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On the asymptotic equivalence of systems of ordinary differential equations.

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Summary. - *In this paper we deal with the asymptotic equivalence of the linear system* (A) *and the "quasi-linear" system* (A_g) *and improve previous results on the subject of H. Weyl* [6], *V. A. Yakubovich* [7] *and R. Conti* [3].

1. – Consider the systems of ordinary differential equations

$$(A) \qquad \dot{y} = A(t)y \qquad\qquad \left(\cdot = \frac{d}{dt}\right).$$

$$(A_g) \qquad \dot{x} = A(t)x + g(t, x)$$

where $A(t)$, $t \in [t_0, +\infty)$ is an $n \times n$ complex matrix with entries summable functions in every finite subinterval of $[t_0, +\infty)$ and $g(t, x)$, $(t, x) \in [t_0, +\infty) \times C^n$ (C is the complex plane) is an n-dimensional complex vector.

Moreover, we suppose that the system (A_g) is "*quasi-linear*", that is g satisfies the following conditions:

$$(1) \qquad \int_{t_0}^{+\infty} |g(s, 0)|\, ds = \gamma < +\infty \ (^1)$$

$$(2) \qquad |g(t, x) - g(t, y)| \leq l(t)\,|x - y|$$

for every (t, x), (t, y) in $[t_0, +\infty) \times C^n$

$$(3) \qquad \int_{t_0}^{+\infty} l(t)\, dt = l < +\infty.$$

Hence, the uniqueness of the solutions of (A_g) is valid and the right end-point of the domain (interval) of the solutions of this system is $+\infty$.

We suppose further that the system (A) is *uniformly stable* which implies that *the solutions of* (A_g) *are bounded and uniformly stable.* This can be easily proved by the argument used in [1; p. 97, lemma].

(¹) By definition, $|x| = \sum_i |x_i|$ and $|X| = \sum_{i,j} |x_{ij}|$, where x is a complex vector and X a complex matrix.

Definition 1. – *The systems* (A) *and* (A_g) *are called asymptotically equivalent if and only if there exists a homeomorphism* $\omega : C^n \to C^n$ *such that*

$$\lim_{t \to +\infty} [x(t;\, \xi) - y(t;\, \omega(\xi))] = 0,$$

where $x(t;\, \xi)$ *is the solution of* (A_g) *with* $x(t_0;\, \xi) = \xi$ *and* $y(t;\, \omega(\xi))$ *the solution of* (A) *with* $y(t_0;\, \omega(\xi)) = \omega(\xi)$.

In a paper by R. Conti [4] it was stated (Teorema I) that quasi-linearity of (A_g) plus uniform stability of (A) are sufficient to insure asymptotic equivalence between (A) and (A_g), but the proof is not correct. While it remains an open question whether this statement is true or not, we are going in what follows to prove it under an additional assumption on $A(t)$ (Theorem 1). Our result includes previous ones by H. Weyl [6] or [5; p. 514], V. A. Yakubovich [7] and R. Conti [3].

2. – Case of $A(t)$ having a Jordan canonical form. Let $J(t)$, $t \in [t_0, +\infty)$ be àn $n \times n$ complex matrix having the Jordan canonical form, that is having blocks $J_r(t)$, $r = 1, \ldots, m$ down the main diagonal and zeros elsewhere, where $J_r(t)$ is an $n_r \times n_r$ matrix of the form $J_r(t) = \lambda_r(t)E_r + Z_r$ with E_r the unit $n_r \times n_r$ matrix and Z_r the $n_r \times n_r$ matrix of the form

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \text{ if } n_r > 1,$$

otherwise the 1×1 null matrix.

Let $A(t) = J(t)$, $t \in [t_0, +\infty)$ and let $Y(t) = (y_{ij}(t))$ be the principal fundamental matrix of (A), i.e.

$$(4) \qquad \begin{cases} y_{ij}(t) = \dfrac{(t - t_0)^{j-i}}{(j - i)!} \exp \displaystyle\int_{t_0}^{t} \lambda_r(s)ds \text{ for } \nu_{r-1} < i \leq j \leq \nu_r, \\ \text{where } \nu_0 = 0,\ \nu_r = n_1 + \ldots + n_r \\ y_{ij}(t) = 0 \text{ for all the other indices } i,\ j. \end{cases}$$

For the matrix $Y(t)Y^{-1}(\tau) = (\psi_{ij}(t, \tau))$, an easy computation leads to

$$(5) \qquad \begin{cases} \psi_{ij}(t, \tau) = \dfrac{(t-\tau)^{j-i}}{(j-i)!} \exp \displaystyle\int_\tau^t \lambda_r(s)ds \text{ for } \nu_{r-1} < i \leqq j \leqq \nu_r \\ \psi_{ij}(t, \tau) = 0 \text{ for all the other indices } i, j. \end{cases}$$

Under the assumptions of this and the previous section the following lemmas hold.

LEMMA 1 - *If* $T = const.$, $T \in [t_0, +\infty)$, *then* $\lim_{t\to+\infty} \psi_{ij}(t, T) = 0$, *otherwise* $i = j$ *and* $\psi_{ii}(t, T)$ *is bounded in* $[t_0, +\infty)$.

PROOF. - It is sufficient, by (5), to prove the lemma for the functions $\psi_{\nu_{r-1}+1, \nu_r}(t, T) = \dfrac{(t-T)^{n_{r-1}}}{(n_r-1)!} \exp \displaystyle\int_T^t \lambda_r(s)\,ds$. The uniform stability of the system (A) means that there exists a constant $c > 0$ such that

$$(6) \qquad |Y(t)Y^{-1}(\tau)| < c \text{ for every } t, \tau \text{ with } t_0 \leqq \tau \leqq t$$

and consequently

$$(7) \qquad 0 \leqq \psi_{\nu_{r-1}+1, \nu_r}(t, T) < c \text{ for every } t \in [T, +\infty).$$

From (6), it follows also that

$$\psi_{\nu_{r-1}+1, \nu_r}\left(t, \frac{t+T}{2}\right) = \frac{1}{2^{n_r-1}} \psi_{\nu_{r-1}+1, \nu_r}(t, T) \exp\left(-\int_T^{\frac{t+T}{2}} \lambda_r(s)ds\right) < c,$$

i.e.

$$\psi_{\nu_{r-1}+1, \nu_r}(t, T) < c2^{n_r-1} \exp \int_T^{\frac{t+T}{2}} \lambda_r(s)ds = (n_r-1)!\, \frac{c4^{n_r-1}}{(t-T)^{n_r-1}} \cdot$$
$$\cdot\ \psi_{\nu_{r-1}+1, \nu_r}\left(\frac{t+T}{2}, T\right).$$

Hence, by (7),

$$0 \leqq \psi_{\nu_{r-1}+1, \nu_r}(t, T) < (n_r-1)!\, \frac{c^2 4^{n_r-1}}{(t-T)^{n_r-1}} \quad \text{for every } t \in (T, +\infty)$$

which proves the lemma.

Let now $I = \{(i, j) : \lim_{t \to +\infty} y_{ij}(t) = 0\}$. Then, by lemma 1 ($T = t_0$), it follows that $(i, j) \notin I$ implies $i = j$. If $U(t) = (u_{ij}(t))$, $t \in [t_0, +\infty)$ is the $n \times n$ matrix with $u_{ij}(t) = 0$ for $(i, j) \in I$ and $u_{ii}(t) = y_{ii}^{-1}(t)$ for $(i, i) \notin I$, then the following lemma holds.

LEMMA 2. - *The integrals* $\int_{t_0}^{+\infty} U(\tau)\, g(\tau, x(\tau; \xi))\, d\tau$ *and* $\int_{t_0}^{+\infty} |U(\tau)|\, l(\tau)\, d\tau$ *exist*

PROOF. - We have that

$$\int_{t_0}^{+\infty} |U(\tau)|\ |g(\tau, x(\tau; \xi))|\ d\tau = \sum_{(i, i) \notin I} \int_{t_0}^{+\infty} y_{ii}^{-1}(\tau)\ |g(\tau, x(\tau; \xi))|\ d\tau$$

By (1), (2), (3), (4), (5) and (6), we get

$$y_{ii}(t) \int_{t_0}^{t} y_{ii}^{-1}(\tau)\ |g(\tau, x(\tau; \xi))|\ d\tau = \int_{t_0}^{t} \psi_{ii}(t, \tau)\ |g(\tau, x(\tau; \xi))|\ d\tau <$$

$$c \int_{t_0}^{t} |g(\tau, x(\tau; \xi)) - g(\tau, 0)|\ d\tau + c \int_{t_0}^{t} |g(\tau, 0)|\ d\tau \leqq c \int_{t_0}^{t} l(\tau)\ |x(\tau; \xi)|\ d\tau +$$

$$+ c \int_{t_0}^{t} |g(\tau, 0)|\ d\tau \leqq c c_\xi l + c\gamma = k,$$

where c_ξ is a bound of $x(t; \xi)$ in $[t_0, +\infty)$.

Hence

$$\int_{t_0}^{t} y_{ii}^{-1}(\tau)\ |g(\tau, x(\tau; \xi))|\ d\tau < k y_{ii}^{-1}(t)$$

and if $\int_{t_0}^{+\infty} y_{ii}^{-1}(\tau)\ |g(\tau, x(\tau; \xi))|\ d\tau = +\infty$, then $\lim_{t \to +\infty} y_{ii}(t) = 0$ which contradicts $(i, i) \notin I$. It is obvious that the first integral exists also when we replace the solution $x(t; \xi)$ by any function which is bounded in $[t_0, +\infty)$.

Similarly we have

$$\int_{t_0}^{+\infty} |U(\tau)|\, l(\tau)\, d\tau = \sum_{(i,i)\notin I} \int_{t_0}^{+\infty} y_{ii}^{-1}(\tau)\, l(\tau)\, d\tau$$

and

$$y_{ii}(t)\int_{t_0}^{t} y_{ii}^{-1}(\tau) l(\tau)\, d\tau = \int_{t_0}^{t} \psi_{ii}(t, \tau)\, l(\tau)\, d\tau < c\int_{t_0}^{t} l(\tau)\, d\tau \leqq cl.$$

Hence

$$\int_{t_0}^{t} y_{ii}^{-1}(\tau)\, l(\tau)\, d\tau < c\, l\, y_{ii}^{-1}(t)$$

and if $\int_{t_0}^{+\infty} y_{ii}^{-1}(\tau)\, l(\tau)\, d\tau = +\infty$, then $\lim_{t\to+\infty} y_{ii}(t) = 0$ which also contradicts $(i,\ i) \notin I$.

Lemma 3. – *The systems* (A) *and* (A_g) *are asymptotically equivalent.*

Proof. – By virtue of lemma 2, we define the homeomorphism $\omega: C^n \to C^n$ by

$$\omega(\xi) = \xi + \int_{t_0}^{+\infty} U(\tau) g(\tau, x(\tau, \xi)) d\tau. \tag{8}$$

The continuity of the function ω can be easily derived from

$$|\omega(\xi) - \omega(\xi^*)| < |\xi - \xi^*| + \int_{t_0}^{+\infty} |U(\tau)|\, l(\tau)\, |x(\tau; \xi) - x(\tau; \xi^*)| d\tau,$$

lemma 2 and the (uniform) stability of the solutions of (A_g). Moreover, we have to prove that the range of ω is the whole space C^n and that ω is one-to-one. To this end we prove first that for any $\eta \in C^n$ there exists a $\xi \in C^n$ such that

$$\begin{aligned} x(t; \xi) = y(t; \eta) - Y(t)\int_{t_0}^{+\infty} U(\tau) g(\tau, x(\tau; \xi))\, d\tau \\ + \int_{t_0}^{t} Y(t)\, Y^{-1}(\tau) g(\tau, x(\tau; \xi)) d\tau. \end{aligned} \tag{9}$$

In fact, without loss of generality we can assume, by (3) and lemma 2. that t_0 is such that

$$c\left[\int_{t_0}^{+\infty} |U(\tau)|\ l(\tau)\,d\tau + \int_{t_0}^{+\infty} l(\tau)\,d\tau\right] = q < 1, \tag{10}$$

since a finite shifting does not affect the substance of the question by virtue of the continuous dependence of the solutions on the initial values.

Now, by means of successive approximations

$$x_0(t) = y(t;\ \eta)$$

$$x_{\nu+1}(t) = y(t;\ \eta) - Y(t)\int_{t_0}^{+\infty} U(\tau)g(\tau,\ x_\nu(\tau))d\tau$$

$$+\int_{t_0}^{t} Y(t)Y^{-1}(\tau)g(\tau,\ x_\nu(\tau))d\tau \quad (\nu = 0,\ 1,\ 2,\ \ldots)$$

it can be easily verified that

$$|x_{\nu+1}(t) - x_\nu(t)| < q^\nu c_\eta \quad \text{for every} \quad t \in [t_0,\ +\infty), \tag{11}$$

where c_η is a bound of the solution $y(t;\ \eta)$. Thus, (9) can be easily derived from (11).

From (8) and (9) it follows immediately that

$$\eta = \omega(\xi), \tag{12}$$

i.e. that the range of ω is the whole space C^n.

On the other hand, supposing $\omega(\xi) = \omega(\xi^*)$, we have by (2), 6 , (9), (10) and (12) that

$$|x(t;\xi) - x(t;\xi^*)| \leq q \sup_{t\in[t_0,+\infty)} |x(t;\xi) - x(t;\xi^*)| \text{ for every } t\in[t_0,+\infty),$$

i.e. $x(t;\xi) = x(t;\xi^*)$ for every $t \in [t_0,\ +\infty)$ and hence $\xi = \xi^*$ which proves that ω in one-to-one.

It remains to prove now that

$$\lim_{t\to+\infty}\ [x(t;\xi) - y(t;\ \omega(\xi))] = 0. \tag{13}$$

It is easy to verify that

$$x_i(t;\xi) - y_i(t;\omega(\xi)) = \sum_j F_{ij}(t,\ T), \tag{14}$$

where $F_{ij}(t.\ T) = \psi_{ij}(t,\ T)[x_j(T;\xi) - y_j(T;\omega(\xi)] + \int_T^t \psi_{ij}(t,\tau)g_j(\tau,\ x(\tau;\xi))d\tau$.

Now, we consider the following two cases

a. $(i,\ j) \in I$. Because of the boundedness of the solutions of the systems (A) and (A_g), there exists a constant $M > 0$ such that

$$|x(t;\xi) - y(t;\omega(\xi))| < M \text{ for every } t \in [t_0,\ +\infty). \tag{15}$$

Let $T,\ T \in [t_0,\ +\infty)$ be chosen so that

$$\int_T^t l(\tau)\,d\tau < \frac{\varepsilon}{2n^2cc_\xi} \text{ and } \int_T^t |g(\tau,\ 0)|\,d\tau < \frac{\varepsilon}{2n^2c} \text{ for every } t \in [T, +\infty\ , \tag{16}$$

where c_ξ is a bound of the solution $x(t;\xi)$ in $[t_0,\ +\infty)$.
By (2), (15) and (16) we obtain

$$|F_{ij}(t,\ T)| \leq \psi_{ij}(t,\ T)M + cc_\xi \int_T^t l(\tau)\,d\tau + c\int_T^t |g(\tau,\ 0)|d\tau < M\psi_{ij}(t,\ T) + \frac{\varepsilon}{n^2},$$

which, by lemma 1, implies that

$$\limsup_{\to+\infty} |F_{ij}(t,\ T)| < \frac{\varepsilon}{n^2}. \tag{17}$$

b. $(i,\ j) \notin I$. In this case $i = j$ and, by virtue of (4) and (5), one can easily verify that

$$F_{ii}(t,\ T) = y_{ii}(t)[\xi_i - \omega_i(\xi) + \int^t y_{ii}^{-1}(\tau)g_i(\tau,\ x(\tau;\ \xi))d\tau]$$

and by (8),

$$|F_{ii}(t,\ T)| = y_{ii}(t)\left|\int_t^{+\infty} y_{ii}^{-1}(\tau)g_i(\tau,\ x(\tau;\ \xi))d\tau\right|$$

which, by virtue of the lemmas 1 and 2, implies that

(18) $$\lim_{t \to +\infty} F_u(t, T) = 0 .$$

Now, from (14), (17) and (18), it follows

$$\limsup_{t \to +\infty} |x(t; \xi) - y(t; \omega(\xi))| < \varepsilon \quad \text{for every } \varepsilon > 0$$

which implies (13).

3. - **General case of** $A(t)$. Let $G(t)$, $Q(t)$, $t \in [t_0, +\infty)$ be $n \times n$ complex matrices having entries summable functions in every finite subinterval of $[t_0, +\infty)$.

DEFINITION 2. - (R. CONTI [2]). *We say that $G(t)$ is t_∞-similar to $Q(t)$ if and only if*

$$\int_{t_0}^{+\infty} |\dot{S}(t) + S(t)G(t) - Q(t)S(t)| \, dt < +\infty$$

for some non-degenerate $n \times n$ complex matrix $S(t)$, $t \in [t_0, +\infty)$ with entries absolutely continuous functions in every finite subinterval of $[t_0, +\infty)$ and such that $S(t)$ and $S^{-1}(t)$ are bounded in $[t_0, +\infty)$. More exactly we say that $G(t)$ is t_∞-similar to $Q(t)$ by means of the matrix $S(t)$

Under the assumptions of section 1 the following theorem holds.

THEOREM 1. - *Let $A(t)$ be t_∞-similar to a matrix $J(t)$ having the Jordan canonical form. Then the systems (A) and (A_g) are asymptotically equivalent.*

PROOF. - If $A(t)$ is t_∞-similar to a matrix $J(t)$ by means of the matrix $S(t)$, then by the substitution

(19) $$y = S(t)w$$

the system (A) is transformed into the system

(J_{g^*}) $$\dot{w} = J(t)w + g^*(t, w),$$

where $g^*(t, w) = -S^{-1}(t)[\dot{S}(t) + S(t)J(t) - A(t)S(t)]w$.

Similarly the substitution

(20) $$x = S(t)z$$

transforms the system (A_g) into the system

$$(J_{g_*}) \qquad \dot{z} = J(t)z + g_*(t, z),$$

where $g_*(t, z) = g(t, S(t)z) - S^{-1}(t)[\dot{S}(t) + S(t)J(t) - A(t)S(t)]z$.

Moreover, we consider the system

$$(J) \qquad \dot{u} = J(t)u$$

which is uniformly stable. since (A) is uniformly stable and $A(t)$ is t_∞-similar to $J(t)$ [2; p. 249].

Now, it is easy to verify that we can apply lemma 3 for the systems (J), (J_{g^*}) and (J), (J_{g_*}) respectively. Hence, the asymptotic equivalence of (J), (J_{g^*}) and of (J), (J_{g_*}) leads to the asymptotic equivalence of the systems (J_{g^*}) and (J_{g_*}). since the relation of the asymptotic equivalence is transitive. Thus the assertion of the theorem follows immediately by virtue of (19) and (20).

Let now consider the systems (A) and

$$(B) \qquad \dot{x} = B(t)x,$$

where $B(t)$, $t \in [t_0, +\infty)$ is an $n \times n$ complex matrix with entries summable functions in every finite subinterval of $[t_0, +\infty)$.

Corollary. - *Let $A(t)$ be t_∞-similar to a matrix $J(t)$ having the Jordan canonical form and let also $A(t)$ be t_∞-similar to $B(t)$ by means of a matrix $T(t)$ for which $T = \lim_{t \to +\infty} T(t)$ exists and is non-degenerate, i.e. det $T \neq 0$. Then the systems (A) and (B) are asymptotically equivalent.*

Proof. - The substitution

$$x = T(t)z$$

transforms the system (B) into the system

$$(A_{\tilde{g}}) \qquad \dot{z} = A(t)z + \tilde{g}(t, z),$$

where $\tilde{g}(t, z) = - T^{-1}(t)[\dot{T}(t) + T(t)A(t) - B(t)T(t)]z$.

An application of Th. 1 leads to the asymptotic equivalence of the systems (A) and $(A_{\tilde{g}})$. On the other hand it is easy to verify that the system (B) is asymptotically equivalent to the system $(A_{\tilde{g}})$ by means of the homemorphism $\omega(\xi) = T^{-1}\xi$.

Hence, by the transitivity of the relation of the asymptotic equivalence, the assertion of the corollary follows.

As we have mentioned in section 1 the well-known theorem of H. WEYL [6] or [5; p. 514] and that of V. A. YAKUBOVICH [7; p. 237] fall into Th. 1 as particular cases, since the case where $A(t)$ is constant or reducible implies that $A(t)$ is t_∞-similar to a matrix $J(t)$ having the JORDAN canonical form. Also, for the same reason, the theorems 1 and 2 of R. CONTI [3; p. 45 and 46] fall into the above corollary as particular cases.

4. – We shall give now a more general formulation of Th. 1 by which the above corollary is obvious.

Consider the "*quasi-linear*" systems (A_g) and

$$(B_f) \qquad \dot{y} = B(t)y + f(t,\, y)$$

and suppose, as in section 1, that the system (A) is uniformly stable.

THEOREM 2. – *Let $A(t)$ be t_∞-similar to a matrix $J(t)$ having the Jordan canonical form and let also $A(t)$ be t_∞-similar to $B(t)$ by means of a matrix $T(t)$ for which $T = \lim_{t \to +\infty} T(t)$ exists and is non-degenerate, i.e. det $T \neq 0$. Then the systems (A_g) and (B_f) are asymptotically equivalent.*

PROOF. – The theorem follows immediately from Th. 1, the corollary and the transitivity of the relation of asymptotic equivalence, by comparing first the systems (A) and (A_g) and then $(B$ and (B_f).

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