstov, *On partial derivatives*, «Amer. Math. Soc. Transl.», N° 69; «Izv. Akad. Nauk SSSR Mat.» vol. 13 (1949) pp. 425-449.

Pervenuta alla Segreteria dell' U.M.I.
il 7 settembre 1964.