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Maximal subgroups of the kernel of a semigroup with left zeroids

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Summary. - *We consider the existence and uniqueness of, and partition by, maximal subgroups of the kernel.*

By a left (right) zeroid of a semigroup S is meant an element λ of S such that for each a in S, λ is in Sa (aS). A zeroid of S is an element of S which is a left zeroid and a right zeroid.

Semigroups with zeroids were investigated by CLIFFORD and MILLER [1]. They found that a semigoup S contains a left zeroid if and only if it contains a universally minimal left ideal L, and then L consists of all the left zeroids of S. Furthermore they showed that if a semigroup S contains a left zeroid and a right zeroid, then each one-sided zeroid is a zeroid, and that if S contains a zeroid, then there in an idempotent element e such that $Se = eS = K = U$, where K denotes the kernel, and U the subgroup of zeroids, of S. It is also true that L is a right ideal of S and that $L = K$ [2, p. 70].

In this paper it is assumed that S contains a left zeroid λ. It follows directly from the above results that $S\lambda$ is the universally minimal left ideal of left zeroids of S, $S\lambda = K$ and $\lambda S \subseteq S\lambda$. Consequently the existence of an idempotent left zeroid e is necessary for eS to be a maximal subgroup of K. We show that it is also sufficient. After considering the uniqueness of maximal subgroups, we establish the existence of a subsemigroup of S of which eS is the zeroid subgroup and find conditions such that K is partitioned by the collection of its maximal subgroups.

It will be convenient to state the following lemma which is adapted from CLIFFORD and PRESTON [2, p. 22].

LEMMA. - If e is an idempotent element of a semigroup S, then (i) if a is in eS, $a = ea$; (ii) if b is in Se, $b = be$; (iii) $eSe = eS \cap Se$.

THEOREM 1. - If S is a semigroup containing a left zeroid λ and an idempotent e, then the following statements are mutually equivalent:

(i) e is a left zeroid of S;

(ii) eS is a maximal subgroup of Se;

(iii) $Se = K$.

PROOF. – Suppose (i) is true. If e is the zero of S, then $e = eS = Se$. If e is not the zero of S, then S contains no zero, since if z is the zero of a semigroup, then z is its only zeroid. Since Se is a minimal left ideal of S, it follows from a theorem of R. J. KOCH [4] (stated in [2, p. 84]) that eSe is a maximal subgroup of S. But $eS \subseteq Se$, so that from the Lemma $eSe = eS$.

Suppose (ii) is true. There is an x in eS such that $x \cdot e\lambda = e$. But $e\lambda$ is a left zeroid of S, so that if $a \in S$, there is a y in S such that $ya = e\lambda$. Hence $xy \cdot a = e$ and e is a left zeroid of S.

If e is a left zeroid of S, we have noted above that $Se = K$. If $Se = K$, then $Se = S\lambda$ and each element of Se is a left zeroid. Therefore (i) and (iii) are equivalent.

If there is only one idempotent left zeroid in S, then the maximal subgroup of the kernel K of S is K; otherwise each two maximal subgroups are isomorphic.

THEOREM 2. – If S is a semigroup which contains among its left zeroids only one, denoted by e, which is idempotent, then eS is the subgroup of zeroids of S.

PROOF. – It is shown by D. F. DAWSON [3] that e is a zeroid of S. Since from Theorem 1 eS is a maximal subgroup of Se and $Se = K$, it is the zeroid subgroup of S [1, p. 120, Theorem 2].

THEOREM 3. – If each of e and f is an idempotent left zeroid of a semigroup S, then eS and fS are isomorphic and disjoint.

PROOF. – Since $Se = Sf$, according to the Lemma $ef = e$ and $fe = f$. Hence $fS = f \cdot eS$ and $eS = e \cdot fS$. If $\lambda \in eS$, then $f\lambda \in fS$, and if $\mu \in fS$, then $e\mu \in eS$, so that $f \cdot e\mu = f\mu = \mu$. Similarly the mapping $e : fS \rightarrow eS$ is onto eS. Now $f\lambda \in Sf$, whence $f\lambda f = f\lambda$. If $\lambda' \in eS$, then $f \cdot \lambda\lambda' = f\lambda \cdot f\lambda'$. That eS and fS are disjoint follows from Theorem 1 and [2, p. 22, Theorem 1.11].

There is however a subsemigroup S_e of S such that eS is the subgroup of zeroids of S_e. With $S_e = \{x : x \in S, xe \in eS\}$ we have the following theorem.

THEOREM 4. – If e is an idempotent left zeroid of a semigroup S, then S_e is a subsemigroup of S and eS is the zeroid subgroup of S_e.

PROOF. – If $x \in S_e$ and $y \in S_e$, then $xy \cdot e = x \cdot ye = x \cdot eye = xe \cdot ye$. Since eS is a group, $xy \cdot e \in eS$ and S_e is a semigroup. If $a \in S_e$, then from the Lemma we have $ae = e \cdot ae = ea \cdot e = ea$.

Hence there is an x in eS such that $xe \cdot a = x \cdot ae = e$. But if $x \in eS$, then $xe \cdot e \in eS$ and $xe \in S_e$. Therefore e is a left zeroid of S_e. Suppose f is a left zeroid of S_e. $f^2 = f$ and $f \neq e$. Since $fe \in eS$ and $fe = f$, we have the contradiction of a group with two idempotents. It follows from Theorem 2 that eS_e is the group of zeroids of S_e. Finally, if $x \in eS$, then $x = ex$ and, since $eS \subseteq Se$, $xe = ex$. Thus $x \in S_e$, whence $x \in eS_e$ and $eS_e = eS$.

There remains for consideration the partition of the kernel K of S by the collection of its maximal subgroups. Let $Q = \{ e : e \in K \text{ and } e^2 = e \}$ and let $P = \{ eS : e \in Q \}$.

THEOREM 5. – If S is a semigroup which contains an idempotent left zeroid e, then the following two statements are equivalent:

(i) $K \subseteq \bigcup_e S_e$;

(ii) P is a partition of K.

PROOF. – Suppose (i) is true. If $e \in Q$ and $f \in Q$, then by Theorem 3 eS and fS are disjoint. If $e \in Q$ and $\lambda \in K$, then $\lambda e = \lambda$, since $K = Se$. There is an e in Q such that $\lambda \in S_e$ and $\lambda = \lambda e \in eS$. Since $eS \subseteq K$, $\bigcup_e eS = K$.

Suppose (i) is false. Then there is a λ in K such that if $e \in Q$, $\lambda e \notin eS$. Since $\lambda = \lambda e$, P is not a partition of K.

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