Bollettino Unione Matematica Italiana

P. J. McCarthy

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **21** (1966), n.3, p. 239–242.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1966_3_21_3_239_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1966.

Note on some arithmetic sums

P J. Mc Carthy (Lawrence, Kans., U.S.A.)

Summary. - *Necessary and sufficient conditions are obtained for certain congruences to hold involving certain arithmetic sums Unilary analogues of some results of Carlitz and Daykin are obtained.*

Let g be an arithmetic function and consider the sum

$$F(g,\ n) = \sum_{d \mid n} \mu(d) g(n/d).$$

In [1] Carlitz obtained necessary and sufficient conditions, involving the fuuction g, that

$$F(g,\ n) \equiv 0 \pmod{n} \quad \text{for all } n \geq 1.$$

Some related results may be found in [4] and [5]. Earlier results concerned with special cases of the sum $F(g, n)$ may be found in Dickson's *History of the Theory of Numbers*, vol. I, pp. 84-86.

In this note we shall discuss several results which are of this same type. We shall consider unitary analogues of $F(g, n)$. Following Cohen [2], we call d a unitary divisor of n if $d \mid n$ and $(d,\ n/d) = 1$. In this case we write $d \| n$. We can consider two analogues of $F(g,\ n)$, namely,

$$F'(g,\ n) = \sum_{d \| n} \mu(d) g(n/d)$$

and

$$F^*(g,\ n) = \sum_{d \| n} \mu^*(d) g(n/d).$$

These sums are over all positive unitary divisors d of n, and the function μ^* is the unitary analogue of the Möbius function μ [2, Theorem 2 5]: $\mu^*(n) = +1$ or -1 accordiug as n has an even or odd number of distinct prime divisors. We shall obtain neccssary and sufficient conditious for $F'(g,\ n) \equiv 0 \pmod{n}$ for all $n \geq 1$ and for $F^*(g,\ n) \equiv 0 \pmod{n}$ for all $n \geq 1$. Actually, we can treat both of these sums at the same time by making use of the techniques of [3]. Thus, we denote by q_k the characteristic function of the set of k-free integers and by μ^*_k the multiplicative function such that for every prime p we have $\mu^*_k(p^e) = -1$ or 0 according

as $1 \leq e < k$ or $e \geq k$.
Set

$$F^*_k(g,\ n) = \sum_{d \mid\mid n} \mu^*_k(d) g(n/d).$$

Then $F'(g,\ n) = F^*_2(g,\ n)$ and $F^*(g,\ n) = \lim_{k \to \infty} F^*_k(g,\ n)$.

THEOREM 1 - *We have*

$$F^*_k(g,\ n) \equiv 0 \pmod{n}$$

for all $n \geq 1$ *if and only if for all primes* p *and all positive integers* e *and* t, *with* t *not divisible by* p,

$$g(p^e t) \equiv 0 \pmod{p^e} \qquad \textit{when } e \geq k,$$

$$g(p^e t) \equiv g(t) \pmod{p^e} \qquad \textit{when } 1 \leq e < k.$$

PROOF. - If $n = p^e m$ where p does not divide m then

$$F^*_k(g\ \ n) = \begin{cases} \sum_{d \mid\mid m} \mu^*_k(d) \{ g(p^e m/d) - g(m/d) \} & \text{if } 1 \leq e < k \\ \sum_{d \mid\mid m} \mu^*_k(d) g(p^e m/d) & \text{if } e \geq k, \end{cases}$$

from which the sufficiency of the condition follows. Now assume that $F^*_k(g, n) \equiv 0 \pmod{n}$ for all $n \geq 1$. By [3, Theorem 2] we have

$$g(n) = \sum_{d \mid\mid n} F^*_k(g,\ d) q_k(n/d).$$

If $n = p^e t$ where p does not divide t then

$$g(p^e t) = \sum_{d \mid\mid t} \{ F^*_k(g,\ d) q_k(p^e t/d) + F^*_k(g, p^e d) q_k(t/d) \}.$$

If $e \geq k$ then $q_k(p^e t/d) = 0$ for each d. Hence, in this case,

$$g(p^e t) = \sum_{d \mid\mid t} F^*_k(g, p^e d) q_k(t/d) \equiv 0 \pmod{p^e}.$$

If $1 \leq e < k$ then $q_k(p^e t/d) = q_k(t/d)$, and so in this case,

$$g(p^e t) - g(t) = \sum_{d \mid\mid t} F^*_k(g,\ p^e d) q_k(t/d) \equiv 0 \pmod{p^e}$$

COROLLARY. - *We have* $F'(g, n) \equiv 0 \pmod{n}$ *for all* $n \geq 1$ *if and only if for all primes* p *and all positive integers* e *and* t, *with* t *not divisible by* p,

$$(A) \qquad \begin{aligned} & g(pt) \equiv g(t) \pmod{p}, \\ & g(p^e t) \equiv 0 \pmod{p^e} \qquad \textit{for } e \geq 2. \end{aligned}$$

We have $F^*(g, n) \equiv 0 \pmod{n}$ *for all* $n \geq 1$ *if and only if for all* p, e, *and* t, *as above,*

(B) $$g(p^e t) \equiv g(t) \pmod{p^e} \text{ for all } e \geq 1.$$

If a is an integer, the function g defined by

$$g(n) = \begin{cases} a^n & \text{if } n \text{ is square free,} \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the conditions (A).

To obtain a function which will satisfy (B), we first define the function Q to be the multipticative function such that for any prime p, $Q(p^e) = p$ or 1 according as $e = 1$ or $e > 1$ Let a be an integer and define the function g by

$$g(n) = a^{Q(n)} \quad \text{for all } n \geq 1.$$

Let $n = p^e t$, where p is a prime and p does not divide t. If $e = 1$ then $g(p^e t) = a^{pQ(t)} \equiv a^{Q(t)} = g(t) \pmod{p}$ by Fermat's Theorem, while if $e > 1$ then $g(p^e t) = a^{Q(t)} = g(t)$. Hence this function g does satisfy (B). With g defined in this way we write $F^*(a, n)$ in place of $F^*(g, n)$.

Let R be a positive integer, w an arithmetic function, a and b integers, and set

$$W^*(w, n; a, b) = \sum_{\substack{d^*e = n \\ (d, R)_* = 1}} w(d)(a^{Q(e)} - b^{Q(e)}),$$

where $d^*e = n$ means that $d \| n$ and $de = n$, and $(d, R)_*$ is the largest divisor of d which is a unitary divisor of R. We shall now obtain a unitary analogue of a theorem of DAYKIN [4].

THEOREM 2. - *We have*

(C) $$\begin{cases} W^*(w, n; a, b) \equiv 0 \pmod{nR} \text{ for all } a \text{ and } b \text{ with} \\ a \equiv b \pmod{R} \text{ and for all } n \geq 1 \text{ such that each prime} \\ \text{divisor of } (n, R) \text{ divides both } n \text{ and } R \text{ only once.} \end{cases}$$

if and only if

(D) $$\sum_{d \| n} w(d) \equiv 0 \pmod{n} \text{ for all } n \geq 1 \text{ such that } (n, R) = 1.$$

PROOF. - Let $r = (n, R)$ and write $n = mr$. Then $(m, R) = 1$ and we have $d \| n$ and $(d, R)_* = 1$ if and only if $d \| m$. Hence

$$W^*(w, n; a, b) = \sum_{d^*e=m} w(d)(a^{Q(e)Q(r)} - b^{Q(e)Q(r)})$$
$$= \sum_{d^*e \| m} w(d) \sum_{s \| e} \{F^*(a^{Q(r)}, s) - F^*(b^{Q(r)}, s)\}$$
$$= \sum_{s^*t=m} \{F^*(a^{Q(r)}, t) - F^*(b^{Q(r)}, t)\} \sum w(d).$$

Now assume (D) and the conditions on a, b, and n of (C). Then (D) and the Corollary to Theorem 1 imply that $W^*(w, n; a, b) = 0 \pmod{m}$. If $p \mid (n, R)$ then $p \mid Q(r)$ and so $a^{Q(e)Q(r)} \equiv b^{Q(e)Q(r)} \pmod{p^2}$. Hence $W^*(w, n; a, b) \equiv 0 \pmod{rR}$, and so (C) holds.

Now assume (C) and let $(n, R) = 1$. Then $d \| n$ and $(d, R)_* = 1$ if and only if $d \| n$. Therefore,

$$\sum_{d^*e=n} \{F^*(a^{Q(r)}, e) - F^*(b^{Q(r)}, e)\} \sum_{s \| d} w(s) \equiv 0 \pmod{nR}.$$

The result now follows by induction on n, arguing as in the last part of the proof of the theorem in [4], and using the Corollary to Theorem 1.

The function μ^* and the unitary analogue of the Euler function, φ^*, are two functions which satisfy (D) [2, Corollaries 2.1.1 and 2.1.2].

Simple examples show that this result is the best possible in the sense that if some prime divisor of (n, R) divides either n or R more than once, then the congruence of (C) may not hold.

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Pervenuta alla Segreteria dell' U. M. I.
il 28 maggio 1966