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Roberto Conti

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RECENT TRENDS IN THE THEORY OF BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

ROBERTO CONTI (Firenze)

PREFACE

The theory of boundary value problems for ordinary differential equations almost exclusively included, until a few years ago, problems with side conditions assigned over a compact interval of the independent variable. By contrast, problems involving the behavior of solutions over non compact intervals were usually labelled as « asymptotic ».

Recent developments of the theory show a tendency to reduce more and more this traditionally accepted distinction. We have tried, by this report, to give an account of such trend and to recognize, at the same time, the common features of the underlying techniques.

Correspondingly our bibliography mainly refers to the literature of the last 15 years; earlier references may be found in the expository papers of W. T. REID [1], G. STAMPACCHIA [2] and W. M. WHYBURN [3] or in books like E. A. CODDINGTON - N. LEVINSON [1] and G. SANSONE [1]

The report is divided into two parts. Part 1, essentially algebraic, is devoted to the vector equation $\dot{x} - Ax = f$, when f does not depend on x.

In Part II, where f depends on x, boundary value problems are first reduced to the search for solution of some functional equation, i.e. for fixed points of suitable mappings. The most frequently used fixed point theorems, like SCHAUDER–TYCHONOV'S, BANACH'S etc, are listed in the Appendix, for the reader's convenience. The LERAY-SCHAUDER topological degree theory has not been considered because, as was noted in J. Cronin's book [1], boundary value problems for ordinary differential equations considered so far involve mappings with topological degree equal to $+1$, so that only a partial corollary of the Leray-Schauder theory is needed, namely Th. A' of the Appendix. There is however an exception at least, represented by a problem studied by G Stampacchia, [1] to which we devote Sec. 19.

Throughout our exposition x denotes a real n-vector function, i.e. a function whose values belong to R^n. A part of the theory can be extended to differential equations in a Banach space, provided that A be bounded. This situation is fairly described in the recent book by J. L. Massera - J. J. Schäffer [3].

To render our exposition as organic as possible we had also to ignore the myriads of particular b.v.p. connected with n.th order scalar differential equations.

Finally another intentional omission refers to b.v.p. with parameters (eigenvalue problems, perturbation theory, etc.). This was due to the fact that in our opinion this kind of problems, and also problems involving arbitrary functions, as in control theory, or differential inequalities, should find their natural and most appropriate place within the theory of «differential relations», $\dot{x} - Ax \in F$, where F represents a set-valued function. This part of differential calculus, essentially a revival of Marchaud-Zaremba's « équations au contingent », is now rapidly developing and most promising, but still fragmentary and incomplete.

Most probably, unintentional omissions should be added to the above mentioned intentional ones, a fact, which, I hope, will stimulate the reader's criticisms and suggestions.

0. - Introduction.

The present report deals with boundary value problems for a differential equation

$$\text{(I)} \qquad dx/dt - A(t)x = y(t)$$

or, more generally,

$$\text{(II)} \qquad dx/dt - A(t)x = f(t, x).$$

In the sequel we shall denote by:

t, a real variable, $t \in J =]\alpha, \omega[$, an open, possibly unbounded, interval of R, the real numbers space;

R^n the space of real n-vectors x with components $x_1, \ldots, x_n$ and any norm $|x|$. (not necessarily the euclidean one);

$\mathcal{A}$ the algebra of real n by n matrices M with the norm $|M| = \sup_{x \in R^n} |Mx| / |x|$;

$t \longrightarrow A(t)$ a function from J into $\mathcal{A}$;

$t \longrightarrow y(t)$ a function from J into R^n;

$t, x \longrightarrow f(t, x)$ a function from $J \times R^n$ into R^n.

If Δ is a subinterval of J, a solution of (I) on Δ will be any function $t \longrightarrow x(t)$ from Δ into R^n locally absolutely continuous on Δ, i.e. absolutely continuous on compact subintervals of Δ, such that

$$dx(t)/dt - A(t)x(t) = y(t) \quad \text{a.e. on} \quad \Delta.$$

A solution of (II) on Δ is defined similarly.

The locution «boundary value problems» is part of a terminology traditionally used for partial differential equations. In that context it refers to determining solutions of a p.d.e. which, in addition, satisfy some conditions on the boundary of a prescribed domain. When transferred to ordinary d.e. this would mean, literally, conditions to be satisfied at the endpoints of some interval, a problem which is a very particular (though important) case of situations encountered in the field of «b.v.p.» for ordinary d.e.

Therefore we prefer to talk about «additional» rather than about «boundary» conditions.

To be more specific, throughout this report, such «additional» conditions will be of the form

$$(C) \qquad x \in \Omega$$

where Ω is a given infinite subset of $C(\Delta, R^n)$, the linear space of continuous functions $t \longrightarrow x(t)$ from a prescribed subinterval Δ of J into R^n.

Therefore a boundary value problem will henceforth consist of equation (I) or (II) plus an additional condition (C) to be specified from time to time.

PART I

EQUATION (I)

1. - The evolution operator.

In what follows we shall constantly assume that $t \longrightarrow A(t)$ is a (LEBESGUE) measurable and locally integrable function of $t \in J$ into $\mathcal{A}$, i.e., that the entries of the matrix A are real measurable functions of $t \in J$, integrable on compact subintervals of J.

This implies that $t \longrightarrow |A(t)|$ is a real function of $t \in J$, measurable and locally integrable on J.

It follows the well known, basic theorem

TH. 1.1. - *There exists only one function* $(t, s) \longrightarrow U(t, s)$ *of* $J \times J$ *into* $\mathcal{A}$ *which is continuous and such that*

$$U(t, s) = I + \int_s^t A(\tau) U(\tau, s) d\tau, \qquad s, t \in J \tag{1.1}$$

$$U(t, s) = I + \int_s^t U(t, \tau) A(\tau) d\tau, \qquad s, t \in J \tag{1.2}$$

where I *denotes the identity of* $\mathcal{A}$.

We sketch the proof in order to put into evidence some properties of U needed later.

Let K be any compact subinterval of J and let s be any point of K. The linear space $C(K, \mathcal{A})$ of continuous $\Phi : K \longrightarrow \mathcal{A}$ normed by

$$|\Phi| = \sup_{t \in K} |\Phi(t)| \exp\left(-\lambda \left| \int_s^t |A(\tau)| d\tau \right|\right), \qquad \lambda \geq 0$$

is complete. If $\lambda > 1$, then

$$\mathcal{T}: \qquad \Phi \longrightarrow I + \int_s^t A(\tau) \Phi(\tau, s) d\tau$$

is a contractive mapping of $C(K, \mathcal{A})$, therefore (See Appendix) there is only one solution of (1.1) in $C(K, \mathcal{A})$. Since K and s are arbitrary, U is defined in $J \times J$.

Similarly one can prove the existence of a unique function $(t, s) \longrightarrow V(t, s)$ of $J \times J$ into $\mathcal{A}$ such that

$$V(t, s) = I + \int_s^t V(t, \tau)A(\tau)dt, \qquad s, t \in J. \tag{1.2'}$$

From (1.1) and (1.2′) we obtain $\partial(V(t, r)U(r, s))/\partial r = 0$, the zero of $\mathcal{A}$, for a.e. $r \in J$ and all $s, t \in J$. Since $r \longrightarrow V(t, r)U(r, s)$ is absolutely continuous it follows $U(t, s) = V(t, s)$ so that (1.2) is proved by (1.2′). Further $\partial(U(t, s)U(s, r))/\partial s = 0$ for a.e. $s \in J$ and all $t, r \in J$. Hence also

$$U(t, s)U(s, r) = U(t, r), \qquad r, s, t \in J \tag{1.3}$$

and since $t \longrightarrow U(t, s)$, $r \longrightarrow U(s, r)$ are continuous and multiplication in $\mathcal{A}$ is a continuous operation it follows that U is continuous on $J \times J$.

DEF. 1.1. - The function U: $(t, s) \longrightarrow U(t, s)$ defined by Th. 1.1 is called the *evolution operator* generated by $t \longrightarrow A(t)$.

REMARK 1.1. - From (1.3) we have, for $r = t$, $U(t, s)U(s, t) = \mathrm{I}$, which means that for every (t, s) there exists the inverse $U^{-1}(t, s)$ and

$$U^{-1}(t, s) = U(s, t) \qquad t, s \in J. \tag{1 4}$$

REMARK 1.2. - For each $(t, s) \in J \times J$, $U(t, s)$ is the limit of the PICARD sequence

$$U_0(t, s) = I \tag{1.5}$$

$$U_{k+1}(t, s) = I + \int_s^t A(\tau)U_k(\tau, s)d\tau$$

so that it is also the sum of the PEANO series

$$U(t, s) = I + \int_s^t A(\tau_1)d\tau_1 + \int_s^t A(\tau_1)\left(\int_s^{\tau_1} A(\tau_2)d\tau_2\right)d\tau_1 + \ldots \tag{1.6}$$

2. - Problem I. - The Cauchy problem for equation (I).

Since A is assumed to be measurable and locally integrable on J, if x is a solution of (I) on any subinterval $\Delta \subset J$, then the function

$t \to dx(t)/dt - A(t)x(t)$ is necessarily measurable and locally integrable on Δ. Therefore, denoting by $L_{loc}(J, R^n)$ the linear space of functions of $t \in J$ into R^n which are measurable and locally integrable on J, it is natural to assume that

H_1) $\quad t \to y(t)$ *belong to* $L_{loc}(J, R^n)$.

In this Sec. conditions (C) is, in a sense, the simplest possible as the Ω of (C) will be the set of functions of $C(J, R^n)$ taking a given value $\xi \in R^n$ at a given $\tau \in J$. We thus have:

PROBLEM I. - *The Cauchy problem for equation* (I).

With given $\tau \in J$, $\xi \in R^n$, *determine the solution of* (I) *on* J *such that*

$$x(\tau) = \xi. \tag{2.1}$$

From Th. 1.1 we have immediately

TH. 2.1. - *For every* $\tau \in J$, *every* $\xi \in R^n$ *and every* $y \in L_{loc}(J, R^n)$ *the unique solution of Problem* I *is*

$$t \to U(t, \tau)\xi + \int_\tau^t U(t, s)y(s)ds, \qquad t \in J. \tag{2.2}$$

We use this classic result for a few remarks.

The linear operator defined by

$$D = d/dt - A(t) \tag{2.3}$$

has domain $\mathfrak{D}(D) \subset C(J, R^n)$ and range $\mathfrak{R}(D) \subset L_{loc}(J, R^n)$. In fact, writing (I) as

$$Dx = y \tag{I'}$$

Th. 2.1 asserts that $\mathfrak{R}(D) = L_{loc}(J, R^n)$.

Further, the null space $\mathfrak{N}(D)$ of D is isomorphic to R^n. For every $\tau \in J$ the isomorphism is given by $\xi \to U(\cdot, \tau)\xi$.

For each $y \in L_{loc}(J, R^n)$ its inverse image $\overset{-1}{D}y$ by D, i.e., the set of solutions of (I), is represented by the linear variety of $C(J, R^n)$

$$\overset{-1}{D}y = \mathfrak{N}(D) + \left\{ \int_\tau^t U(t, s)y(s)ds \right\} \tag{2.4}$$

and using (1.3) one easily sees that the right hand side does *not* depend on τ.

The operator D, being onto, has right inverses, i.e. there are linear operators D^+ of $L_{loc}(J, R^n)$ into $C(J, R^n)$ such that, DD^+ is the identity operator on $L_{loc}(J, R^n)$. For each $\tau \in J$ a right inverse of D is

$$D_\tau^+ : \ y \to \int_\tau^t U(t, s) y(s) ds \tag{2.5}$$

so that the solutions of (I'), i.e. of (I), are represented by

$$x = \chi + D_\tau^+ y, \qquad \chi \in \mathfrak{N}(D). \tag{2.6}$$

REMARK 2.1. – All the solutions of equation (I) are defined for $t \in J$.

3. - Problem II.

We consider now

PROBLEM II. – *Let Λ be a real linear space and let L be a linear operator with $\mathfrak{D}(L) = C(J, R^n)$, $\mathfrak{R}(L) \subset \Lambda$, $\mathfrak{N}(L) \neq \{0\}$. Given $l \in \mathfrak{R}(L)$, determine the solutions of equation* (I) (*on J*) *such that*

$$Lx = l. \tag{3.1}$$

Problem I is a particular case of Problem II ($\Lambda = R^n$, $L : x \to x(\tau)$) and Problem II is in turn a particular case of (I)-(C), corresponding to $\Omega = \breve{L} l$, the inverse image of l by L.

Since $l \in \mathfrak{R}(L)$, $\mathfrak{N}(L) \neq \{0\}$, $\breve{L} l$ is a linear variety of $C(J, R^n)$ having dimension greater than zero.

Using the same notations of Sec. 2, we look for solutions $x \in C(J, R^n)$ of

$$Dx = y, \qquad Lx = l. \tag{3.2}$$

By virtue of (2.6), having fixed $\tau \in J$ arbitrarily $\chi + D_\tau^+ y$ will be a solution of (3.1) if and only if $\chi \in \mathfrak{N}(D)$ is a solution of

$$L_0 \chi = l - L D_\tau^+ y,$$

having denoted by L_0 the restriction of L to $\mathfrak{N}(D)$.

Since $\chi = U(\cdot, \tau)\xi$, $\xi \in R^n$,

$$L_U = L_0 U$$

is a linear operator of R^n into Λ and $U(\cdot, \tau)\xi + D_\tau^+ y$ will be a solution of (3.1) if and only if $\xi \in R^n$ is a solution of

$$L_U \xi = l - L D_\tau^+ y. \tag{3.3}$$

Since $\mathfrak{D}(L_U) = R^n$ we have $\dim \mathfrak{R}(L_U) = m \leq n$, so that L_U (can be represented by an $m \times n$ matrix and it) has generalized inverses. This means that there are linear operators L_U^g of $\mathfrak{R}(L_U)$ into R^n such that

$$L_U L_U^g L_U = L_U, \tag{3.4}$$

and it means, also, that (3.3) will have solutions if and only if

$$(I_\Lambda - L_U L_U^g)\,(l - L D_\tau^+ y) = 0, \tag{3.5}$$

where I_Λ is the identity operator on Λ. In fact if ξ is a solution of (3 3) we have $(I_\Lambda - L_U L_U^g)\,(l - L D_\tau^+ y) = (I_\Lambda - L_U L_U^g) L_U \xi = 0$ by (3.4). Conversely if (3.5) holds, (3.3) can be written $L_U \xi = L_U L_U^g (l - - L D_\tau^+ y)$, i.e. $L_U(\xi - L_U^g(l - L D_\tau^+ y)) = 0$, so that $\xi_0 + L_U^g(l - L D_\tau^+ y)$ is a solution of (3.3) for each $\xi_0 \in \mathfrak{N}(L_U)$.

In general there will be infinitely many L_U^g satisfying (3.4), but it is readily seen that (3.5) is either valid for all of them or for none.

In particular (3.4) (3.5) hold when L_U has a right inverse L_U^+: $L_U L_U^+ = I_\Lambda$, and, more in particular, when L_U has the inverse L_U^{-1}: $L_U^{-1} L_U = I_{R^n}$, $L_U L_U^{-1} = I_\Lambda$. In this last case $\mathfrak{N}(L_U) = \{0\}$. Thus:

TH. 3.1. - *A necessary and sufficient condition in order that Problem II have solutions is that* (3.5) *hold for any* L_U^g *satisfying* (3.4). *Solutions are given by*

$$U L_U^g l + U\xi_0 + [D_\tau^+ - U L_U^g L D_\tau^+] y, \qquad \xi_0 \in \mathfrak{N}(L_U) \tag{3.6}$$

If L_U *has a right inverse* L_U^+ *then* ((3.4) *holds and*) (3.5) *holds for any* $y \in L_{\mathrm{loc}}(J, R^n)$. *Solutions are given by* (3.6) *with* L_U^g *replaced by* L_U^+.

If L_U has the inverse L_U^{-1} then for each $y \in L_{\mathrm{loc}}(J, R^n)$ the unique solution of Problem II *is*

$$(3.7) \qquad UL_U^{-1}l + [D_\tau^+ - UL_U^{-1}LD_\tau^+]y.$$

NOTE: About generalized inverses of linear bounded operators between finite dimensional or hilbert spaces see, also for the bibliography: W.V. PETRYSHYN, *Jour. of Math. Anal. & Appls.*, 18 (1967).

4. - Some remarks about Problem II.

REMARK 4 1. - Problem II is said to be *incompatible* when

$$(4.1) \qquad Dx = 0,\ Lx = 0 \Rightarrow\ x = 0,$$

compatible otherwise.

It should be noted that (4.1) is equivalent to the existence of left inverses L_U^- of L_U, i.e. of linear operators $L_U^- : \Lambda \to R^n$, $L_U^- L_U = I_{R^n}$, so that, in general, incompatibility is *not* enough to insure the existence of solutions of Problem II for all $l \in \Lambda$, $y \in L_{\mathrm{loc}}(J, R^n)$. According to (3.5) solutions will exist only for those $l \in \Lambda$, $y \in L_{\mathrm{loc}}(J, R^n)$ such that

$$(I_\Lambda - L_U L_U^-)(l - LD_\tau^+ y) = 0$$

and for any such pair l, y, there will be a unique solution, namely

$$UL_U^- l + [D_\tau^+ - UL_U^- LD_\tau^+]y.$$

It is easily verified, however, that if $\dim \Lambda = n$ then the existence of L_U^- (or of L_U^+) is equivalent to that of the inverse L_U^{-1}. Therefore if $\dim \Lambda = n$, Problem II is incompatible if and only if the inverse L_U^{-1} exists, i.e. if and only if Problem II has a unique solution for each $l \in \Lambda$, $y \in L_{\mathrm{loc}}(J, R^n)$.

REMARK 4.2. - To verify the validity of (3 5) is not easy in general. The most favorable situation is encountered when L_U has the inverse. This suggests to write equation (I) as

$$(4.2) \qquad dx/dt - B(t)x = [A(t) - B(t)]x + y(t)$$

and then look for some B whose evolution operator V has the property that L_V^{-1} exists. If it is so then Problem II can be dealt with by the methods which will be described in Sec. 16 for Problem V.

Such methods require some conditions which will involve not only B and l but also the right hand side of (4.2), hence A and y.

These conditions are only sufficient ones, while (3.5) is also necessary.

REMARK 4.3. – The existence of the inverse L_U^{-1} of L_U depends on L and, via U and the PEANO series (1.6), on A. Sufficient criteria involving A directly can be given for particular L.

REMARK 4.4. – For particular operators L it is possible to represent solutions in a more compact form than (3.6) by defining an appropriate *Green operator* (See Secs. 6-9).

Again, for particular L it is also possible to define a pair of linear operators D^*, L^* such that the solvability of Problem II can be expressed by means of a relationship between the pair l, y, and the solutions z of

$$D^*z = 0,\ L^*z = 0. \tag{4.3}$$

Such a relationship is, necessarily, a disguised version of (3.4) (3.5).

System (4.3) is called the *adjoint* of

$$Dx = 0,\ Lx = 0 \tag{4.4}$$

and reciprocally.

Extensive investigations have been made in order to define a sufficiently «satisfactory» adjoint of (4.4) with several particular L. (See Secs 6-9).

5. – Examples. - The generalized Cauchy problem for equation (I).

Let $\tau \in J$, $l \in \Lambda = R^m$ and let L be defined as $x \to Mx(\tau)$ where M is an $m \times n$ real matrix, so that (3.1) becomes

$$Mx(\tau) = l, \qquad l \in R^m, \tag{5.1}$$

and we have a «generalized CAUCHY problem» for equation (I).

Now L_U can be represented by the matrix M itself. Since $LD_\tau^+ y = 0$ for all y, then if M^g is any generalized inverse of M, $MM^gM = M$, there will be solutions of equation (I) satisfying (5.1) if and only if $(I_{R^n} - MM^g)l = 0$, and these solutions are given by

$$(5.2) \qquad t \to U(t, \tau)[M^g l + \xi_0] + \int_{\tau}^{t} U(t, s)y(s)ds,$$

with $M\xi_0 = 0$.

The CAUCHY problem corresponds to $m = n$, $M = I_{R^n}$ and (5.2) then reduces to (2.1).

6. - Examples. - Two-point b.v.p. for equation (I).

a) Let again $l \in \Lambda = R^m$, let $\tau_1, \tau_2 \in J$, $\tau_1 \neq \tau_2$ and let L be defined as $x \to M_1 x(\tau_1) + M_2 x(\tau_2)$ where M_1, M_2 are real $m \times n$ matrices.

Condition (3.1) then becomes

$$(6.1) \qquad M_1 x(\tau_1) + M_2 x(\tau_2) = l$$

and we have a «two-point b.v.p.» for equation (I).

Let $\tau_1 < \tau_2$ and take $\tau = \tau_1$. Then L_U can be represented by the $m \times n$ matrix $M_1 + M_2 U(\tau_2, \tau_1)$ and $LD_\tau^+ y = \int_{\tau_1}^{\tau_2} M_2 U(\tau_2, s)y(s)ds$.

If $[M_1 + M_2 U(\tau_2, \tau_1)]^g$ satisfies

$$(6.2) \qquad [M_1 + M_2 U(\tau_2, \tau_1)][M_1 + M_2 U(\tau_2, \tau_1)]^g [M_1 + M_2 U(\tau_2, \tau_1)] = M_1 + M_2 U(\tau_2, \tau_1)$$

then for all $l \in R^m$, $y \in L_{\mathrm{loc}}(J, R^n)$ such that

$$(6.3) \qquad \{ I_{R^m} - [M_1 + M_2 U(\tau_2, \tau_1)][M_1 + M_2 U(\tau_2, \tau_1)]^g \} [l - \int_{\tau_1}^{\tau_2} M_2 U(\tau_2, s)y(s)ds] = 0$$

the solutions of (I)-(6.1) are given by

$$(6.4) \qquad t \to U(t, \tau_1) \{ [M_1 + M_2 U(\tau_2, \tau_1)]^g l + \xi_0 \} + \int_{\tau_1}^{t} U(t, s)y(s)ds - \int_{\tau_1}^{\tau_2} U(t, \tau_1)[M_1 + M_2 U(\tau_2, \tau_1)]^g M_2 U(\tau_2, s)y(s)ds$$

where ξ_0 is any solution of $[M_1 + M_2 U(\tau_2, \tau_1)]\xi_0 = 0$.

b) While (6.4) is valid for all $t \in J$, a simpler form of (6.4) restricted to $t \in [\tau_1, \tau_2]$ is

$$t \to U(t, \tau_1) \{ [M_1 + M_2 U(\tau_2, \tau_1)]^g l + \xi_0 \} + \int_{\tau_1}^{\tau_2} G(t, s) y(s) ds, \tag{6.5}$$

where $t, s \to G(t, s)$ is the «generalized» GREEN operator of problem (I)-(6.1) defined by taking

$$G(t, s) = \begin{cases} - U(t, \tau_1) [M_1 + M_2 U(\tau_2, \tau_1)]^g M_2 U(\tau_2, s) + \\ \qquad + U(t, s), & \tau_1 \leq s < t \leq \tau_2 \\ - U(t, \tau_1)[M_1 + M_2 U(\tau_2, \tau_1)]^g M_2 U(\tau_2, s), & \tau_1 \leq t < s \leq \tau_2. \end{cases}$$

The «ordinary» GREEN operator corresponds to the case $m = n$ when $[M_1 + M_2 U(\tau_2, \tau_1)]^{-1}$ exists, and (6.5) is replaced by

$$t \to U(t, \tau_1) [M_1 + M_2 U(\tau_2, \tau_1)]^{-1} l + \int_{\tau_1}^{\tau_2} G(t, s) y(s) ds \tag{6.6}$$

where G can be defined more symmetrically by taking

$$G(t, s) = \begin{cases} U(t, \tau_1) [M_1 + M_2 U(\tau_2, \tau_1)]^{-1} M_1 U(\tau_1, s), & \tau_1 \leq s < t \leq \tau_2 \\ - U(t, \tau_1) [M_1 + M_2 U(\tau_2, \tau_1)]^{-1} M_2 U(\tau_2, s), & \tau_1 \leq t < s \leq \tau_2. \end{cases}$$

c) When $l = 0$, denoting as usual by M^* the transpose of a matrix M, (6.3) can be written

$$\int_{\tau_1}^{\tau_2} z^*(s) y(s) ds = 0 \tag{6 7}$$

with

$$z(t) = U^{*-1}(t, \tau_2) M_2^* H^* \xi, \qquad \xi \in R^m \tag{6.8}$$

$$H = I_{R^m} - [M_1 + M_2 U(\tau_2, \tau_1)] [M_1 + M_2 U(\tau_2, \tau_1)]^g.$$

It is readily verified that z defined by (6.8) is a solution of the equation

$$dz/dt + A^*(t) z = 0. \tag{I$_0^*$}$$

If N_1, N_2 are two real $n \times k$ matrices such that

$$N_1 M_1 = N_2 M_2 \tag{6.9}$$

then by (6.2) we see that the z represented by (6.8) are the solutions of $(\mathrm{I})_0^*$ satisfying

$$N_1^* z(\tau_1) + N_2^* z(\tau_2) = 0. \tag{6.1$)_0^*$}$$

If rank $(M_1 : M_2) = m < 2n$ and $k = 2n - m$ it can be proved that there is a unique $(2n - m) \times n$ matrix $(N_1^* : N_2^*)$ of rank $= 2n - m$, such that (6.9) holds and we say that $(\mathrm{I})_0^*$-$(6.1)_0^*$ is the adjoint of

$$(\mathrm{I})_0 \qquad dx/dt - A(t)x = 0$$

$$(6.1)_0 \qquad M_1 x(\tau_1) + M_2 x(\tau_2) = 0$$

and reciprocally.

Therefore we can state the classical result: the two-point b.v. problem (I)-$(6.1)_0$ has solutions if and only if y satisfies the « orthogonality » condition (6.7) for all the solutions z of the adjoint problem $(\mathrm{I})_0^*$-$(6.1)_0^*$.

d) A case of special interest is $m = n$, $M_1 = - M_2 = I_{R^n}$, $l = 0 \in R^n$, i.e.

$$x(\tau_1) - x(\tau_2) = 0. \tag{6.10}$$

When $t \to A(t)$ and $t \to y(t)$ are periodic functions with period $\tau_2 - \tau_1$, then problem (I)-(6.10) is equivalent to that of determining the « harmonic » solutions of (I), i.e. solutions of (I) with period $\tau_2 - \tau_1$. In this case we can take $N_1 = - N_2 = I_{R^n}$ so that (I) will have harmonic solutions if and only if the « perturbing term » y is « orthogonal » in the sense of (6.9) to all the harmonic solutions of equation $(\mathrm{I})_0^*$.

References: J. S. Bradley [1]; W. J. Coles [1]; J. B. Garner [1]; J. B. Garner-L. P. Burton [1]; A. Ja. Hokryakov [1]; W. S. Loud [1]; M. I. Urbanovic [1]; W. M. Whyburn [3]; O. Wyler [1].

7. – Examples. - k-point b.v.p. for equation (I).

A generalization of the two-point b.v.p. is the following.

Let $l \in \Lambda = R^m$, let $\tau_1, \ldots, \tau_k \in J$, $\tau_1 < \ldots < \tau_k$, and let L be defined as $x \to M_1 x(\tau_1) + \ldots + M_k x(\tau_k)$, where $M_1, \ldots, M_k$ are k real $m \times n$

matrices. A «k-point b.v.p.» (also known as «NICOLETTI b.v.p.» for equation (I) is then that of finding solutions of (I) such that

$$M_1 x(\tau_1) + \dots + M_k x(\tau_k) = l.$$

Taking $\tau = \tau_1$, L_U can be represented by the $m \times n$ matrix $M_1 + M_2 U(\tau_2, \tau_1) + \dots + M_k U(\tau_k, \tau_1)$ and

$$LD_\tau^+ y = \int_{\tau_1}^{\tau_2} M_2 U(\tau_2, s) y(s) ds + \dots + \int_{\tau_1}^{\tau_k} M_k U(\tau_k, s) y(s) ds.$$

Instead of going into further details as in Sec. 6 we wish to point out that a k-point b.v.p. is, for instance, that of finding an integral curve of (I) intersecting k given linear varieties lying in the hyperplanes $t = \tau_1, \dots, t = \tau_k$ respectively.

References: L. N. ESHUKOV [1]; W. M. WHYBURN [3].

For $k = 3$: W.J. COLES [1]; J.B. GARNER [1], [2], [3]; J.B. GARNER-L. P. BURTON [1].

8. - **Examples. - B.v.p. for equation (I) with conditions at a countable set of points.**

We have so far considered b.v.p. with conditions bearing on the value of the solutions at a *finite* number of points of J. The case of conditions at a *countable* set of points of J can be dealt with along similar lines as follows.

Let Δ be a compact subinterval of J, let $\{\tau_h\}$ be a sequence in Δ, and let $\{M_h\}$ be a sequence of real $m \times n$ matrices such that

$$\sum_1^\infty {}_h |M_h| < \infty, \quad |M_h| = \sup_{x \in R^n} |M_h x| / |x|.$$

If $x \in C(\Delta, R^n)$ we have $|\sum_1^\infty {}_h M_h x(\tau_h)| \leq \max_{t \in \Delta} |x(t)| \sum_1^\infty {}_h |M_h| < \infty$, so that it makes sense to define L as $x \to \sum_1^\infty {}_h M_h x(\tau_h)$, and (3.1) will become

$$\sum_1^\infty {}_h M_h x(\tau_h) = l$$

with $l \in R^m$.

L_U is represented by $\sum_1^\infty{}_h M_h U(\tau_h, \tau_1)$ and $(\tau = \tau_1)$

$$LD_\tau^+ y = \sum_2^\infty{}_h \int_{\tau_1}^{\tau_h} M_h U(\tau_h, s) y(s) ds.$$

References: R. H. Cole [1]; W. M. Whyburn [4].

9. – Examples. - B.v.p. for equation (I) with integral conditions.

There are also b.v p. with side condition (3.1) involving an integral over a compact subinterval Δ of J and this, in turn, may involve the values of solutions at *all* points of Δ.

a) For instance, let $t \to M(t)$ be a given real $m \times n$ matrix whose entries are integrable functions of $t \in \Delta$. Then (3.1) can be written

$$\sum_1^k{}_h M_h x(\tau_h) + \int_\Delta M(\sigma) x(\sigma) d\sigma = l$$

with $l \in R^m$, and τ_h, M_h as in Sec. 7, or, more generally,

$$\sum_1^\infty{}_h M_h x(\tau_h) + \int_\Delta M(\sigma) x(\sigma) d\sigma = l$$

again with $l \in R^m$, but with τ_h, M_h as in Sec. 8.

References: R. H. Cole [1]; W. R. Jones [1]; A. M. Krall [1]; M. Pagni [1]; W. M. Whyburn [3], [4].

b) Conditions containing Stieltjes integrals arise when L is represented by $x \to \int_\Delta dF(\sigma)x(\sigma)$ where $t \to F(t)$ is a given real $m \times n$ matrix whose entries are functions of bounded variation on $\Delta = [\tau_1, \tau_2] \subset J$.

Condition (3.1) becomes then

$$\int_\Delta dF(\sigma) x(\sigma) = l, \tag{9 1}$$

with $l \in R^m$. The operator L_U can be represented by the $m \times n$

matrix $\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)$ and $(\tau = \tau_1)$

$$LD_{\tau}^{+} y = \int\limits_{\tau_1}^{\tau_2}\left[\int\limits_{s}^{\tau_2} dF(\sigma)U(\sigma,\ s)\right] y(s)ds.$$

If $\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]^g$ is an $n \times m$ matrix such that

$$\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]^g\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right] =$$

$$= \int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1),$$

then for all $l \in R^m$, $y \in L_{\mathrm{loc}}(J,\ R^n)$ such that

$$\left\{ I_{R^m} - \left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]^g \right\}\left\{ l - \right.$$

$$\left. - \int\limits_{\Delta}\left[\int\limits_{s}^{\tau_2} dF(\sigma)U(\sigma,\ s)\right] y(s)\,ds \right\} = 0$$

the solutions of (I)-(9.1) are given by

$$t \to U(t,\ \tau_1)\left\{\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]^g l + \xi_0\right\} + \int\limits_{\tau_1}^{t} U(t,\ s)y(s)ds -$$

$$- \int\limits_{\tau_1}^{\tau_2} U(t,\ \tau_1)\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]^g\left[\int\limits_{s}^{\tau_2} dF(\sigma)U(\sigma,\ s)\right] y(s)ds$$

where ξ_0 is any solution of $\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]\xi_0 = 0.$

For $t \in \Delta = [\tau_1 .\ \tau_2]$ the solutions can be represented by

$$t \to U(t,\ \tau_1)\left\{\left[\int\limits_{\Delta} dF(\sigma)U(\sigma,\ \tau_1)\right]^g l + \xi_0\right\} + \int\limits_{\tau_1}^{\tau_2} G(t,\ s)y(s)ds$$

where $t, s \to G(t, s)$ is the GREEN operator defined by taking

$$G(t, s) = \begin{cases} -U(t, \tau_1)\left[\int\limits_{\Delta} dF(\sigma)U(\sigma, \tau_1)\right]^g \int\limits_s^{\tau_2} dF(\sigma)U(\sigma, s) + U(t, s), \\ \hfill \tau_1 \leq s < t \leq \tau_2 \\ -U(t, \tau_1)\left[\int\limits_{\Delta} dF(\sigma)U(\sigma, \tau_1)\right]^g \int\limits_s^{\tau_2} dF(\sigma)U(\sigma, s), \quad \tau_1 \leq t < s \leq \tau_2. \end{cases}$$

When $\left[\int\limits_{\Delta} dF(\sigma)U(\sigma, \tau_1)\right]^{-1}$ exists, G can be defined by taking

$$G(t, s) = \begin{cases} -U(t, \tau_1)\left[\int\limits_{\Delta} dF(\sigma)U(\sigma, \tau_1)\right]^{-1} \int\limits_s^{\tau_1} dF(\sigma)U(\sigma, s), & \tau_1 \leq s < t \leq \tau_2 \\ -U(t, \tau_1)\left[\int\limits_{\Delta} dF(\sigma)T(\sigma, \tau_1)\right]^{-1} \int\limits_s^{\tau_2} dF(\sigma)U(\sigma, s), & \tau_1 \leq t < s \leq \tau_2. \end{cases}$$

References: A. SMOGORSHEWSKY [1]; W. M. WHYBURN [3].

10. - Problem III.

As we stated at the beginning of Sec. 3, Problem II is a particular case of (I)-(C) corresponding to $\Omega = \check{L}l$. a linear variety of $C(J, R^n)$ defined by means of the linear operator L whose domain $\mathfrak{D}(L)$ is the whole of $C(J, R^n)$. There are also problems where Ω, although a linear variety of $C(J, R^n)$, is not represented this way.

For instance the set of $x \in C(J, R^n)$, $J =]\alpha, \omega[$, for which the limit $x(w -) = \lim\limits_{t \to \omega -} x(t)$ exists is clearly a proper subspace of $C(J, R^n)$ so that a condition like

$$x(\omega -) = \xi, \qquad \xi \in R^n$$

cannot be written as $Lx = \xi$ with $\mathfrak{D}(L) = C(J, R^n)$.

We are thus led to consider

PROBLEM III. – *To determine the solutions x of equation* (I) *such that*

$$x \in V \tag{10.1}$$

where V is a given linear variety of $C(J, R^n)$, such that $V \cap \mathfrak{D}(D)$ has dimension > 0.

REMARK 10 1. – It can always be assumed that V is a subspace of $C(J, R^n)$, otherwise taking any $v_0 \in V \cap \mathfrak{D}(D)$ we would have $V = \{ v_0 \} + W$, W being a subspace of $C(J, R^n)$ and the translation $z = x - v_0$ in $C(J, R^n)$ would replace the original problem, i.e. $Dx = y$, $x \in V$ by another problem of the same kind, $Dz = y - Dv_0$, $z \in W$. Therefore Problem III is equivalent to

PROBLEM III'. – *Let V be a given subspace of $C(J, R^n)$, such that $V \cap \mathfrak{D}(D) \neq \{ 0 \}$. We wish to determine the solutions of equation* (I) *satisfying the side condition* (10.1).

REMARK 10.2. – Owing to the linearity Problem III', i.e. $Dx = y$, $x \in V$, V a subspace of $C(J, R^n)$, will have solutions for the y belonging to a certain subspace B of $L_{\text{loc}}(J, R^n)$. Using the terminology of J. L. MASSERA–J. J. SCHÄFFER [3] we say that the pair (B, V) is *admissible for* D, if for every $y \in B$, $Dx = y$ has solutions $x \in V$. (V-solutions).

Then problem III' can be reformulated in a more exhaustive way as

PROBLEM III''. – *Let V be a given subspace of $C(J, R^n)$ such that $V \cap \mathfrak{D}(D) \neq \{ 0 \}$. It is required* a) *to determine the maximal subspace $B \subset L_{\text{loc}}(J, R^n)$ such that the pair (B, V) is admissible for D;* b) *for each $y \in B$ to determine all $\chi \in \mathfrak{N}(D)$ such that $\chi + D_\tau^+ y \in V$.*

11. – Examples.

Clearly Problem III is extremely general. It includes Problem II, hence practically all traditional b.v.p. for equation (I), as well as a quantity of other problems which are far apart from that class or halfway.

We are going to illustrate this point by a number of examples.

a) Let V be the subspace $C^r(J, R^n)$ of functions x with continuous r^{th} derivative $(r \geq l)$. Problem III is then that of determining the solutions of (I) with a prescribed «degree of regularity».

b) Let us denote by $L^p(\omega, R^n)$, $1 \leq p < \infty$, the space of functions x of $t \in J =]\alpha, \omega[$ into R^n, such that $\int_\beta^\omega |x(t)|^p dt < \infty$ for some $\beta \in J$, and let $V = L^p(\omega, R^n) \cap C(J, R^n)$.

Symmetrically we can consider $L^p(\alpha, R^n)$, or else $L^p(J, R^n)$, the space of x such that $\int_J | x(t) |^p \, dt < \infty$.

c) Define V as the intersection of $C(J, R^n)$ with the space of n-vector functions $t \rightarrow x(t)$ whose first $k(< n)$ components are in L^p on $[\beta, \omega[$ for some $\beta \in J$, or on $]\alpha, \beta]$ for some $\beta \in J$, or on J. This includes for instance the well known problems of determining $L^2(0, \infty)$ or $L^2(-\infty, \infty)$ solutions of a second order linear differential equation.

d) Let $L^\infty(\omega, R^n)$ be the space of x which are essentially bounded on $[\beta, \omega[$, $\beta \in J$, i e. such that $\operatorname{ess\,sup}_{t \in [\beta, \omega[} |x(t)| < \infty$.

Then $V = L^\infty(\omega, R^n) \cap C(J, R^n)$ is the space of n-vector functions continuous on J, bounded at ω.

Similarly we can consider $L^\infty(\alpha, R^n)$, and $L^\infty(J, R^n)$ and their intersections V with $C(J, R^n)$.

e) In particular given a sequence of real $m \times n$ matrices $\{ M_h \}$ such that $\sum_1^\infty{}_h |M_h| < \infty$ as in Sec. 8, let $\{ \tau_k \}$ be a sequence on J having ω as a limit point. Then take as V the linear manifold of $x \in L^\infty(\omega, R^n) \cap C(J, R^n)$ such that

$$\sum_1^\infty{}_k M_h x(\tau_h) = l, \qquad l \in R^m.$$

f) The «asymptotic» CAUCHY problem

$$x(\omega -) = \xi, \qquad \xi \in R^n \tag{11.1}$$

already considered in Sec. 10 is a Problem III where V is the linear variety of $x \in L^\infty(\omega, R^n) \cap C(J, R^n)$ which satisfy (11.1).

It should be noted that, contrary to the ordinary CAUCHY problem, the «asymptotic» version may have no solution or infinitely many: for instance take $dx/dt + x = 0$, $n = 1$, with $\xi \neq 0$ or $\xi = 0$ respectively.

g) Without going into details we shall indicate problems analogous to those considerèd in Secs. 6, 7, 8, where one or both limits $x(\alpha +)$, $x(\omega -)$ enter together with values of x at given points of J.

Again, we could consider problems similar to those of Sec. 9, in which the integrals (LEBESGUE or STIELTJES) are extended over an interval $[\beta, \omega[$, or $]\alpha, \beta]$, or over J.

h) The problem indicated at the end of Sec. 7, of finding solutions of (I) «intersecting» k given linear varieties of the (t, x)-space is no more a Problem II, but a Problem III, with $J = R$, if one at least of those varieties is not contained in a hyperplane $t =$ const.

i) Another problem III with $J=R$ is that of finding solutions of (I) which are periodic with a given period T, i.e. $x(t)-x(t+T)=0$.

If also A and y have period T, this will be a Problem II (Sec. 6, d) but not otherwise.

j) The problem of finding solutions of equation (I) which are almost periodic is again a Problem III, with $J=R$.

12. – **Non linear problems for equation** (I).

In all the problems which have been considered so far, the set Ω of condition (C) is a linear variety of $C(J, R^n)$. There are other problems in which the linearity of Ω is replaced by the weaker assumption of convexity. For instance let $\rho>0$ and let Ω be defined as

$$\Omega = \{ x \in C(J,\ R^n) : |x(t)| \leq \rho,\ \beta \leq t < \omega \}.$$

PART II

EQUATION (II)

13 – **General remarks about problem** (II)-(C).

Before we turn our attention to problems of type (II)-(C) there are a few general remarks to be made.

First of all, while the solutions of equation (I) are all defined over the whole interval $J =]\alpha, \omega[$ and, for a given y, they form an n-dimensional linear variety of $C(J, R^n)$, this is no longer true for equation (II). In fact, to every solution x of (II) there corresponds a (maximal) interval of existence, which, in general, is a proper subinterval of J, depending on x. From this follows that, when dealing with equation (II), Ω will no longer be a subset of $C(J, R^n)$, but rather of $C(\Delta, R^n)$ where Δ is a prescribed subinterval of J.

Secondly, the treatment of problem (I)-(C), at least as long as Ω is a linear variety, is almost entirely algebraic. The only point requiring some topology is the existence of the evolution operator U (Th. 1.1). On the contrary to solve problem (II)-(C) systematically requires such tools, as fixed point theorems, which are essentially topological. This will require, in turn, the introduction of a topology into the linear space $C(\Delta, R^n)$ and this is done in different ways according to whether Δ is compact or not.

If Δ is compact one usually uses the norm topology with the norm $|x| = \sup_{t \in \Delta} |x(t)|$ and then $C(\Delta, R^n)$ is a BANACH space (= linear, normed, complete space).

If $\Delta = [\beta, \omega[$, or $\Delta =]\alpha, \beta]$, or $\Delta = J$, one can use the compact topology, i.e. the topology of uniform convergence on compact intervals $K \subset \Delta$, and $C(\Delta, R^n)$ then becomes a FRÉCHET space (= linear metric, locally convex, complete space).

We are now going to look more closely at the connection between fixed point theorems and problem (II)-(C)

Given $\Delta \subset J$, define first the operator F which transforms $x: t \to x(t)$ of $C(\Delta, R^n)$ into $Fx: t \to f(t, x(t))$, so that equation (II) can be written $Dx = Fx$. It is clear that if Fx does not belong to the image $D\Omega$ of Ω by D there is no solution of problem (II)-(C), so that a necessary condition for (II)-(C) to be solved is

$$F\Omega \cap D\Omega \neq \emptyset, \tag{13.1}$$

where $F\Omega$ is the image of Ω by F. Let us replace (13.1) by the stronger assumption

$$F\Omega \subset D\Omega, \tag{13.2}$$

which means that for each $w \in \Omega$ there are $x \in \Omega$ such that $Fw = Dx$.

Therefore (13.2) means that

$$Dx = Fw, \qquad x \in \Omega, \tag{13.3}$$

a problem of type (I)-(C), has solutions for all $w \in \Omega$.

Take $w \in \Omega$, then Fw, then its inverse image $\breve{D}Fw$ by D, and finally the intersection set $\breve{D}Fw \cap \Omega$, i.e. the set of solutions of (13.3). Since, generally, this set contains more than one point, we have thus defined a set-valued mapping

$$\mathcal{C}: \quad w \to \mathcal{C}w = \breve{D}Fw \cap \Omega$$

of Ω into the class $P(\Omega)$ of subsets of Ω. Its fixed points, $x \in \mathcal{C}x$, if there are any, are solutions of

$$Dx = Fx, \qquad x \in \Omega, \tag{13.4}$$

i.e. of problem (II)-(C) and conversely, so that every fixed point theorem for $\mathcal{C}$ will also be an existence theorem for (II)-(C). It does not seem, however, that this procedure has been followed so far, probably because the existing fixed point theorems for set-valued mappings are not suitable. The current trend is rather that of replacing $\mathcal{T}$ by an ordinary, point-to-point mapping T of Ω into itself, obtained by selecting a single point Tw from $\breve{D}Fw \cap \Omega$. We thus have

$$(13.5) \qquad w \in \Omega \Rightarrow DTw = Fw, \qquad Tw \in \Omega$$

so that every fixed point $w = Tw$ is a solution of (13 4) i.e. of (II)-(C). In other words one has to define a convenient selection mapping S of $P(\Omega)$ into Ω and then replace $\mathcal{T}$ by the composition $T = S\mathcal{T}$.

It should be noted that since $\breve{D}Fw = \{ D_\tau^+ Fw \} + \mathfrak{N}(D)$, the effect of S will be that of singling a certain $U\xi_w$ out of $\mathfrak{N}(D)$ and, ultimately, a certain ξ_w out of R^n, such that $U\xi_w + D_\tau^+ Fw \in \Omega$ and T will be given by

$$(13.5) \qquad T: \quad w \to Tw = U\xi_w + D_\tau^+ Fw.$$

Finally, to prove the existence of fixed points of T one can either use theorems based on «compactness» properties (such as Ths. A, B of the Appendix) or on «contractivity» (such as Ths. C, D, E of the same App.).

14. - Problem IV. - The Cauchy problem for equation (II). Existence.

a) The simplest b.v.p. for equalion (II) is

PROBLEM IV. - *The Cauchy problem for equation* (II).

Given an interval $\Delta \subset J$, $\tau \in \Delta$, $\xi \in R^n$, *determine the solutions of* (II) *on* Δ *such that*

$$x(\tau) = \xi.$$

Contrary to what happens for Problem I, there are $\Delta \subset J$ where Problem IV has no solution. For instance, the equation ($n = 1$, $J = R$)

$$(14.1) \qquad dx/dt = |x|^q$$

with $q > 1$ and the condition $x(0) = \xi > 0$ has no solution on Δ if $\Delta \supset [0, (q-1)^{-1}\xi^{1-q}]$.

b) In accordance with (13.2) we have to impose conditions on the data Δ, A, f, τ, ξ in order that

$$Dx = Fw, \qquad x(\tau) = \xi \tag{14.2}$$

have solutions on Δ for all $w \in C(\Delta, R^n)$ such that $w(\tau) = \xi$. By virtue of Th. 2.1 if $w \in C(\Delta, R^n)$ implies $Fw \in L_{\text{loc}}(\Delta, R^n)$, i.e. if we assume that

H_2) $\quad t \to f(t, w(t))$ *belong to* $L_{\text{loc}}(\Delta, R^n)$ *for all* $w \in C(\Delta, R^n)$,

then (14.2) has a unique solution, namely

$$Tw = U\xi + D_\tau^+ Fw, \tag{14.3}$$

so that we have a point-to-point mapping T of

$$\Omega = \{ w \in C(\Delta, R^n) : w(\tau) = \xi \}$$

into itself whose fixed points are solutions of Problem IV, and no selection is needed.

However assumption H_2) does not insure the existence of fixed points of T, as is shown by the example of equation (14.1), so one has to impose on f stronger assumptions.

c) The simplest set of such conditions is represented by CARATHEODORY'S ones, namely

α) *for each* $t \in \Delta$ *let* $x \to f(t, x)$ *be continuous on* R^n;

β) *for each* $x \in R^n$ *let* $t \to f(t, x)$ *be (Lebesgue) measurable on* Δ;

γ) *let*

$$|f(t, x)| \leq \beta(t), \qquad x \in R^n, \quad \text{a.e.} \quad t \in \Delta,$$

for some $t \to \beta(t)$ *belonging to* $L_{\text{loc}}(\Delta, R)$.

It is readily seen that α), β), γ) imply H_2) so that T can be defined by (14.3), i.e. by

$$T: w \to Tw = U(t, \tau)\xi + \int_\tau U(t, s) f(s, x(s)) ds. \tag{14.4}$$

If Δ is compact, $C(\Delta, R^n)$ is a BANACH space in the norm topology (Sec. 13). To prove the existence of fixed points of T we can use SCHAUDER'S theorem (Appendix, Th. A), since T turns out to be continuous, $T\Omega$ is bounded and the $x \in T\Omega$ are equicontinuous on Δ, hence, by ASCOLI's theorem, $\overline{T\Omega}$ is compact.

Alternatively, since, again by ASCOLI's theorem, T turns out to be compact, hence completely continuous, we can apply Th. B of the Appendix by showing that there are sequences $\{ w_k \}$ in Ω which are bounded and such that $Tw_k - w_k \to 0$ in $C(\Delta, R^n)$. Assuming, for instance, $\Delta = [\tau, \tau + \delta]$, $\delta > 0$, TONELLI's sequence

$$w_k(t) = \begin{cases} U(t, \tau)\xi, & t \in [\tau, \tau + \delta/k] \\ U(t, \tau)\xi + \int\limits_{\tau}^{t-\delta/k} U(t, s)f(s, w_k(s))ds, & t \in [\tau + \delta/k, \tau + \delta] \end{cases}$$

has both properties. It should be noted that Th. B does not insure the convergence of $\{ w_k(t) \}$ but only the existence of a subsequence which converges to a fixed point.

When Δ is non-compact, for instance $\Delta = [\beta, \omega[$ then $C(\Delta, R^n)$ can be made into a FRÉCHET space (Sec. 13) and we may replace SCHAUDER'S theorem by TYCHONOV'S one (Appendix, Th. A).

Summing up we have:

TH. 14.1. - *If Caratheodory conditions* α), β, γ) *are satisfied on* Δ, *then Problem* IV *has solutions for each* $\tau \in \Delta$ *and each* $\xi \in R^n$.

d) When $\Delta = J$ Th. 2.1 is a Corollary of Th. 14.1 as far as existence is concerned. However assumption γ) is exceedingly restricted. For instance it does not apply to equation (14.1) with $0 < q \leq 1$ for which Problem IV has solutions on $J = R$ for all possible τ and ξ. This example suggest to replace γ) by the weaker

γ') *let*

$$| f(t, x) | \leq \beta(t) + \gamma(t) \, | x |^q, \qquad x \in R^n, \qquad \text{a.e. } \; t \in \Delta$$

for some pair of functions $t \to \beta(t)$, $t \to \gamma(t)$ *belonging to* $L_{\mathrm{loc}}(\Delta, R)$ *and* $0 \leq q \leq 1$.

By using an artifice it is then possible to deduce from Th. 14.1 the two extensions represented by Ths. 14.2 and 14.3.

TH. 14.2. - *If assumptions* α), β), γ') *are satisfied on* Δ *and* $0 \leq q < 1$, *then problem* IV *has solutions for each* $\tau \in \Delta$ *and each* $\xi \in R^n$.

PROOF. - Let Δ be compact. Then $L_{\mathrm{loc}}(\Delta, R) = L(\Delta, R)$. Since $0 \leq q < 1$ it follows that there are $r > 0$ such that

$$r - M_\Delta r^q \int\limits_{\Delta} \gamma(s)ds \geq M_\Delta \Big[\, | \xi | + \int\limits_{\Delta} \beta(s)ds \Big] \tag{14.5}$$

where

$$M_\Delta = \sup | U(t, s) |, \qquad (t, s) \in \Delta \times \Delta. \tag{14.6}$$

The artifice, largely used in the literature, consists of replacing f by

$$\bar{f}(t, x) = \begin{cases} f(t, x), & \text{if } |x| \leq r. \\ f(t, r|x|^{-1}x), & \text{if } |x| > r. \end{cases}$$

Clearly $\bar{f}$ satisfies α), β) and also γ) since from γ') follows

$$|\bar{f}(t, x)| \leq \beta(t) + r^q\gamma(t), \qquad x \in R^n, \qquad \text{a.e. } t \in \Delta. \tag{14.7}$$

By virtue of Th. 14.1 there are fixed point of the mapping

$$\bar{T}\colon w \to \bar{T}w = U(t, \tau)\xi + \int_\tau^t U(t, s)\bar{f}(s, w(s))ds.$$

But if $x = \bar{T}x$ from (14.7) and (14.5) follows $|x(t)| \leq r$, $t \in \Delta$, hence $\bar{f}(s, x(s)) = f(s, x(s))$ a.e. $s \in \Delta$. Therefore $x = Tx$, i.e. x is a fixed point also for T defined by (14.4), hence a solution of Problem IV.

If $\Delta = [\beta, \omega[$ let $\{\tau_k\}$ be a sequence in Δ, $\tau_k \to \omega$, $\tau < \tau_1 < \tau_2 < \dots$ and let x^1 be a solution of (II) on $[\beta, \tau_1]$ such that $x^1(\tau) = \xi$, let x^2 be a solution on $[\tau_1, \tau_2]$ such that $x^2(\tau_1) = x^1(\tau_1)$, etc. The x defined by x^1 on $[\beta, \tau_1]$, by x^2 on $[\tau_1, \tau_2]$, etc. will be a solution on Δ hence a solution of Problem IV.

e) When $q = 1$, (14.5) would become

$$\left(1 - M_\Delta \int_\Delta \gamma(s)ds\right) r \geq M_\Delta\left[|\xi| + \int_\Delta \beta(s)ds\right]$$

and to insure the existence of such an $r > 0$ we have to make the additional assumption

$$1 - M_\Delta \int_\Delta \gamma(s)ds > 0, \quad \text{for every compact } \Delta \subset J, \tag{14.8}$$

with M_Δ defined by (14.6) We thus have

TH. 14.3. - *If assumptions* α), β), γ') *are satisfied on* Δ, *with* $q = 1$, *and if* (14.8) *holds, then Problem* IV *has solutions for each* $\tau \in \Delta$ *and each* $\xi \in R^n$.

REMARK 14.1. - When $q = 1$ the inequality in $\gamma')$

$$|f(t, x)| \leq \beta(t) + \gamma(t)|x|$$

could be replaced by the stronger one

$$|f(t, x) - \gamma(t)x| \leq \beta(t) \tag{14.9}$$

without any further assumption on $\gamma(t)$ like (14.8). In fact if (14.9) holds we can write equation (II) as

$$dx/dt - [A(t) + \gamma(t)I_{R^n}]x = f(t, x) - \gamma(t)x$$

and then apply Th. 14.1.

f) The same artifice used to deduce Ths. 14.2 and 14.3 from Th. 14.1 can be applied to prove

TH. 14.4. - *Let Caratheodory hypotheses* α), β) *hold and let* γ) *be replaced by the assumption that there is some pair* $r > 0$ *and* $t \to \beta_r(t)$ *belonging to* $L(\Delta, R)$ *such that*

$$t \in \Delta \quad |x| \leq r \Rightarrow |f(t, x)| \leq \beta_r(t)$$

$$M_\Delta \int_\Delta \beta_r(s)ds \leq r$$

with M_Δ *defined by* (14.6).

Then Problem IV *has solutions for each* $\tau \in \Delta$ *and each* $\xi \in R^n$ *such that*

$$|\xi| \leq M_\Delta^{-1}r - \int_\Delta \beta_r(s)ds.$$

This Theorem covers such cases as that of equation (14.1) with $q > 1$.

g) All the existence theorems considered in this Sec. are based on a majoration of the norm $|f|$ of f. Other theorems, insuring the existence only on the left or on the right of τ, can be obtained by using instead a minoration or a majoration, respectively, of the inner product x^*f. This is the starting point of a comparison principle which is largely used in various branches of the theory of ordinary d.e. A presentation of this principle would go beyond the scope of this report and we refer the reader to the exposition of F. BRAUER [1] and to R. CONTI [3], [4].

15. - Problem IV: the Cauchy problem for equation (II). Uniqueness.

a) Another remarkable feature of Problem IV is that, contrary to Problem I, it may admit more than one solution. For instance $x = 0$ and $x = (1 - q)^{1/(1-q)} t^{1/(1-q)}$ for $t > 0$ are distinct solutions of equation (14.1), $0 < q < 1$, such that $x(0) = 0$.

The structure and the properties of the set of solutions of Problem IV with given Δ, were investigated in detail by G. Peano, H. Kneser, M. Fukuhara, M. Nagumo, E Kamke and others. For more complete references and more recent results see: K. Hayashi [1], Ch. C. Pugh [1], G. R. Sell [1].

When there is only one solution of Problem IV on Δ it is customarily denoted as $x = x(t, \tau, \xi)$. The continuity of the mapping $\xi \longrightarrow x(\cdot, \cdot, \xi)$ of R^n into $C(\Delta, R^n)$ can be proved by observing that it transforms sets of R^n which are relatively compact (i.e. bounded) into sets of $C(\Delta, R^n)$ which are also relatively compact (i.e. bounded and equicontinuous), while the inverse mapping $x(\cdot, \cdot, \xi) \longrightarrow \xi$ has a closed graph.

See for instance A. F. Filippov [1], [2].

b) The theorems of Sec. 14 not only do not insure the uniqueness of the solution but they are not of a constructive kind. Both these disadvantages are eliminated when the assumptions on f allow the use of Banach contraction principle in a form or another (See Ths. C, D, E of the Appendix), so that a unique fixed point of T is obtained by the «method of successive approximations», i.e. as the limit of a sequence of iterations x_0, Tx_0, T^2x_0, ... Since all solutions of Problem IV are also solutions of $x = U\xi + D_\tau^+ Fx$, i.e. fixed point of the mapping T defined by (14.3) we have, for instance

Th. 15.1. - *Let Δ be a compact subinterval of J, let Caratheodory hypotheses α), β) hold and let, further*

$$|f(t, x) - f(t, y)| \leq \lambda(t)\,|x - y|, \quad x, y \in R^n, \quad \text{a.e. } t \in \Delta \tag{15.1}$$

for some $t \longrightarrow \lambda(t)$ belonging to $L(\Delta, R)$ If, either

$$1 - M_\Delta \int_\Delta \lambda(s)ds > 0 \tag{15.2}$$

with M_Δ defined by (14.6), *or Caratheodory's γ) holds, then Problem IV has a unique solution. This solution is the limit in* $C(\Delta, R^n)$

of the sequence

$$x_{k+1}(t) = U(t,\ \tau)\xi + \int_{\tau}^{t} U(t,\ s)f(s,\ x_k(s))ds \tag{15.3}$$

with x_0 *any function of* $C(\Delta,\ R^n)$.

Proof. – Let (15.2) hold. Then by (14.4) we have

$$|Tx - Ty| \leq \left(M_\Delta \int_\Delta \lambda(s)ds\right)|x - y|$$

where $|\ |$ denote norms in $C(\Delta,\ R^n)$, and (15.2) means that T is a contractive mapping of $C(\Delta,\ R^n)$.

Let now Caratheodory's γ) hold. Then T is continuous and it is easy to prove that there is a positive integer ν such that T^ν is contractive. The result follows from Th. D of the Appendix. Alternatively we could use the device we already used in the proof of Th. 1.1, due to A. Bielecki [1], consisting of rendering T a contraction by introducing a suitable norm into $C(\Delta,\ R^n)$ and then apply Th. C of the Appendix.

c) The inequality (15.1) does not hold for «strongly» nonlinear f such as $|x|^q$ with $q > 1$. In cases like this one has to assume the validity of (15.1) for x, y restricted to a certain ball of R^n and to make use of «locally» contractive mappings, like that of Th. E of the Appendix.

d) A remark analogous to that made in Sec. 14, *g*) holds. Namely Th. 15.1 is based on a majoration of the norm $|\Delta f|$ of $\Delta f = f(t,\ x + \Delta x) - f(t,\ x)$ by means of a linear function of Δx. More general uniqueness theorems like Perron's or Kamke's criterion (See C. Olech [2]) using an inequality of the form $|\Delta f| \leq \leq \varphi(t,\ |\Delta x|)$ are known. However, unilateral uniqueness (i.e. only on one side of τ) can be obtained by using a majoration or a minoration of the inner product $(\Delta x)^*(\Delta f)$, or, more in general, by means of the comparison principle already mentioned. We refer, also for the literature to R. Conti [3], R. D. Moyer [1] and C. Ciliberto [1].

e) It is quite natural to ask whether there are conditions under which the sequence of iterations can be used to obtain a solution of Problem IV when there is no uniqueness. This is still an open question, apparently (See E. A. Coddington - N. Levinson [1]; also R. M. Bianchini [1]).

16. - Problem V.

a) We consider now for equation (II) the corresponding of Problem II, namely:

PROBLEM V. - *Let Δ be a given compact subinterval of J. Let Λ be a real linear normed space and let L be a linear bounded operator with $\mathfrak{D}(L) = C(\Delta, R^n)$, $\mathfrak{R}(L) \subset \Lambda$ and $\mathfrak{N}(L) \neq \{0\}$. Given an $l \in \mathfrak{R}(L)$, determine the solutions of equations* (II) *on Δ such that*

$$Lx = l. \tag{16.1}$$

We proceed, as for Problem IV, along the lines sketched in Sec. 13.

In accordance to (13.2) we have to put conditions on Δ, A, f, L, l, in order that

$$Dx = Fw, \qquad Lx = l \tag{16.2}$$

have solutions on Δ for all w such that $Lw = l$.

To apply Th. 3.1, we define the operators L_U, L_U^g as in Sec. 3. Then (16.2) will have solutions for all w, $Lw = l$, if and only if

$$Lw = l \Rightarrow (I_\Lambda - L_U L_U^g)(l - LD_\tau^+ Fw) = 0, \tag{16.3}$$

with $\tau \in \Delta$ arbitrarily fixed. According to (3.6) the set of solutions of (16.2) is

$$\mathcal{T}w = U\mathfrak{N}(L_U) + \{ UL_U^g(l - LD_\tau^+ Fw) + D_\tau^+ Fw \}$$

so that $w \longrightarrow \mathcal{T}w$ is a point-to-set mapping of $\Omega = \breve{L}l$ into $P(\Omega)$, unless $\mathfrak{N}(L_U) = \{0\}$, i e. unless L_U has a left inverse. But, obviously, $\mathfrak{N}(L_U)$ does not depend on w so that to replace $\mathcal{T}$ by a point-to-point mapping T of $\breve{L}l$ into itself all we have to do is to select $\xi_0 \in \mathfrak{N}(L_U)$, for instance $\xi_0 = 0$, and define

$$T\colon w \longrightarrow Tw = U\xi_0 + UL_U^g(l - LD_\tau^+ Fw) + D_\tau^+ Fw. \tag{16.4}$$

b) *A* set of conditions insuring the existence of fixed points of T is, again, represented by CARATHEODORY'S α), β), γ) of Sec. 14. To see this note first that $\Omega = \breve{L}l$ is closed since L is continuous, by assumption. Next, L_U^g is also continuous since it maps $\mathfrak{R}(L_U)$, a finite dimensional space, into R^n. Then writing T as

$$Tw = U(\xi_0 + L_U^g l) - UL_U^g LD_\tau^+ Fw + D_\tau^+ Fw$$

it is not difficult to prove, using α), β), γ), that $w_k \longrightarrow w$ in $C(\Delta, R^n)$ implies $Tw_k \longrightarrow Tw$ in $C(\Delta, R^n)$, so that T is continuous.

Again by α), β), γ), we have that the functions $t \longrightarrow Tw(t)$ are uniformly bounded and equicontinuous on Δ, hence by ASCOLI'S theorem it follows that $\overline{TLl}$ is a compact subset of $C(\Delta, R^n)$. Therefore SCHAUDER-TYCHONOV'S theorem (Th. A of the Appendix) insures the existence of fixed points of T. Summing up we have:

TH. 16.1. - *Problem V has solutions if i) f satisfies Caratheodory's conditions* α), β), γ), *of Sec.* 14, *and ii) the problem*

$$dx/dt - A(t)x = f(t, w(t)), \qquad Lx = l \tag{16.5}$$

has solutions for each $w \in C(\Delta, R^n)$, $Lw = l$.

c) Let Δ be a compact subinterval of J and assume, in addition to the assumptions of Th. 16.1, that the CAUCHY problem

$$Dx = Fx, \qquad x(\tau) = \xi$$

has a *unique* solution for each $\xi \in R^n$. Denote this solution by x_ξ and define the mapping

$$S: \xi \longrightarrow S\xi = \xi_0 + L_U^g l - L_U^g L D_\tau^+ F x_\xi$$

of R^n into itself. Since $\xi \longrightarrow x_\xi$ is a continuous mapping of R^n into $C(\Delta, R^n)$ (Sec. 15, *a*)), it follows, under CARATHEODORY conditions, that S is continuous and it maps R^n into a bounded set. BROUWER'S theorem then insures the existence of fixed points $\xi = S\xi$.

But if $\xi = S\xi$ then

$$x_\xi = U\xi + D_\tau^+ F x_\xi = U(\xi_0 + L_U^g l - L_U^g L D_\tau^+ F x_\xi) + D_\tau^+ F x_\xi = T x_\xi ,$$

i.e x_ξ is a fixed point of T.

Therefore, the additional assumption of uniqueness for the CAUCHY problem allows to replace SCHAUDER'S theorem by the more elementary BROUWER'S theorem in the proof of Th. 16.1.

d) To prove Th. 16.1 we could also have used Th. B of the Appendix, but to construct a suitable sequence $\{w_k\}$ would require, in general, the application of BROUWER'S theorem *for each k*, which is substantially equivalent to a *unique* application of SCHAUDER'S theorem.

e) As was already observed in Sec. 14. *d*), assumption γ) is a very restrictive one. It can be replaced however by more general assumptions like γ') (Sec. 14, *d*)) and existence theorems similar to Ths. 14.2, 14.3 and 14.4 can be obtained through the artifice used in the proof of Th. 14.2. We have, for instance:

TH. 16.2. - *Problem V has solutions if i) f satisfies conditions* α), β) *of Sec.* 14, *ii) there are two functions $t \to \beta(t)$, $t \to \gamma(t)$ integrable on Δ such that*

$$(16.6)\qquad |f(t, x)| \leq \beta(t) + \gamma(t)|x|, \qquad x \in R^n, \qquad \text{a.e. } t \in \Delta$$

$$1 - M_\Delta(1 + |L_U^q|\,|L|\,M_\Delta)\int_\Delta \gamma(s)ds > 0$$

with M_Δ defined by (14.6), *and iii) the problem* (16.5) *has solutions for each $w \in C(\Delta, R^n)$, $Lw = l$.*

f) It is easily verified that, provided (16.3) hold, not only every fixed point of T is a solution of Problem V, but also, conversely, every solution of Problem V is a fixed point of T. Therefore conditions on f which insure some kind of contractivity for T will also insure uniqueness of the solution of Problem V. Further, such conditions will also insure that the solution can be obtained as the limit of a sequence of iterations $\{T^k x_0\}$ starting from some $x_0 \in C(\Delta, R^n)$. arbitrarily chosen.

The dependence of the solution from the data A, f, L, l would certainly deserve further investigation.

g) When $l = 0$, taking $\xi_0 = 0$ the mapping T is defined by

$$Tw = KFw$$

where $K = -L_U^q L D_\tau^+ + D_\tau^+$ is a linear operator of $L_{loc}(\Delta, R^n)$ into $C(\Delta, R^n)$ Therefore the fixed points of T are the solutions of a HAMMERSTEIN equation $w = KFw$ Such equations have been extensively investigated: we refer to the expository papers of H. EHRMANN [2], [3], also for the bibliography, and to M. M. VAINBERG'S book [1].

h) We repeatedly observed that $\mathfrak{R}(L_U)$ is a subspace of Λ of dimension $m \leq n$. If $m > n$ the incompatibility of Problem II (Remark 4.1), i.e.

$$Dx = 0, \;\; Lx = 0 \Rightarrow 0,$$

or, equivalently, the existence of a left inverse L_U^- of L_U does not insure per se that (16.2) i.e. (16.5), has solutions for each w, $Lw=l$. However, this will be the case when $m=n$, since then $L_U^- = L_U^{-1}$ and (16.3) is satisfied for all $w \in C(\Delta, R^n)$, even those for which $Lw \neq l$.

This case is the one most frequently encountered in the literature. For instance, when L is defined by $Lx = x(\tau_1) - x(\tau_2)$ as in the problem of harmonic solutions (Sec. 6, d)) it is called the «non resonance» case.

When $\Lambda = R^n$, so that $m = n$, and L is into $\Lambda = R^n$, it can be proved (A. Lasota-Z. Opial [2]) that there is a $t \longrightarrow \tilde{A}(t)$ from Δ into $L(\Delta, \mathcal{C})$ such that denoting by $\tilde{U}$ the evolution operator associated with $\tilde{A}$, the corresponding operator $L_{\tilde{U}}$ has the inverse $L_{\tilde{U}}^{-1}$. Writing equation (II) as

$$dx/dt - \tilde{A}(t)x = \tilde{f}(t, x)$$

with $\tilde{f}(t, x) = f(t, x) + [A(t) - \tilde{A}(t)]x$, if suitable assumptions are satisfied by f such, for instance, as Caratheodory's α), β), γ), the existence of solutions of Problem V can be proved. It must be noted however that imposing conditions like γ) on $\tilde{f}$ implies a restriction both on A and L, hence on L_U.

i) A case which frequently occurs is that of Problem V for an equation (II) of the form

$$\text{(II}')\qquad dx/dt = \Phi(t, x)x + \varphi(t, x)$$

where φ satisfies Caratheodory's α), β), γ) and Φ is a function (t, x) into $\mathcal{C}$ which also satisfies Caratheodory's assumptions. More precisely Φ is continuous (in the norm topology of $\mathcal{C}$) with respect to $x \in R^n$ for each $t \in \Delta$, (Lebesgue) measurable (in the same topology) with respect to t for each $x \in R^n$, and $|\Phi(t, x)| \leq \gamma(t)$, $x \in R^n$, a.e. $t \in \Delta$, for some $\gamma \in L_{\text{loc}}(\Delta, R)$.

Equation (II$'$) can be considered as a special case of (II) with $A(t) = 0$, $f(t, x) = \Phi(t, x)x + \varphi(t, x)$,

$$|f(t, x)| \leq |\Phi(t, x)|\,|x| + |\varphi(t, x)| \leq \beta(t) + \gamma(t)|x|.$$

Since $A(t) = 0$ the corresponding evolution operator is the identity I in R^n and $L_U = L_I$ will be the restriction of L to R^n, considered as the subspace of constant $x \in C(\Delta, R^n)$. Since $M_\Delta = 1$, we derive from Th. 16.2 the conclusion that Problem V for equation

(II′), i.e.

$$(16.7) \qquad dx/dt = \Phi(t,\ x)x + \varphi(t,\ x), \qquad Lx = l$$

has solutions if, in addition to the assumptions made on Φ and φ, we have

$$(16.8) \qquad 1 - (1 + |L_U^g||L|)\int_\Delta \gamma(s)ds > 0$$

and

$$(16.9) \qquad dx/dt = \Phi(t,\ w(t))w(t) + \varphi(t,\ w(t)), \qquad Lx = l$$

has solutions for each w, $Lw = l$. In particular, if $\Lambda = R^n$ and $\det L_I \neq 0$ there will be a solution of (16.7) provided that (16.8) holds, with L_I^g replaced by L_I^{-1}.

j) Another way of dealing with Problem (16.7) is the following. For each w, $Lw = l$, the function

$$A_w:\ t \longrightarrow A_w(t) = \Phi(t,\ w(t))$$

of Δ into $\mathcal{C}$, is integrable by virtue of the assumptions on Φ.

Instead of (16.9) let us consider the problem

$$(16.10) \qquad dx/dt - A_w(t)x = \varphi(t,\ w(t)), \qquad Lx = l$$

and assume that it has solutions for each w, $Lw = l$. Next we define (Th. 1.1) the evolution operator U_w associated with A_w for each w, the restriction $L_{w,0}$ of L to the null space $\mathfrak{N}(D_w)$ of $D_w = d/dt - A_w(t)$, the composition $L_{U_w} = L_{w,0}U_w$, and its generalized inverse $L_{U_w}^g$.

The fixed points of

$$w(t) \longrightarrow U_w(t,\ \tau)L_{U_w}^g\left(1 - L\int_\tau^t U_w(t,\ s)\varphi(s,\ w(s))ds\right) + $$
$$+ \int_\tau^t U_w(t,\ s)\varphi(s,\ w(s))ds$$

will then be solutions of problem (16.7). To insure the existence of such fixed points one has to assume that the set spanned by $|L_{U_w}^g|$ for $Lw = l$ is bounded.

This assumption will be satisfied, in particular, if $\Lambda = R^n$,

$$dx/dt - A_w(t)x = 0, \qquad Lx = 0 \Rightarrow x = 0$$

for each w, $Lw = l$, so that $L\overset{-1}{U_w}$ exists and, further,

$$Lw = l \Rightarrow |\det LU_w| \geq \delta > 0$$

for some $\delta > 0$.

References: H. A. Antosiewicz [1], [2]; L. Barbalat - A. Halanay [1]; R. Conti [1], [2], [5], [6], [7], [8]; L. N. Eshukov [2]; S. N. Hilt [1]; A. Lasota [1]; A. Lasota - C. Olech [1]; A. Lasota - Z. Opial [1], [2], [3], [4]; Z. Opial [1]; P. Santoro [1]; V. P. Skripnik [1]; G. Villari [1]; W. M. Whyburn [1], [2], [5].

17. - Problem VI.

a) The linear variety $\widetilde{L}l$ defined by $Lw = l$ can be considered as the l-level set of the linear operator L of $C(\Delta, R^n)$ into the space Λ. This remark suggests the following generalization of Problem V.

Problem VI. - *Let Δ be a given compact subinterval of J. Let Λ be a real linear normed space and let Q be a continuous, non necessarily linear, mapping of $C(\Delta, R^n)$ into Λ such that the set*

$$\Omega = \{x \in C(\Delta. R^n) : Qx = 0\}$$

is infinite. It is required to determine the solutions x of equation (II) *belonging to Ω, i.e. such that*

$$(\overset{\circ}{1}7.1) \qquad Qx = 0.$$

b) We assume (Sec. 13) that

$$Dx = Fw, \qquad Qx = 0$$

has solutions for each w, $Qw = 0$. Next we define a function Φ of $(w, \chi) \in \Omega \times \mathfrak{N}(D)$ into Λ by taking $\tau \in \Delta$ and

$$\Phi(w, \chi) = Q(\chi + D_\tau^+ Fw).$$

The assumption just made means that to each $w \in \Omega$ there corresponds a non empty subset S_w of $\mathfrak{N}(D)$ such that $\chi \in S_w \Rightarrow \Rightarrow \Phi(w, \chi) = 0$.

A criterion to select a single χ_w out of S_w will then be represented by any implicit function theorem insuring the existence of one mapping $w \longrightarrow \chi_w$ of Ω into $\mathfrak{N}(D)$ such that $\Phi(w, \chi_w) = 0$.

A fixed point $w = Tw$ of the mapping $T: w \longrightarrow Tw = \chi_w + D_\tau^+ Fw$ of Ω into Ω is a solution of Problem VI.

Reference: C. Avramescu [1].

c) Problem VI has also been treated by a different method which is closer to the one used for Problem V.

Write $Q = L - H$, with $H = L - Q$ and L linear, $\mathfrak{D}(L) = C(\Delta, R^n)$, $\mathfrak{N}(L) \neq \{0\}$, such that L_U has right inverses L_U^+ (or, in particular the inverse L_U^{-1}). Then

$$T: x \longrightarrow Tx = U\xi_0 + UL_U^+(Hx - LD_\tau^+ Fx) + D_\tau^+ Fx$$

is a mapping of $C(\Delta, R^n)$ into itself such that

$$DTx = Fx, \qquad QTx = Hx - HTx.$$

Therefore, in general, T does not map Ω into Ω, but anyway its fixed points, if there are any, belong to Ω and are solutions of Problem VI.

References: R. Conti [6]; H. Ehrmann [1], [3]; G. Pulvirenti [1]; G. Santagati [1], [2]; E. Scrucca [1].

18. - Problem VII.

a) Problem V can now be generalized in another direction. Observe that the linear variety $\tilde{L}l$ defined by $Lw = l$ is a convex and also (due to the assumptions on L) infinite and closed subset of $C(\Delta, R^n)$. This suggests:

Problem VII. - *Let Δ be a given subinterval of J. Let Ω be an infinite, convex closed subset of $C(\Delta, R^n)$. We want to determine the solutions x of equation* (II) *such that*

$$x \in \Omega. \tag{18.1}$$

It should be noted that Problem VII does not include Problem VI because the set $\{x \in C(\Delta, R^n): Qx = 0\}$ needs not to be convex.

This time, as we shall see, the selection criterion is provided, under suitable assumptions, by a fundamental lemma due to Massera and Schäffer.

b) In accordance with Sec. 13 we should assume that $Dx = Fw$, $x \in \Omega$ has solutions for each $w \in \Omega$. In fact we shall assume, more restrictively, that the span of Ω (i.e. the set of linear combinations of elements in Ω) and the span of $F\Omega$, are subspaces $V \subset C(\Delta, R^n)$ and $B \subset L_{loc}(\Delta, R^n)$, respectively, such that the pair (B, V) is admissible with respect to D in the sense of MASSERA-SCHÄFFER (Sec. 10). This means that

$$Dx = y, \qquad x \in V$$

has solutions for each $y \in B$.

Having fixed $\tau \in \Delta$ let $X_0 \subset R^n$ be the subspace of n-vectors $\chi(\tau)$ corresponding to $\chi \in \mathfrak{N}(D) \cap V$, let X_1 be any complement of X_0 to R^n and let P be the projection of R^n onto X_1. Since X_0, X_1 are both finite dimensional, hence closed, it can be proved:

Massera-Schaffer's lemma. If the pair (B, V) is admissible for D and if $\xi_0 \in X_0$, then to each $y \in B$ the recorresponds a unique x_y such that

$$Dx_y = y, \qquad x_y \in V, \qquad Px_y(\tau) = \xi_0 .$$

Moreover the mapping $y \longrightarrow x_y$ of B into $\breve{D}B \cap V$ is continuous.

This is a selection criterion which allows to define a point-to-point mapping T of Ω into Ω (Sec. 13) and the existence of fixed points can then be proved under suitable assumptions.

This technique has been successfully applied both for compact and non compact Δ.

References: H. A. ANTOSIEWICZ [2]; W. A. COPPEL [1]; C. CORDUNEANU [2], [3], [4]; P. HARTMAN [1]; P. HARTMAN - N. ONUCHIC [1]; J. L. MASSERA [1]; J. L. MASSERA - J. J. SCHÄFFER [1], [2], [3]

c) A remark analogous to the one made in Sec. 16, *d*) about the inconvenience of using Th. *B* of the Appendix instead of SCHAUDER-TYCHONOV'S or BANACH'S theorems to prove the existence of solutions of Problem V is valid also for Problem VII, in general. However Th. B suits well to solve the asymptotic CAUCHY problem (11.1) for equation (II) by using a modified version of TONELLI'S sequence (Sec. 14) (See for instance IA. D. MAMEDOV [1]).

19. - A problem of G. Stampacchia.

The following problem considered by G. STAMPACCHIA [1] is an extension to equation (II) of the one considered in Sec. 11, h) for equation (I).

Let Δ be a compact subinterval of J. Let $V_1, \ldots, V_n$ be given subsets of $\Delta \times R^n$. It is required to determine solutions of equation (II) whose graph has a non empty intersection with each set V_i.

Therefore in this problem Ω will be the subset of $x \in C(\Delta, R^n)$ whose graph intersects each set V_i, hence it needs not to be convex.

Take $\tau \in \Delta$ and let U_i denote the set of points $\chi(\tau) \in R^n$ corresponding to $\chi \in \mathfrak{N}(D)$ whose graph intersects V_i. In other words each set V_i is projected into the hyperplane $t = \tau$ along the integral curves of $Dx = 0$ and the projection is U_i.

Then let $\theta_i : x \longrightarrow \theta_i(x)$ be defined on R^n by

$$\theta_i(x) = d(x, U_i), \qquad x \in R^n.$$

Assumptions are made about the existence of solutions of the system $\theta_i(x) = 0$, $i = 1, \ldots, n$, so that $\bigcup_i U_i \neq \emptyset$ or, equivalently

$$Dx = 0, \qquad x \in \Omega$$

has solutions.

Finally equation (II), $Dx = Fx$, is imbedded in the family $Dx = = \lambda Fx$, $\lambda \in [0, 1]$ and it is assumed that the CAUCHY problem with arbitrary initial data has a unique solution on Δ for each λ. This allows to define n sets $U_{i,\lambda}$ for each λ by projecting the sets V_i along the integral curves of $Dx = \lambda Fx$ and to define n functions $\theta_{i,\lambda}$ in R^n, by

$$\theta_{i,\lambda}(x) = d(x, U_{i,\lambda}), \qquad x \in R^n, \qquad \lambda \in [0, 1].$$

The problem is thus transformed into that of proving $\bigcup_i U_{i,1} \neq \emptyset$, or equivalently the existence of solutions of $\theta_{i,1}(x) = 0$, $i = 1, \ldots, n$.

This is done under additional assumptions on the $U_{i,0} = U_i$ which insure that the mapping $x \longrightarrow \theta_{i,\lambda}(x)$ has an odd topological degree (not necessarily equal to $+1$), by applying BROUWER's invariance theorem.

The assumption about the uniqueness for the CAUCHY problem can be replaced by using TONELLI's sequence $\{x_{k,\lambda}\}$ but this requires an application of BROUWER's theorem for each k. It seems likely that a unique application of LERAY-SCHAUDER's theory would be equivalent, a remark analogous to that of Sec. 16, *d*).

20. - Generalized solutions and interface conditions.

In all what precedes solutions are always supposed to be absolutely continuous. Technical applications require however to extend the definition of solution so as to include functions of bounded variation, possibly discontinuous. Such discontinuities will then necessarily be of the first kind (both limits $x(\tau -)$, $x(\tau +)$ exist for each τ) and they will be countably many at most.

It makes sense therefore to ask for such generalized solutions which in addition to side conditions like those entering the various b v.p., also satisfy a finite or countable set of equalities

$$M_k x(\tau_k -) + N_k x(\tau_k +) = c_k \tag{20.1}$$

where the τ_k are prescribed discontinuity points on J, the M_k and N_k are given matrices, and the c_k are given vectors. Equalities (20.1) are called interface conditions and problems (I)-(C) or (II)-(C) with (20.1) are interface problems.

A natural development of this kind of problem is represented by distribution differential equations, i.e. by equations $\dot{x} - Ax = f$ with f a distribution.

References: A. Gonnelli [1]; C. Olech [1]; D. Pham - D. Weiss [1]; T. J. Pignani - W. M. Whyburn [1]; D. Weiss - D. Pham [1]; D. Wexler [1], [2]; W. M. Whyburn [5].

APPENDIX

ABOUT FIXED POINT THEOREMS

The most currently used fixed point theorems in the theory of ordinary d.e. refer to a mapping T of a metric space X. They can be roughly divided into two categories, the first based on « compactness » assumptions, the second based on « contractivity ». Of the following, Ths. A, A′ and B belong to the first class, Ths. C, D and E, to the second.

Th. A. - *Let X be a Fréchet* (= linear metric, locally convex, complete) *space. Let Ω be a convex, closed subset of X. If T is continuous and $\overline{T\Omega}$ is a compact subset of Ω, then there exists at least one $x = Tx \in \Omega$.*

This is a particular case of Tychonov's theorem (Math. Annalen 111, (1935), 767-776). When X is a Banach (= linear, normed, complete) space Th. A reduces to Schauder's theorem (Studia Math. 2, (1930), 171-180) and, for $X = R^n$, to the classical Brouwer's theorem.

Extensions of Brouwer's, Schauder's and Tychonov's theorems to set-valued mappings were given respectively by S. KAKUTANI (Duke Math. Jour., 8 (1941), 457-459), H. F. BOHNENBLUST - S. KARLIN (Contributions to the Theory of Games Princeton 1950, 155-160) and KY FAN (Proc. Nat. Acad. Sci. U.S.A., 38 (1952), 121-126).

TH. A′ - *Let X be a Fréchet space. Let T be completely continuous, i.e. compact* ($=\overline{TS}$ is a compact subset of X for each bounded subset S of X) *and continuous. Then, either there are $x=\lambda Tx$ for each $\lambda\in[0,1]$, or the set $\{x: x=\lambda Tx,\ \lambda\in\,]0,1[\,\}$ is unbounded.*

This is a particular case of H. SCHAEFER's theorem (Math. Annalen, 129 (1955), 415-416).

TH. B. - *Let X be a metric space with distance d. If T is completely continuous there is at least one $x=Tx$ if* (and only if) *there is a sequence $\{x_k\}$ in X which is bounded and such that $d(Tx_k, x_k)\to 0$. Moreover there is a subsequence $\{x_{k'}\}$ such that $d(x_{k'}, x)\to 0$.*

See: R. CONTI, Le Matematiche, 15 (1960), 92-97; A. HAIMOVICI, Analele Stiint. Univ. Al. I. Cuza, 7 (1961), 65–76; K. ISEKI, Math. Japon., 7 (1962), 203-204.

TH. C. - *Let X be a complete metric space with distance d. Let T be a contractive mapping (i.e. $d(Tx', Tx'')\leq \alpha d(x', x'')$, with $0<$ $<\alpha<1$, for all $x', x''\in X$). Then T has a unique fixed point $x=Tx$, and $x=\lim T^k x_0$, for any $x_0\in X$, where $T^1=T$, $T^2=TT, \ldots$.*

Th. C is known as BANACH's (or BANACH-CACCIOPPOLI-TYCHONOV's) contraction principle. Easy to prove and useful consequences of Th. C are the following Th. D and E.

TH. D. - *Let X be a complete metric space with distance d. Let there exist a positive integer ν such that T^ν is contractive. Then T has a unique fixed point $x=Tx$. If, further, T is continuous, then $x=\lim T^k x_0$, for any $x_0\in X$.*

TH. E - *Let X be a Banach space, with norm $|\ |$. Let T be a contractive mapping of the ball $|x|\leq\rho$ into X. If 0 is the zero of X and $|T0|\leq\rho(1-\alpha)$, there is a unique fixed point $x=Tx$ and $x=\lim T^k 0$.*

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