Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **22** (1967), n.1, p. 15–24.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1967_3_22_1_15_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1967.

Ordered operations in linearly ordered systems

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Summary. - *An algebra of ordered operations operating inter-systems in a class of linearly ordered systems is presented and applied to limiting processes of integration and derivation.*

Introduction. - An algebra of ordered operations in linearly ordered systems was formulated in reference [1], and it was applied in [1], [2] and [3], to the development and application of a theory of tensors defined over sets. Papers on set tensors will not ordinarily draw the attention of readers interested in algebra or in ordered systems; for that reason the author has thought it may be convenient to make a separate presentation of that algebra.

Some analytical applications are included. The author considers that the interconnection of this algebra with analytical processes offers interesting problems for investigation.

The paper is not solely expository; besides correcting an error of omission in [1] and [2], some of the material is presented here for the first time. This applies in particular to the last part.

Algebra of Ordered Operations in Linearly Ordered Systems.

The values of the sequence $\{A_n\}$ may be considered ordered by the natural, induced ordering: $A_m \rightarrow A_n$ iff $m \rightarrow n$. A similar ordering may be considered in the terms of the series $\sum\limits_n A_n$. But the sequence differs from the series in that the second one has an algebraic structure as well as the order structure on its range, while the first one has only the order structure. Both are linearly ordered systems, but the series includes, besides the linearly ordered structure, an algebraic operation that operates *within* the linearly ordered (l.o.) system; that operation is an *intra*-system operation. Our present objective, on the other hand, is an algebra that operates *inter*-systems in the class of all l.o. systems.

Consider the class C of all l.o. systems c, each c being composed of elements belonging to a field F. The *internal* structure of each of these systems $c \in C$ may include more than the order relation between its elements; it may also include operations of F

and passages to the limit (these passages to the limit preserving the order structure). In particular, these l.o. systems may be sequences, finite sums, series, functions (the domains of which are l.o. systems, the range being ordered by the induced ordering), finite and infinite products, etc. Each of these particular structures will be said to belong to a *type* of system of the class C; systems of the same type possess the same internal algebraic structure.

We wish to emphasize that the elements of each $c \in C$ are the *range* values of the sequences, function, series terms, etc., these range values being ordered by the linear ordering of the domain– through the induced ordering. Conversely, any l.o. system can be considered as the range values of a function on a domain that is l.o., with the induced ordering holding.

Each element $c_x \in F$ (x being an ordering index) of c can be multiplied by an element σ of the field F, the same σ multiplying each c_x of c. In this way we make up the class of ordered pairs $(c, \sigma) = CXF$, cartesian product of C and F. To each $\sigma_0 \in F$ corresponds (C, σ_0), a subclass of CXF, and $(c_x\sigma_0)$ will denote a generic element of the system (c, σ_0). For example, if the system (c, σ_0) is of the type of a series, then $(c, \sigma_0) = c_1\sigma_0 + c_2\sigma_0 + c_3\sigma_0 + \dots + c_n\sigma_0 + \dots$ Systems of the same type and similar ordering indeces (i.e., isomorphic with respect to order) will be called *homologus*.

Definition 1. - Let (a, σ) and (b, σ) be arbitrary homologous systems of a subclass (C, σ). To that pair of elements will be associated two other elements (homologous to the original pair) of the same subclass: the ordered sum $(a \oplus b, \sigma)$ and the ordered product $(a \odot b, \sigma)$ defined by

$$(a_x\sigma) \oplus (b_y\sigma) \equiv ([a_x + b_x]\sigma)\,;\ (a_x\sigma) \odot (b_y\sigma) \equiv (a_xb_x\sigma),$$

x and y being ordering indices.

The following are important particular cases:

$$(a_n\sigma) \oplus (b_m\sigma) \equiv ([a_n + b_n]\sigma) \tag{1}$$

$$(\sum_i a_i\sigma) \odot (\sum_k b_k\sigma) \equiv \sum_i a_ib_i\sigma \tag{2}$$

$$[f_1(x)\sigma] \oplus [f_2(y)\sigma] \equiv [f_1(x) + f_2(x)]\sigma$$

$$[f_1(x)\sigma] \odot [f_2(y)\sigma] \equiv (f_1(x)f_2(x))\sigma.$$

When these special cases are taken in the subclass $\sigma = 1$, abbreviation of $(C, 1)$, the first of them coincides with the usually considered sum of two sequences, and the third and fourth with the usually considered sum aud product of two functions.

THEOREM 1. – When systems of the subclass $\sigma = 1$ consist of only one element, their ordered sum and product coincide with their ordinary sum and product in F.

If in $(a, \sigma) \oplus (b, \sigma)$, or in $(a, \sigma) \odot (b, \sigma)$, the F operations that correspond to the internal structure of (a, σ) and (b, σ) are performed *before* carrying out the operations of definition 1, the induced order structure of the systems (in C) will be destroyed, and the ordered operations reduced to the degenerate and trivial case corresponding to Theorem 1. In consequence we introduce the following:

ORDER RULE. – When an expression or a relation involving l.o. systems contains ordered operations, those ordered operations are to be carried out previously to the F operations internal to the systems, whenever these F operations would affect the order structure of the l.o. systems.

As example, application of the order rule implies that, if $\sum_i a_i\sigma = A$ and $\sum_j b_j\sigma = B$, then in general

$$A + B \neq (\sum_i a_i\sigma) \oplus (\sum_j b_j\sigma) = \sum_i (a_i + b_i)\sigma$$

The subclass (c, σ) with the ordered operations defined in definition 1 will be denoted by $\vec{F}_\sigma$. When the calculations do not take us outside of a single subclass, the symbol $\vec{F}$ will suffice. Later on when considering limiting processes, we shall handle sets of subclasses (generated by $\sigma \to 0$); in those cases we must carefully distinguish the subclasses, and the more simplified symbol $\vec{F}$ will not suffice. The symbol $\vec{F}$, however, will be employed to denote the class CXF with the ordered operations of definition 1.

The order rule is but a consequence of the dual nature of a l.o. system that has algebraic operations in its internal structure. Because of these internal algebraic operations, the l.o. system can be reduced to a single number – and this is what is to be done when the l.o. system enters in the expression *as an element in F*. In those cases we operate in F with that number that results from operating with the F operations internal to the l.o. system. But, when the l.o. system enters as an element in $\vec{F}_\sigma$ operations, the order rule states that then it is its nature as a l.o. system (i.e. its ordered structure) that is to be taken into account. In F the algebraic structure dominates; in $\vec{F}_\sigma$ the order structure dominates.

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We may now state the order rule in a different way:

ORDER RULE. – When operating in $\overrightarrow{F_\sigma}$, the only F operations that are permissible are those that do not change the order structure of the l o. systems. Invariance of the order structure is the criterion to decide which F operations are permissible when operating within $\overrightarrow{F}$.

It is evident that each $\overrightarrow{F_\sigma}$ constitutes a ring with divisors of zero. Thus, for example: $[(1, 0)\sigma] \odot [(0, 1)\sigma] = [(0, 0)\sigma]$ which is the neutral element for the additive group in the subclass σ of sequences of two elements.

To eliminate the rather serious limitations that the existence of divisors of zero would bring upon our ordered algebra, we shall impose upon the subclass (C, σ) either one of the two following conditions:

Condition I: Each $c \in C$ possesses as first element an element of F different from 0.

Condition M: There exists one value i_0 of the ordering index such that $c_{i_0} \neq 0$ for every $c \in C$.

When the ordering index is of the nature of a linear interval. condition I requires that the interval be closed on the left. Thus, in $(f(x)\sigma) \odot (g(x)\sigma)$, condition I would demand that the domain of definition of $f(x)$ and $g(x)$ be closed on the left.

If it is desired to operate with l.o. systems that do not possess a first element (like when handling functions with the whole real line as domain of definition), then condition M should be imposed. While condition M is more general than condition I, this last one is of much easier verification.

From now on it will be considered that subclass (C, σ) obeys *either* condition I *or* condition M. Such subclasses will be called *admissible*, and they will be the only ones handled in this study, unless otherwise stated.

The restriction to admissible subclasses is not strongly limiting, since any finite set of bounded l.o. systems can be transformed into an admissible one by a translation. Condition M (or its special case, condition I) plays in this study a role similar, but not equal, to that played in the theory of matrices by the condition of non-vanishing determinant.

THEOREM 2. – Every admissible class (C, σ) constitutes a field $\overrightarrow{F_\sigma}$.

For future reference we shall introduce the following notation:

Identity system of the subclass (C, σ) for the ordered sum:

$$(0, \sigma) = \overrightarrow{0_\sigma} .$$

Identity system of the subclass (C, σ) for the ordered product:

$$(1, \sigma) = \overrightarrow{1_\sigma} .$$

Inverse to the system (a, σ) for the ordered sum:

$$(-a, \sigma) = \ominus (a, \sigma).$$

Inverse to the system (a, σ) for the ordered product;

$$\left(\left(\frac{1}{a}\right), \sigma\right) = (a, \sigma)^{-1} = (a^{-1}\sigma).$$

THEOREM 3. - In any subclass (C, σ) of CXF, the operation of ordered multiplication is distributive with respect to the operation of ordered addition.

PROOF. - $$\begin{aligned}(c_z\sigma) \odot \{(a_x\sigma) \oplus (b_y\sigma)\} &= c_z\sigma \odot ([a_x + b_x]\sigma) = \\ &= (c_x[a_x + b_x]\sigma) = ([c_x a_x + c_x b_x]\sigma) = \\ &= (c_x a_x \sigma) \oplus (c_y b_y \sigma) = \\ &= \{(c_z\sigma) \odot (a_x\sigma) \oplus (c_z\sigma) \odot (b_y\sigma)\}.\end{aligned}$$

DEFINITION 2. - If k is arbitrary element of F, and (a, σ) an arbitrary element of $\overrightarrow{F_\sigma}$, an operation fom $FX\overrightarrow{F_\sigma}$ to $\overrightarrow{F_\sigma}$ will be defined by $k(a, \sigma) \equiv (ka, \sigma)$. This operation will be called multiplication by a scalar.

THEOREM 4. - For any arbitrary fixed σ, the subclass (C, σ) with the operations of definitions 1 and 2 constitutes an algebra.

PROOF. - The two distributivity laws follow from definition 2, Theorem 3 and the commutativity of ordered multiplication, since $(ka, \sigma) = (k, \sigma) \odot (a, \sigma) = (k_x\sigma) \odot (a_y\sigma)$ with $k_x = k$ for all x. Also because of these equalities, the associativity of multiplication by a scalar follows from the associativity of the ordered product. Finally, $0(x, \sigma) = (0, \sigma) = \overrightarrow{0_\sigma}$, $1(x, \sigma) = (x, \sigma)$ and $k[(a, \sigma) \odot (b, \sigma)] = (ka, \sigma) \odot (b, \sigma) = (a, \sigma) \odot (k, \sigma) \odot (b, \sigma) = (a, \sigma) \odot (kb, \sigma)$ by the commutativity of ordered multiplication.

Consider the following system of two equations in two unknowns:

$$(3)\qquad \begin{cases} (A_a\sigma) \odot (X_x\sigma) \oplus (B_b\sigma) \odot (Y_y\sigma) = (H_h\sigma) \\ (C_c\sigma) \odot (X_x\sigma) \oplus (D_d\sigma) \odot (Y_y\sigma) = (K_k\sigma) \end{cases}$$

THEOREM 5. – The set of equations (3) prossesses a unique solution iff

$$[(A_a\sigma) \odot (D_d\sigma) \ominus (B_b\sigma) \odot (C_c\sigma)] \neq \vec{0}.$$

PROOF. – The demonstration follows easily from preceding material; we shall limit ourselves to deriving the actual form of the solution, mostly as manipulative exercise in the algebra of ordered operations.

The inverse of $[(D_d\sigma) \odot (Y_y\sigma)]$ under addition is $[(-D_d\sigma) \odot (Y_y\sigma)]$, and, referring definition 2 with $k = -1$:

$$[(D_d\sigma) \odot (Y_y\sigma)] \equiv [(-D_d\sigma) \odot (Y_y\sigma)] \equiv \ominus [(D_d\sigma) \odot (Y_y\sigma)].$$

Applying that to solve for $(X_x\sigma)$ in the second equation of (3):

$$(X_x\sigma) = [(K_k\sigma) \ominus (D_d\sigma) \odot (Y_y\sigma) \odot (C_c^{-1})].$$

Substituting in the first equation of (3) and solving:

$$\begin{aligned} (Y_y\sigma) = {} & \{ (H_h\sigma) \ominus [(A_a\sigma) \odot (K_k\sigma) \odot (C_c^{-1}\sigma)] \} \\ & \odot \{ [(A_a\sigma) \odot (D_d\sigma) \odot (C_c^{-1}\sigma)] \oplus (B_b\sigma) \}^{-1}. \end{aligned}$$

Multiplying the right hand side by $(C_c\sigma)\ (C_c\sigma)^{-1}$ it is finally obtained

$$\left.\begin{aligned} (Y_y\sigma) = {} & [(A_a\sigma) \odot (K_k\sigma) \ominus (H_h\sigma) \odot (C_c\sigma)] \\ & \odot [(A_a\sigma) \odot (D_d\sigma) \ominus (B_b\sigma) \odot (C_c\sigma)]^{-1}. \end{aligned}\right\} \qquad (4a)$$

Similarly:

$$\left.\begin{aligned} (X_x\sigma) = {} & [(B_b\sigma) \odot (K_k\sigma) \ominus (H_h\sigma) \odot (D_d\sigma)] \\ & \odot [(A_a\sigma) \odot (D_d\sigma) \ominus (B_b\sigma) \odot (C_c\sigma)]^{-1}. \end{aligned}\right\} \qquad (4b)$$

Because $0 \odot (x, \sigma) = (0, \sigma) \odot (x, \sigma) = (0, \sigma) = \vec{0}$ for any (x, σ), it is seen that for (4a) and (4b) to have meaning it is necessary and sufficient that

$$[(A_a\sigma) \odot (D_d\sigma) \ominus (B_b\sigma) \odot (C_c\sigma)]^{-1} \neq \vec{0}.$$

The procedure and theorem are extensible without difficulty, except for notational inconvenience, to the case of n equations in n unknowns.

In (b, σ) it may happen that b is a l.o. system of l.o. systems. It will then be linearly ordered by two indices, the two orderings being independent. Then we may assert

THEOREM 6. – Suppose that (b_{xy}, σ) is homologous to (a, σ) with respect to x, and is homologous to (c, σ) with respect to y, then

$$(a_u\sigma) \odot [(b_{xy}\sigma) \odot (c_\iota\sigma)] = [(a_u\sigma) \odot (b_{xy}\sigma)] \odot (c_v\sigma) = (a_x b_{xy} c_y \sigma).$$

The demonstration is immediate.

From (1) and (2) it is of easy verification that the convergence of $(a_n\sigma)$, $(b_m\sigma)$, $(\sum_{i=1}^{n} b_h\sigma)$ and $(\sum_{k=1}^{m} b_k\sigma)$, as $n, m \to \infty$, implies the convergence of $(a_n\sigma) \oplus (b_m\sigma)$ and of $(\sum_i a_i\sigma) \odot (\sum_k b_k\sigma)$. The corresponding assertions can be made for $(a_n\sigma) \odot (b_m\sigma)$ and for $(\sum_i a_i\sigma) \oplus (\sum_k b_k\sigma)$.

Limiting processes.

Limiting processes are introduced in $\overrightarrow{F}$ by considering sequences of expressions in $\overrightarrow{F}$, each element of the sequence being in correspondence with a value of σ, and making σ pass to a limit.

More generally, we may consider the set of expressions (u, σ), where u belongs to some infinite set $U \subset C$. Then and

DEFINITION 3. – $(u_0, \sigma_0) = \lim_{\sigma \to \sigma_0} (u, \sigma)$ when $\sigma \to \sigma_0$ implies that $u \to u_0$.

Interesting limiting processes in $\overrightarrow{F}$ are the following.

Consider an amorphous space of points in which we introduce a coordinate system X. The coordinates of a point P will be denoted by x_P^j $(j = 1 \dots n)$, or by x_P^J, the capital upper J standing for $j (j = 1 \dots n)$. In this space we shall consider measurable sets S, V, etc. Let $\omega(S)$ represent the measure of S.

DEFINITION 4. – We shall say that we have associated to S, by means of X, a *regular partition* (R.P.) when X generates a family ${}_xS_T$ of intervals with the following properties:

1. the intersection of any two intervals of ${}_xS_T$ is empty.

2. all intervals have the same measure $\omega(S_\iota)$.

3. the ratio of the one-dimensional measures of any two edges of any interval is bounded.

4. S is contained in the union $\bigcup_{\iota \in T} S_\iota$.

5. for no value of i is the intersection $(S \cap S_i)$ empty.

6. the intervals of the family ${}_x S_T$ are ordered.

Now consider the expression

$$I_S(f) = \lim_{\omega(S_k) \to 0}(\text{R.P.}) \sum_{k \in T} [f(x^J)]_{P_k} \omega(S_k),$$

where

P_k represents an arbitrary point of S_k;

f_{P_k} is the value, at the point P_k, of the function $f(x^J)$;

lim(R.P.) means that the passage to the limit is carried out under the conditions imposed by regular partitions.

Observe that $\lim_{\omega(S_k) \to 0}(\text{R.P.}) \sum_{k \in T} \omega(S_k) = \omega(S)$.

In terms of the elements of $\vec{F}$, $(\sum_{k \in T} [f(x^J)_{P_k}]) \in C$, while $\omega(S_k) = \sigma \in F$. I_S is a l.o. system of l.o. systems.

Theorem 7. – If $f(x^j)$ is integrable, then

$$I_S(f) = \int_S f(x^J) d\omega.$$

Theorem 8. – The following relation holds:

$$I_S(f) \odot I_S(g) = I_S(f \odot g) = I_S(fg).$$

Proof. – Consider the expressions

$$\sum_{k \in T} [f(x^J)]_{P_k} \omega(S_k)$$

and

$$\sum_{l \in T} [g(x^J)]_{Q_l} \omega(S_l).$$

Operating upon sums corresponding to the same partition we have that $\omega(S_k) = \omega(S_l)$, and both sums belong to the same subclass $\sigma = \omega(S_k)$.

We may hence apply the ordered product, using as ordering index for the product the same one that orders the intervals in the R.P. Remembering then that, according to the Order Rule

$$I_S(f) \odot I_S(g) =$$

$$= \lim_{\omega(S_k) \to 0} \{ (\text{R.P.}) \sum_{k \in T} [f(x^J)]_{P_k} \omega(s_k) \odot (\text{R.P.}) \sum_{l \in T} [g(x^J)]_{Q_l} \omega(S_l) \};$$

on passing to the limit the thesis of the theorem follows.

COROLLARY. - If $f(x^J)$ and $g(x^J)$ are integrable, then

$$\int_S f(x^J)d\omega \odot \int_S g(x^J)d\omega = \int_S (f \odot g)d\omega.$$

PROOF. - Consequence of theorem 7 and 8.

The preceding theorems have been used in [1] and [2] to formulate a theory of tensors defined over arbitrary mesurable sets.

Consider now the $l.$o. set the elements of which are $\Delta_i f_x = [f(x_i) - f(x)]\frac{1}{x_i - x}$, $i \in R$, (R set of real numbers), ordered by the induced ordering $\Delta_i f_x \rightarrow \Delta_j f_x$ iff $(x_i - x) > (x_j - x)$. Each $\Delta_i f_x$ itself is a $l.$o. system of two elements: the terms $f(x_i)$ and $f(x)$ in the subclass $\sigma = \frac{1}{x_i - x}$. In terms of the elements of $\overrightarrow{F}$, $[f(x_i) - f(x)] \in C$ while $\frac{1}{x_i - x} = \sigma \in F$.

It is immediate that, if the function f possesses an ordinary derivative $D_x f$ at the point x, then $\lim\limits_{x_i \to x} \Delta_i f_x = D_x f$.

THEOREM 9. - If the functions f and g possess ordinary derivatives at the point x, then the following relations hold in $\overrightarrow{F}$:

$$D_x F \odot D_x g = D_x(f \odot g).$$

PROOF. - Applying the order rule, in the subclass $\sigma = \frac{1}{x_i - x}$:

$$\begin{aligned} D_x f \odot D_x g &= \lim_{x_i \to x} \left\{ [f(x_i) - f(x)] \frac{1}{x_i - x} \odot [g(x_i) - g(x)] \frac{1}{x_i - x} \right\} \\ &= \lim_{x_i \to a} [f(x_i)g(x_i) - f(x)g(x)] \frac{1}{x_i - x} \\ &= \lim_{x_i \to x} [f(x_i) \odot g(x_i) - f(x) \odot g(x)] \frac{1}{x_i - x} = D_x(f \odot g) \end{aligned}$$

Observe that, while $fg = f \odot g$ in the subclass $\sigma = 1$, it does not follow that in general $D_x(fg) = D_x(f \odot g)$. This should not be surprising because, in the first place, we are operating in the subclass $\sigma = \frac{1}{x_i - x}$. But more important is the fact that, to obtain the well known formula $D_x(fg) = fD_x(g) + gD_x(f)$, we perform operations in F that change the order structure in $[f(x_i) \odot g(x_i) - f(x) \odot g(x)]\frac{1}{x_i - x}$; indeed, those operations add terms to the order structure. To evaluate $D_x(f \odot g)$ the order rule must be followed, and the order structure must be left *invariant* by the ordered operations.

THEOREM 10. - If the functions f and g possess derivative functions that are integrable over the interval $[x_1, x_2]$, then

$$\int_{x_1}^{x_2} D_x(f \odot g)dx = f(x_2)g(x_2) - f(x_1)g(x_1).$$

PROOF. - By direct application of preceding theorems:

$$\int_{x_1}^{x_2} D_x(f \odot g)dx = \int_{x_1}^{x_2} [D_x f \odot D_x g]dx = \int_{x_1}^{x_2} D_x f dx \odot \int_{x_1}^{x_2} D_x g dx$$
$$= [f(x_2) - f(x_1)] \odot [g(x_2) - g(x_1)] = [f(x_2) \odot g(x_2)]$$
$$- [f(x_1) \odot g(x_1)] = f(x_2)g(x_2) - f(x_1)g(x_1).$$

As an interesting application of the preceding consider the following function in $\vec{F}_1$:

$$E^x \equiv 1 \odot \frac{x}{1!} \odot \frac{x^2}{2!} \odot \frac{x^3}{3!} \odot \ldots \odot \frac{x^n}{n} \odot \ldots = \lim_{n \to \infty} \Pi \frac{x^n}{n!}.$$

If R represents the real line, the following results are of immediate demonstration:

THEOREM 11. - $D_x(E^x) = E^x$ and $I_R(E^x) = E^x$.
Observe that in general $E^{x \oplus y} \neq E^x \odot E^y$. On the other hand

THEOREM 12. - $E^{x \odot y} = E^x \odot E^y$.

An interesting area of investigation may be the possible relations between differential and integral equations in $\vec{F}$ and the corresponding equations in F.

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Pervenuta alla Segreteria dell' U.M.I.
il 3 giugno 1965