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Quasi-Symmetrization of Hyperbolic Systems and Propagation of the Analytic Regularity.

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Sunto. – *Dopo aver introdotto la nozione di quasi-simmetrizzatore per sistemi del prim'ordine debolmente iperbolici, si dimostra che ad ogni sistema di tipo Sylvester, cioè proveniente da un'equazione scalare di ordine superiore, si può associare in modo regolare un quasi-simmetrizzatore. Come applicazione di questo risultato si prova che, per qualunque sistema semi-lineare $N \times N$ debolmente iperbolico, le soluzioni Gevrey in x di ordine $s < N/(N-1)$ restano analitiche non appena lo siano all'istante iniziale.*

1. – Introduction.

The main purpose of this paper is to investigate the notion of *smooth quasi-symmetrizer* for a hyperbolic system

$$\partial_t u + A(t, x, D)\, u = f\,,$$

where $D = i^{-1}\partial_x$, $t \geq 0$, $x \in \boldsymbol{R}^n$, and $A(t, x, \xi)$ is a $N \times N$ matrix-valued symbol, homogeneous of order one in ξ. By hyperbolic we mean here weakly hyperbolic, i.e.,

$$A(t, x, \xi) \quad \text{has pure imaginary eigenvalues} \tag{1.1}$$

for all t, x and $\xi \in \boldsymbol{R}^N$.

We call *quasi-symmetrizer* for A any family of $N \times N$ matrix-valued symbols

$$Q_\varepsilon(t, x, \xi) \in C^1([0, T]; S^0)\,, \qquad \varepsilon > 0\,,$$

with the following properties: for some positive constants $\delta_\varepsilon > 0$, C independent of t, x, ξ,

$$\delta_\varepsilon I \leq Q_\varepsilon = Q_\varepsilon^* \leq I\,,$$

$$AQ_\varepsilon + Q_\varepsilon A^* \leq C\varepsilon\langle\xi\rangle\, Q_\varepsilon\,,$$

where $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$, I is the identity matrix, and $A \leq B$ (with A, B matrices) means $(Av, v) \leq (Bv, v)$ for all $v \in \boldsymbol{C}^N$.

This is clearly an extension of the classical concept of symmetrizer for a system of first order, which corresponds to the case when $Q_\varepsilon \equiv Q$ is independent of ε, so that $AQ + QA^* = 0$. A similar notion was introduced for the first time by E. Jannelli in [J], where he constructed a non-smooth quasi-symmetrizer for systems depending only on time (see also [DS]).

The notion of quasi-symmetrizer is connected to energy estimates in spaces of analytic or Gevrey functions. We mention, e.g., the local stability of the analytic solutions to nonlinear weakly hyperbolic systems (see [DS], [Ki]).

In the first part of the paper we shall construct a smooth quasi-symmetrizer for a special class of hyperbolic systems, namely the systems of *Sylvester* type (or, more generally, of *block Sylvester* type); these are the systems arising after reduction of a higher order scalar equation to a system of first order, see Section 2 for a precise definition. This result is contained in Proposition 1 below.

In the second part we shall apply the preceding construction to prove the analytic regularity for a class of semilinear weakly hyperbolic systems. More precisely, we consider the systems

$$\partial_t u + A(t, D)\, u = f(t, x, u) \tag{1.2}$$

where $A(t, \xi)$ are $N \times N$ matrix-valued symbols of order one in ξ and of class C^N in t, hyperbolic in the sense of (1.1). Denote by $\gamma^s_{L^2}(\boldsymbol{R}^n)$ the space of the so called uniformly Gevrey functions on $\boldsymbol{R}^n$, i.e., the functions $u(x)$: $\boldsymbol{R}^n \to \boldsymbol{C}^N$ satisfying

$$\|D^\alpha u\|_{L^2} \leqslant C\Lambda^{|\alpha|}\, \alpha!^s$$

for some $C, \Lambda \geqslant 0$; for $s = 1$ we obtain the space $\mathcal{A}_{L^2}(\boldsymbol{R}^n)$ of *uniformly analytic* functions. Moreover, we assume that $f(t, x, u)$ is of class C^N in t, with time derivatives $\partial_t^j f(t, x, u)$ uniformly analytic in x and entire analytic in u, for $j = 0, 1, \ldots, n$. Then we shall prove:

THEOREM 1. – *Let $u(t, x)$ be a solution of the system* (1.2), *belonging to the Gevrey class $C^N([0, T], \gamma^s_{L^2}(\boldsymbol{R}^n))$ for some $1 \leqslant s < N/(N-1)$, and assume that $u(0, \cdot) \in \mathcal{A}_{L^2}(\boldsymbol{R}^n)$. Then $u(t, x)$ is uniformly analytic in x for all t, more precisely $u \in C^N([0, T], \mathcal{A}_{L^2}(\boldsymbol{R}^n))$.*

The condition $s < N/(N-1)$ is connected to the fact that for these values of s, system (1.4), and more generally any nonlinear hyperbolic first order system, is well-posed in the Gevrey class $\gamma^s_{L^2}(\boldsymbol{R}^n)$ (see [B], [Ka]), while for $s > 1/(N-1)$ the local existence may not hold. Compare with the case of strictly hyperbolic nonlinear systems, for which the analytic regularity propagates for any C^∞ solution ([AM]).

We mention that a result similar to Theorem 1, concerning second order scalar equations, was proved in [S].

2. – Construction of a quasi-symmetrizer.

In the following we shall construct a smooth quasi-symmetrizer for a special class of hyperbolic operators A, which we shall call operators of *Sylvester type*. By this we mean that $A(t, x, \xi) = \langle\xi\rangle B(t, x, \xi)$ where B is a *Sylvester matrix*, i.e.,

$$B = \begin{bmatrix} 0 & 1 & & \cdots & & 0 \\ & 0 & 1 & & & \\ & & 0 & & & \\ \vdots & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ b_1 & b_2 & & \cdots & & b_N \end{bmatrix}. \tag{2.1}$$

Here $b_j(t, x, \xi)$ are symbols of order 0. Typically such matrices are obtained after reduction of a higher order scalar equation to a first order system.

More generally, we shall say that B is a *N-block Sylvester type* matrix if it is a block matrix of dimension νN of the form

$$B = \begin{bmatrix} B_0 & & 0 \\ & \ddots & \\ 0 & & B_0 \end{bmatrix}$$

where the ν blocks B_0 are $N \times N$ identical matrices and have the form (2.1).

Then we can prove:

PROPOSITION 1. – *Assume that* $A(t, x, \xi) \in C^k([0, T]; S^1)$ $(k \geq 2)$ *and that*

$$A(t, x, \xi) \quad \text{has only pure imaginary eigenvalues}\,; \tag{2.2}$$

$$\langle\xi\rangle^{-1} A(t, x, \xi) \quad \text{is a uniformly bounded } N\text{-block Sylvester matrix}. \tag{2.3}$$

Then there exists a quasi-symmetrizer

$$Q_\varepsilon = Q_\varepsilon(t, x, \xi) \in C^k([0, T]; S^0)$$

satisfying the following conditions for all $(t, x, \xi) \in [0, T] \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ *and all* $\varepsilon > 0$*:*

$$\varepsilon^{2(N-1)} I \leq Q_\varepsilon = Q_\varepsilon^* \leq I\,, \tag{2.4}$$

$$AQ_\varepsilon + Q_\varepsilon A^* \leq C\varepsilon\langle\xi\rangle\, Q_\varepsilon\,, \tag{2.5}$$

$$-C\varepsilon^{1-N} Q_\varepsilon \leq Q_\varepsilon' \leq C\varepsilon^{1-N} Q_\varepsilon\,, \tag{2.6}$$

where Q' is the time derivative of Q and C a positive constant independent of ε, t, x, ξ.

More precisely, Q_ε has the form

$$Q_\varepsilon = \sum_{h=0}^{N-1} \varepsilon^{2h} q_h(a_1(t, x, \xi)\langle\xi\rangle^{-1}, \dots, a_N(t, x, \xi)\langle\xi\rangle^{-1}) \tag{2.7}$$

where $q_h(z_1, \dots, z_N)$ are matrix-valued polynomials on $\boldsymbol{C}^N$, depending only on N.

REMARK 1. – We do not know if Proposition 1 can be extended to any hyperbolic matrix $A(t, x, \xi)$ not of Sylvester type. However, the nonsmooth quasi-symmetrizer constructed in [J], [DS] is sufficient to handle analytic solutions for nonlinear systems with constant coefficients (but is not useful to get Theorem 1).

The core of the proof of Proposition 1 is the following algebraic lemma.

LEMMA. – *Let N be an integer greater than* 1. *There exists a $N \times N$ matrix-valued polynomial in N complex variables, $P_N(z_1, \dots, z_N)$ such that*

(i) $\det P_N = (-1)^{N-1}$;

(ii) *for any $N \times N$ Sylvester matrix B, denoting with $(\lambda_1, \dots, \lambda_N)$ its (repeated) eigenvalues in any order, the matrix $P_N(\lambda) = P_N(\lambda_1, \dots, \lambda_N)$ triangulates B, more precisely,*

$$P_N(\lambda)\, B P_N(\lambda)^{-1} = \operatorname{diag}[\lambda_1, \dots, \lambda_N] + K \tag{2.8}$$

with

$$K = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ & 0 & -1 & & \\ & & 0 & & \\ & & & \ddots & -1 \\ & & & & 0 \end{pmatrix}. \tag{2.9}$$

We remark that, by (i), also the inverse P^{-1} is polynomial in $z_1, \dots, z_N$.

PROOF OF THE LEMMA. – The first part of the proof is inspired by the paper of E. Jannelli [J2]. Let

$$\pi_h(z_1, \dots, z_N) = \sum_{\substack{|\alpha| = h \\ \alpha_i \leq 1}} z_1^{\alpha_1} \dots z_N^{\alpha_N}, \qquad h = 1, \dots, N,$$

be the elementary symmetric polynomials, and $e_j = (0, \dots, 0, 1, 0, \dots, 0)$, $j = 1, \dots, n$, the canonical row vectors of $\boldsymbol{C}^N$; it is convenient to set $\pi_0 = 1$. Then, if

B is given by (2.1) and $\lambda_1, \ldots, \lambda_N$ are its (repeated) eigenvalues, we have, setting $b_{N+1} = -1$,

$$b_h = (-1)^{N-h} \pi_{N+1-h}(\lambda_1, \ldots, \lambda_N), \qquad 1 \leqslant h \leqslant N+1 .$$

This shows in particular that there is a unique Sylvester matrix B with given eigevalues. On the other hand, if τ is any one of such eigenvalues the row vector

$$w = \sum_{j=1}^{N} \sum_{h=0}^{N-j} \tau^h b_{j+h+1} e_j$$

is a left eigenvector of B for λ, i.e., $wB = \lambda w$, as it can be easily checked. Thus, the vector

$$v(z) \equiv v(z_1, \ldots, z_N) = \sum_{j=1}^{N} \sum_{h=0}^{N-j} z_N^h (-1)^{N-(j+h+1)} \pi_{N-(j+h)}(z_1, \ldots, z_N)\, e_j , \tag{2.10}$$

which is a polynomial in $z \in \boldsymbol{C}^N$, has the following property: if B is the (unique) Sylvester matrix with repeated eigenvalues $z_1, \ldots, z_N$, then $v(z)$ is a left eigenvector of B with eigenvalue z_N.

We now argue by induction on N. The conclusion of the Lemma is trivial in the case $N = 1$, where we can take simply $P_1 \equiv 1$. Let $P_{N-1}(z_1, \ldots, z_{N-1})$ be the $(N-1) \times (N-1)$ matrix given by the Lemma at step $N-1$, and let

$$\widetilde{P}_{N-1}(z_1, \ldots, z_{N-1}) = \begin{bmatrix} P_{N-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

We define now the matrix-valued polynomial

$$S(z_1, \ldots, z_N) = \mathrm{row}\,[e_1, \ldots, e_{N-1}, v(z_1, \ldots, z_N)],$$

with $v(z)$ given by (2.10); we claim that the matrix

$$P_N(z_1, \ldots, z_N) = \widetilde{P}_{N-1}(z_1, \ldots, z_{N-1})\, S(z_1, \ldots, z_N)$$

is the required triangulator for B. To prove this, we first remark that $\det S = -1$, since the last entry of the vector v is $v_N = -1$ (see (2.10)). Moreover we have $S^{-1} = S$, indeed, by the definition of S we have

$$e_i S = e_i \qquad \text{for } i = 1, \ldots, N-1 ,$$

$$e_N S = v$$

and hence

$$e_i S^2 = e_i \qquad \text{for } i \leqslant N-1 , \qquad e_N S^2 = vS = \sum_{j=1}^{N-1} v_j e_j + v_N v = e_N .$$

Now, if B has eigenvalues $z_1, \ldots, z_N$, then $S(z)\,BS(z) \equiv S(z)\,BS(z)^{-1}$ has the form

$$S(z)\,BS(z) = \begin{pmatrix} 0 & 1 & & & & 0 \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \vdots \\ & & & & 1 & 0 \\ v_1(z) & & \cdots & & v_{N-1}(z) & v_N(z) \\ 0 & & \cdots & & 0 & z_N \end{pmatrix} = \begin{pmatrix} & & 0 \\ & B'(z) & \\ & & -1 \\ 0 & 0 & z_N \end{pmatrix}.$$

Indeed, recalling that $e_i B = e_{i+1}$ for $i \leqslant N-1$, and $v(z)B = \lambda_N v(z)$, we have

$$e_i S(z)\,BS(z) = e_i BS(z) = e_{i+1} S(z) = e_{i+1}\,, \qquad i = 1, \ldots, N-2\,,$$

$$e_{N-1} S(z)\,BS(z) = e_{N-1} BS(z) = e_N S(z) = v(z)\,,$$

$$e_N S(z)\,BS(z) = Bv(z)\,S(z) = z_N v(z)\,S(z) = z_N e_N\,.$$

But the $(N-1)\times(N-1)$ matrix $B'(z)$ is of Sylvester type and its repeated eigenvalues are exactly $z_1, \ldots, z_{N-1}$ (since $S(z)\,BS(z)^{-1}$ has the same eigenvalues as B); thus $P_{N-1}(z_1, \ldots, z_{N-1})$ triangulates $B'(z)$ in the sense of (2.8). This implies that the $N \times N$ matrix $\widetilde{P}_{N-1}(z_1, \ldots, z_{N-1})$ triangulates $S(z)\,BS(z)$ in the sense of (2.8), and hence $\widetilde{P}_{N-1}(z_1, \ldots, z_{N-1})\,S(z)$ triangulates B.

Proof of Proposition 1. – The matrix-valued polynomial $P(z) \equiv P_N(z_1, \ldots, z_N)$ constructed in the Lemma, when applied to the eigenvalues of a given Sylvester matrix B, gives a matrix $P(z)$ which is not, in general, a regular function of the entries of B. To overcome this difficulty, we apply a procedure of symmetrization, based on the fact that any symmetric, polynomial function of the eigenvalues of B can be also written as a polynomial function of the entries.

We consider the diagonal matrix

$$H_\varepsilon = \begin{pmatrix} \varepsilon^{1-N} & & & 0 \\ & \varepsilon^{2-N} & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

and we define

$$Q_\varepsilon(z) = P(z)^* H_\varepsilon^{-2} P(z)\,.$$

From (2.8)-(2.9), we have for any $z = (z_1, \ldots, z_N)$ and for the Sylvester matrix B

with repeated eigenvalues $z_1, \ldots, z_N$ the following identity

$$H_\varepsilon^{-1} P(z)\, BP(z)^{-1} H_\varepsilon = D(z) + \varepsilon K, \qquad \text{where } D(z) = \operatorname{diag}[z_1, \ldots, z_N],$$

since $H_\varepsilon^{-1} K H_\varepsilon = \varepsilon K$, and hence also

$$Q_\varepsilon(z)\, B = P(z)^* H_\varepsilon^{-1} D(z)\, H_\varepsilon^{-1} P(z) + P(z)^* H_\varepsilon^{-1} (\varepsilon K)\, H_\varepsilon^{-1} P(z) = D_\varepsilon(z) + \varepsilon R_\varepsilon(z). \tag{2.11}$$

We remark that D_ε is anti-Hermitian whenever B is hyperbolic, i.e, $z \in i\boldsymbol{R}^N$; moreover,

$$(R_\varepsilon v, v) = (K H_\varepsilon^{-1} P v, H_\varepsilon^{-1} P v) \leqslant |H_\varepsilon^{-1} P v|^2 = (Q_\varepsilon v, v). \tag{2.12}$$

Now, denoted by $\mathcal{S}_N$ the class of permutations over N indices, let us define

$$\overline{Q}_\varepsilon(z) = \sum_{\sigma \in \mathcal{S}_N} Q_\varepsilon(\sigma z)$$

and $\overline{R}_\varepsilon(z)$, $\overline{D}_\varepsilon(z)$ analogously. Since $H_\varepsilon^{-2} = \operatorname{diag}[\varepsilon^{2N-2}, \ldots, \varepsilon^2, 1]$, we see that $\overline{Q}_\varepsilon$ has the form

$$\overline{Q}_\varepsilon(z) = \sum_{h=0}^{N-1} \varepsilon^{2h} q_h(z), \tag{2.13}$$

where $q_h(z)$ are symmetric polynomials in $z_1, \ldots, z_N$; hence the q_h can be expressed as polynomials in the fundamental symmetric polynomials $\pi_1, \ldots, \pi_N$. If B is the Sylvester matrix with repeated eigenvalues $z_1, \ldots, z_N$, this implies that each $q_h(z)$ can be expressed as a polynomial in the entries of B (see (2.1)):

$$q_h(z) = \tilde{q}_h(b_1, \ldots, b_N). \tag{2.14}$$

Moreover, by (2.11) we have

$$\overline{Q}_\varepsilon(z) B = \overline{D}_\varepsilon(z) + \varepsilon \overline{R}_\varepsilon(z), \tag{2.15}$$

with $\overline{D}_\varepsilon$ anti-Hermitian and

$$(\overline{R}_\varepsilon v, v) \leqslant (\overline{Q}_\varepsilon v, v).$$

Finally, since $P(z)$, $P(z)^{-1}$ are polynomials in z, recalling the definition of $\overline{Q}_\varepsilon(z)$ we have easily, for some pair of continuous functions $C_1(r)$, $C_2(r)$,

$$C_1(|z|)\, \varepsilon^{2(N-1)} I \leqslant \overline{Q}_\varepsilon(z) \leqslant C_2(|z|) I. \tag{2.16}$$

Let us now go back to the matrix $A(t, x, \xi)$. Clearly, it is sufficient to prove the Proposition in the special case when A is made of a unique Sylvester block, i.e., $A = \langle \xi \rangle B(t, x, \xi)$ with B a Sylvester matrix as in (2.1), with uniformly bounded entries $b_j(t, x, \xi)$. We then define

$$Q_\varepsilon(t, x, \xi) = \sum_{h=0}^{N-1} \varepsilon^{2h} \tilde{q}_h (b_1(t, x, \xi), \ldots, b_N(t, x, \xi))$$

where $\tilde{q}_h$ are given by (2.14). With this definition, (2.4), (2.5) and (2.7) follow immediately (after some rescaling) from the properties (2.13), (2.15), (2.16) proved above. In order to prove (2.6), we resort to Glaeser's inequality

$$|a'(t)|^2 \leq 2 \|a''(t)\|_{L^\infty(0, T)} a(t),$$

valid for any $a(t) \geq 0$. We apply it to the C^2 function $a(t) = (Q_\varepsilon(t, x, \xi) v, v)$. Thanks to the explicit expression (2.7), we see that Q_ε'' is bounded independently of ε, thus using (2.4) we obtain (2.6). This concludes the proof.

3. – The regularity result for a block Sylvester system.

In this section, we investigate the special case of Theorem 1 when the operator $A(t, D)$ is of block Sylvester type. More generally, we consider the pseudo-differential system

$$\partial_t u + A(t, D) u = g(t, x, K(D) u), \tag{3.1}$$

where

(3.2) $A(t, \xi)$ is a N-block Sylvester matrix, homogeneous in ξ of order 1,

$$A(t, \xi) \in C^1([0, T], S^1(\boldsymbol{R}^n)), \tag{3.3}$$

(3.4) $K(\xi)$ is a $m \times N$ matrix-valued symbol of order 0,

and $g(t, x, v): [0, T] \times \boldsymbol{R}^n \times \boldsymbol{C}^m \to \boldsymbol{C}^N$ is continuous in t, uniformly analytic in x and entire analytic in v, so that there is some function $\Phi_\varepsilon(s)$ for which

$$|D_x^\alpha D_v^\beta g(t, x, v)|^2 \leq \Phi_\varepsilon(|v|) \Lambda^{|\alpha|} \varepsilon^{|\beta|} \alpha! \beta! \quad \text{for all } \varepsilon > 0. \tag{3.5}$$

We have then:

PROPOSITION 2. – *Let $u \in C^1([0, T]; \gamma^s_{L^2}(\boldsymbol{R}^n))$, with $1 \leq s < N/(N-1)$, be a solution of* (3.1) *such that $u(0, x) \in \mathcal{A}_{L^2}(\boldsymbol{R}^n)$, and assume that* (3.2)-(3.5) *are satisfied. Then $u \in C^1([0, T]; \mathcal{A}_{L^2}(\boldsymbol{R}^n))$.*

PROOF. – First of all, we fix three positive constants L_0, ϱ_0, M_0 in such a way that

$$\|K(\xi)\| \leq L_0. \tag{3.6}$$

$$(3.7) \quad \int_{\boldsymbol{R}^n} e^{\varrho_0 \langle \xi \rangle^{1/s}} |\widehat{u}(t, \xi)| \langle \xi \rangle^2 d\xi \leq M_0 , \qquad \|u(t, \cdot)\|_{L^\infty(\boldsymbol{R}^n)} \leq M_0 , \qquad \text{on } [0, T],$$

We note that (3.7) can be assumed withot loss of generality by replacing s with some s' such that $s < s' < N/(N-1)$.

In the course of the proof we shall use two kinds of Gevrey energies. To introduce them, we apply Proposition 1 and we take a quasi-symmetrizer Q_ε, of $A(t, \xi)$. Then, we choose

$$\varepsilon = \langle \xi \rangle^{-1/N}$$

and we define the matrix

$$(3.8) \qquad Q(t, \xi) = Q_\varepsilon(t, \xi) \,|_{\varepsilon = \langle \xi \rangle^{-1/N}} ,$$

which satisfies, for some constant C_0,

$$(3.9) \qquad \langle \xi \rangle^{-2(1-1/N)} I \leq Q(t, \xi) \leq I ,$$

$$(3.10) \qquad A(t, \xi)\, Q(t, \xi) + Q(t, \xi)\, A^*(t, \xi) \leq C_0 \langle \xi \rangle^{1-1/N} Q(t, \xi) ,$$

$$(3.11) \qquad |(Q'(t, \xi)\, v, v)| \leq C_0 \langle \xi \rangle^{1-1/N} (Q(t, \xi)\, v, v) .$$

Finally, we fix the *radius function*

$$(3.12) \qquad \varrho(t) = \varrho_0 - \frac{\varrho_0}{2T} t ,$$

which is positive on $[0, T]$, and we define for any vector function $w(x)$ with Fourier transform $\widehat{w}(\xi)$ ($w \in \boldsymbol{C}^N$ in (3.13)), the s-Gevrey energies

$$(3.13) \qquad \mathcal{E}(t, w) = \int e^{\varrho(t) \langle \xi \rangle^{1/s}} (Q(t, \xi)\, \widehat{w}, \widehat{w})^{1/2} d\xi ,$$

$$(3.14) \qquad \tilde{\mathcal{E}}(t, w) = \int e^{\varrho(t) \langle \xi \rangle^{1/s}} |\widehat{w}| d\xi ,$$

the integrals being extended over all $\boldsymbol{R}^n$. If $w = w(t, x)$ depends also on t, we denote by $\widehat{w}$ the Fourier transform with respect to x and we write

$$\mathcal{E}(t, w) = \mathcal{E}(t, w(t, \cdot)) , \qquad \tilde{\mathcal{E}}(t, w) = \tilde{\mathcal{E}}(t, w(t, \cdot)) .$$

We divide the proof into several steps.

A) Linear Gevrey estimate.

Let $w \in C^1([0, T]; \gamma^s_{L^2}(\boldsymbol{R}^n))$ be an arbitrary solution to the linear system

$$(3.15) \qquad \partial_t w + A(t, D)\, w = B(t, x)\, K(D)\, w + f(t, x) ,$$

where $A(t, \xi)$, $K(\xi)$ satisfy (3.2)-(3.4), and $B(t, x)$ is $N \times m$ matrix belonging to the space $C([0, T]; \gamma^s_{L^\infty}(\boldsymbol{R}^n))$. Let us fix a constant M such that

$$\tilde{\mathcal{E}}(t, B) \leqslant M \quad \text{on} \quad [0, T]. \tag{3.16}$$

If $Q = Q(t, \xi)$ is the matrix given by (3.8), we have by (3.9)-(3.11)

$$\frac{d}{dt}[(Q\widehat{w}, \widehat{w})^{1/2}] \leqslant C_0 \langle \xi \rangle^{1-1/N} (Q\widehat{w}, \widehat{w})^{1/2} + |(BK(D)\, w)\widehat{\ }| + |\hat{f}|,$$

thus, by differentiating $\mathcal{E}(t, w)$ with respect to t, we find

$$\mathcal{E}'(t, w) \leqslant$$

$$\int e^{\varrho(t)\langle\xi\rangle^{1/s}} (Q(t, \xi)\, \widehat{w}, \widehat{w})^{1/2} [\langle \xi \rangle^{1/s} \varrho' + C_0 \langle \xi \rangle^{1-1/N}]\, d\xi + \tilde{\mathcal{E}}(t, BK(D)\, w) + \tilde{\mathcal{E}}(t, f).$$

Now for any pair of scalar functions $\phi(x)$, $\psi(x)$, we have the estimate

$$\tilde{\mathcal{E}}(t, \phi \cdot \psi) = \int e^{\varrho(t)\langle\xi\rangle^{1/s}} |\widehat{\phi} * \widehat{\psi}|\, d\xi \leqslant \tilde{\mathcal{E}}(t, \phi)\, \tilde{\mathcal{E}}(t, \psi) \tag{3.17}$$

since $\langle \xi \rangle^{1/s} \leqslant \langle \xi \rangle - \eta^{1/s} + \langle \eta \rangle^{1/s}$, and this implies, by (3.4), (3.6), (3.16) and (3.9),

$$\tilde{\mathcal{E}}(t, BK(D)\, w) \leqslant ML_0 \tilde{\mathcal{E}}(t, w) \leqslant ML_0 \int e^{\varrho(t)\langle\xi\rangle^{1/s}} (Q\widehat{w}, \widehat{w})^{1/2} \langle \xi \rangle^{1-1/N}\, d\xi\, .$$

Thus

$$\mathcal{E}'(t, w) \leqslant \int e^{\varrho(t)\langle\xi\rangle^{1/s}} (Q(t, \xi)\, \widehat{w}, \widehat{w})^{1/2} [\varrho' + (C_0 + L_0 M) \langle \xi \rangle^{1-1/N-1/s}] \langle \xi \rangle^{1/s}\, d\xi + \tilde{\mathcal{E}}(t, f).$$

But $1 - 1/N - 1/s < 0$, hence we can find a constant $R = R(T, C_0, L_0, \varrho_0, M)$ so large that

$$\varrho'(t) + (C_0 + L_0 M) \langle \xi \rangle^{1-1/N-1/s} = -\frac{\varrho}{2T} + (C_0 + L_0 M) \langle \xi \rangle^{1-1/N-1/s} \leqslant 0 \quad \text{if } \langle \xi \rangle \geqslant R\,,$$

which gives

$$\mathcal{E}'(t, w) \leqslant \int_{|\xi| \leqslant R} e^{\varrho(t)\langle\xi\rangle^{1/s}} (Q(t, \xi)\, \widehat{w}, \widehat{w})^{1/2} (C_0 + L_0 M) \langle \xi \rangle^{1-1/N}\, d\xi + \tilde{\mathcal{E}}(t, f).$$

In conclusion we obtain, for any solution w of (3.15), the a priori estimate

$$\mathcal{E}'(t, w) \leqslant C_1\, \mathcal{E}(t, w) + \tilde{\mathcal{E}}(t, f). \tag{3.18}$$

where C_1 depends on T, C_0, L_0, ϱ_0 and on M, defined in (3.16).

We remark that (3.18) implies the global well-posedness in $\gamma^s_{L^2}(\boldsymbol{R}^n)$,

$s < N/(N-1)$, to the Cauchy problem for the linear system (3.15). Moreover, by standard arguments one can prove the local well-posedness in the same space for the nonlinear system (3.1).

B) Uniqueness in the Gevrey class for the nonlinear problem.

Let $u, v \in C^1([0, T]; \gamma^s_{L^2}(\boldsymbol{R}^n))$ be two solutions of (3.1), $s < N/(N-1)$, and assume that $u(0, x) = v(0, x)$. Then $w = u - v$ satisfies a linear system of the form (3.15) with $f = 0$, thus using the linear estimate (3.18) we obtain that $u = v$ on $[0, T] \times \boldsymbol{R}^n$.

C) Gevrey estimates of the nonlinear term.

Let us go back to the nonlinear system (3.1), where $u(t, x)$ is a given solution satisfying (3.7). In the following we shall assume, for sake of simplicity, that the nonlinear term $g(t, x, v) = g(v)$ is independent of (t, x), so that (3.1) becomes

$$\partial_t u + A(t, D)\, u = g(K(D)\, u)\,, \tag{3.19}$$

where $g\colon \boldsymbol{C}^m \to \boldsymbol{C}^N$ is an entire function. The general case presents only some additional technical complications. By applying the operator $D^\alpha = D^\alpha_x$, with $|\alpha| > 0$, to both members of (3.19), we obtain

$$(\partial_t + A(t, D)) D^\alpha u = B(t, x)\, K(D)\, D^\alpha u + f_\alpha(t, x)\,, \tag{3.20}$$

where B is the $N \times m$ matrix given by

$$B(t, x) = Dg \circ (K(D) u) \tag{3.21}$$

and

$$f_\alpha(t, x) = D^\alpha (g \circ (K(D)\, u)) - B(t, x)\, K(D)\, D^\alpha u\,. \tag{3.22}$$

We remark that $Dg(v)$ is entire analytic, so it admits a Taylor expansion $Dg(v) = \sum G_\beta v^\beta$; thus using (3.17) and recalling (3.4), (3.6), we find

$$\tilde{\mathcal{E}}(t, B) \leqslant \Phi(\tilde{\mathcal{E}}(t, K(D)\, u)) \leqslant \Phi(L_0 \tilde{\mathcal{E}}(t, u)) \leqslant \Phi(L_0 M_0) \qquad \text{on } [0, T], \tag{3.23}$$

for some function $\Phi(s)$ depending on g.

On the other hand, for any function $v = (v_1(t, x), \ldots, v_m(t, x))$ and $\alpha \in \boldsymbol{N}^n$, $\alpha > 0$, the chain rule gives

$$D^\alpha (g \circ v) = \alpha! \sum_{1 \leqslant |\beta| \leqslant |\alpha|} \frac{D^\beta g \circ v}{\beta!} \sum_{\substack{\alpha(1) + \ldots + \alpha(|\beta|) = \alpha \\ |\alpha(i)| > 0}} \frac{D^{\alpha(1)} v_1 \ldots D^{\alpha(|\beta|)} v_m}{\alpha(1)! \ldots \alpha(|\beta|)!}\,,$$

where $\beta \in \boldsymbol{N}^m$, $\alpha(i) \in \boldsymbol{N}^n$. Hence, if we introduce the *higher order energies*

$$\mathcal{E}_j(t, w) = \sum_{|\alpha| = j} \frac{j!}{\alpha!} \mathcal{E}(t, D^\alpha w), \qquad \tilde{\mathcal{E}}_j(t, w) = \sum_{|\alpha| = j} \frac{j!}{\alpha!} \tilde{\mathcal{E}}(t, D^\alpha w), \qquad j = 1, 2, \ldots,$$

we find, using (3.17), (3.5), (3.7), that for all $\varepsilon > 0$ there is some constant C_ε depending on M_0 for which

$$\tilde{\mathcal{E}}_j(t, g \circ v) \leqslant C_\varepsilon j! \sum_{\nu=1}^{j} \varepsilon^\nu \sum_{\substack{h_1 + \ldots + h_\nu = j \\ h_i \geqslant 1}} \frac{\tilde{\mathcal{E}}_{h_1}(t, v) \ldots \tilde{\mathcal{E}}_{h_\nu}(t, v)}{h_1! \ldots h_\nu!} .$$

In a similar way we can estimate the higher order energies of f_α. Indeed from (3.22) we derive, replacing v with $K(D)\, u$ and isolating the terms corresponding to $\nu = 1$,

$$\sum_{|\alpha| = j} \frac{1}{\alpha!} \tilde{\mathcal{E}}(t, f_\alpha) \leqslant C_\varepsilon j! \sum_{\nu=2}^{j} \varepsilon^\nu \sum_{\substack{h_1 + \ldots + h_\nu = j \\ h_i \geqslant 1}} \frac{\tilde{\mathcal{E}}_{h_1}(t, u) \ldots \tilde{\mathcal{E}}_{h_\nu}(t, u)}{h_1! \ldots h_\nu!} \qquad (j \leqslant 2). \tag{3.24}$$

But

$$\tilde{\mathcal{E}}_j(t, u) \leqslant \mathcal{E}_{j+1}(t, u),$$

since

$$|\hat{u}| \leqslant \langle \xi \rangle^{1 - 1/N} (Q\hat{u}, \hat{u})^{1/2} \leqslant \langle \xi \rangle (Q\hat{u}, \hat{u})^{1/2}.$$

Hence (3.24) implies, for $j \geqslant 2$,

$$\sum_{|\alpha| = j} \frac{1}{\alpha!} \tilde{\mathcal{E}}(t, f_\alpha) \leqslant C_\varepsilon j! \sum_{\nu=2}^{j} \varepsilon^\nu \sum_{\substack{h_1 + \ldots + h_\nu = j \\ h_i \geqslant 1}} \frac{\mathcal{E}_{h_1+1}(t, u) \ldots \mathcal{E}_{h_\nu+1}(t, u)}{h_1! \ldots h_\nu!} . \tag{3.25}$$

Summing up, if we apply the a priori estimate (3.18) to the system (3.20), and use (3.23) and (3.25), we obtain that our solution satisfies, for all $\varepsilon > 0$ and $j \geqslant 2$,

$$\mathcal{E}_j'(t, u) \leqslant C_1 \mathcal{E}_j(t, u) + C_\varepsilon j! \sum_{\nu=2}^{j} \varepsilon^\nu \sum_{\substack{h_1 + \ldots + h_\nu = j \\ h_i \geqslant 1}} \frac{\mathcal{E}_{h_1+1}(t, u) \ldots \mathcal{E}_{h_\nu+1}(t, u)}{h_1! \ldots h_\nu!} , \tag{3.26}$$

with constants C_1, C_ε depending on T, C_0, L_0, ϱ_0, M_0.

D) Superenergy of Gevrey type.

In the following, to denote the s-energy of order j of the given solution

$u(t, x)$, we shall write simply

$$\mathcal{E}_j = \mathcal{E}_j(t) = \mathcal{E}_j(t, u).$$

Let us fix some $r_0 > 0$, strictly smaller than the radius of analyticity of $u(0, x)$, so that we have in particular

$$\sum_{j=2}^{\infty} \frac{r_0^{j-2}}{(j-2)!} \mathcal{E}_j(0) = M_1 < \infty . \tag{3.27}$$

Then we define, for any $\sigma \in]1, s[$, the (σ, s)-Gevrey energies

$$\mathcal{F}_\sigma(t) = \sum_{j=2}^{\infty} \frac{r^{j-2}}{(j-2)!^\sigma} \mathcal{E}_j , \qquad \mathcal{F}_\sigma^1(t) = \sum_{j=2}^{\infty} \frac{r^{j-2}}{(j-2)!^\sigma (j-1)^{\sigma-1}} \mathcal{E}_{j+1} ,$$

where

$$r = r(t) = r_0 e^{-\mu t} ,$$

with $\mu > 0$ to be defined later. We remark that, for all $\sigma > 1$

$$\mathcal{F}_\sigma(t) \leqslant \mathcal{F}_\sigma^1(t) , \qquad \mathcal{F}_\sigma^1(0) \leqslant M_1 .$$

Finally we assume that the given solution u satisfies

$$\mathcal{F}_\sigma^1(t) < \infty \qquad \text{on } [0, \tau[. \tag{3.28}$$

As it will be precised at the step E), condition (3.28) is always fulfilled, for some $\tau \leqslant T$, by the Cauchy-Kovalewski theorem. By differentiating $\mathcal{F}_\sigma$ and using (3.26) we obtain

$$\mathcal{F}_\sigma' \leqslant r' \mathcal{F}_\sigma^1 + C_1 \mathcal{F}_\sigma + \mathcal{G}_{\sigma, \varepsilon} \tag{3.29}$$

for all $\varepsilon > 0$, where

$$\mathcal{G}_{\sigma, \varepsilon} = C_\varepsilon \sum_{j=2}^{\infty} \frac{j(j-1) r^{j-2}}{(j-2)!^{\sigma-1}} \sum_{\nu=2}^{j} \varepsilon^\nu \sum_{\substack{h_1 + \ldots + h_\nu = j \\ h_i \geqslant 1}} \frac{\mathcal{E}_{h_1+1} \ldots \mathcal{E}_{h_\nu+1}}{h_1! \ldots h_\nu!} .$$

Now, grouping together the terms with $2 \leqslant \nu \leqslant j-1$, we write

$$\mathcal{G}_{\sigma, \varepsilon} = \mathcal{G}_{\sigma, \varepsilon}^{I} + \mathcal{G}_{\sigma, \varepsilon}^{II} .$$

The last term, where $\nu = j$, can be easily estimated thanks to the main assumption (3.7). Indeed, $\nu = j$ implies $h_i = 1$ for all i, thus we have, for $\varepsilon \leqslant 1/(2r_0 M_0)$,

$$\mathcal{G}^{II}_{\sigma,\,\varepsilon} = C_\varepsilon \sum_{j=2}^{\infty} \frac{j(j-1)\,r^{j-2}}{(j-2)!^{\sigma-1}}\,\varepsilon^j\,\mathcal{E}_2^{j+1} \leqslant C_\varepsilon \sum_{j=2}^{\infty} j^2 r_0^{j-2}\,\varepsilon^j M_0^{j+1} \leqslant C_2\,,$$

where C_2 is a constant depending only on M_0, r_0.

As to $\mathcal{G}^{I}_{\sigma,\,\varepsilon}$, we observe that it contains only terms with $\max\{h_i\} \geqslant 2$. Thus, after some reordering, we find

$$\mathcal{G}^{I}_{\sigma,\,\varepsilon} \leqslant C_\varepsilon \sum_{j=2}^{\infty} \frac{j(j-1)\,r^{j-2}}{(j-2)!^{\sigma-1}} \sum_{\nu=2}^{j-1} \nu\varepsilon^\nu \sum_{\substack{h_1+\ldots+h_\nu=j\\ h_1 \geqslant h_i \geqslant 1,\ h_1\geqslant 2}} \frac{\mathcal{E}_{h_1+1}\cdots\mathcal{E}_{h_\nu+1}}{h_1!\ldots h_\nu!}\,.$$

Introducing the notations

$$\eta_h = \frac{r^{h-2}}{(h-2)!^\sigma}\,\mathcal{E}_h, \qquad \eta^1_h = \frac{r^{h-2}}{(h-2)!^\sigma (h-1)^{\sigma-1}}\,\mathcal{E}_{h+1}\,, \qquad h \geqslant 2\,,$$

and exchanging the order of summation between j and ν, the last inequality yields

$$\mathcal{G}^{I}_{\sigma,\,\varepsilon} \leqslant C_\varepsilon \sum_{\nu=2}^{j-1} \nu\varepsilon^\nu \sum_{j=\nu}^{\infty} \sum_{\substack{h_1+\ldots+h_\nu=j\\ |h_1|\geqslant 2,\ h_i\geqslant 1}} H(j,h)\,\eta^1_{h_1}\eta_{h_2+1}\cdots\eta_{h_\nu+1}\,r^{p(j,h)}, \tag{3.30}$$

where

$$H(j,h) = \left[\frac{h_1-1}{j-2}\right]^{\sigma-1}\cdot\frac{j(j-1)}{h_1(h_1-1)h_2\ldots h_\nu}\cdot\left[\frac{(h_1-2)!(h_2-1)!\ldots(h_\nu-1)!}{(j-3)!}\right]^{\sigma-1}$$

and

$$p(j,h) = (j-2) - [(h_1-2)+(h_2-1)+\ldots+(h_\nu-1)]\,.$$

Now, $h_1 \equiv \max\{h_1,\ldots,h_\nu\} \geqslant j/\nu$ thus for $j > \nu$ we have $j(j-1)/[h_1(h_1-1)] \leqslant \nu^3$. Hence, using that $h_1+\ldots+h_\nu = j$ and $\nu \geqslant 2$, we find that

$$H(j,h) \leqslant \nu^3, \qquad p(j,h) \geqslant 1\,.$$

As a consequence, (3.30) gives

$$\mathcal{G}^{I}_{\sigma,\,\varepsilon} \leqslant C_\varepsilon r \sum_{\nu=2}^{\infty} \nu^4\varepsilon^\nu \sum_{\substack{h_1+\ldots+h_\nu=j\\ |h_1|\geqslant 2,\ h_i\geqslant 1}} \eta^1_{h_1}\eta_{h_2+1}\cdots\eta_{h_\nu+1} = C_\varepsilon r \sum_{\nu=2}^{\infty} \nu^4\varepsilon^\nu\,\mathcal{F}^1_\sigma(\mathcal{F}_\sigma)^{\nu-1}\,,$$

since

$$\mathcal{F}_\sigma^1 = \sum_{j=2}^{\infty} \eta_j^1 , \qquad \mathcal{F}_\sigma = \sum_{j=2}^{\infty} \eta_j .$$

If we introduce in (3.29) the above estimates of $\mathcal{G}^I_{\varepsilon,\sigma}$, $\mathcal{G}^{II}_{\varepsilon,\sigma}$, we find

$$\mathcal{F}'_\sigma \leq \{r' + \psi_\varepsilon(\mathcal{F}_\sigma)\, r\}\, \mathcal{F}^1_\sigma + C_1 \mathcal{F}_\sigma + C_2 , \tag{3.31}$$

for $\varepsilon \leq 1/(2 r_0 M_0)$, where

$$C_1 = C_1(T, \varrho_0, C_0, L_0, M_0), \qquad C_2 = C_2(M_0, r_0), \qquad \psi_\varepsilon(s) = C_\varepsilon \sum_{\nu=2}^{\infty} \nu^4 \varepsilon^\nu s^{\nu-1} .$$

Now we split (3.31) into the couple of inequalities

$$\mathcal{F}'_\sigma \leq C_1 \mathcal{F}_\sigma + C_2 , \qquad r' + \psi_\varepsilon(\mathcal{F}_\sigma)\, r \leq 0 .$$

The first inequality gives

$$\mathcal{F}_\sigma(t) \leq \left(\mathcal{F}_\sigma(0) + \frac{C_2}{C_1} \right) e^{C_1 T} \leq C_3 , \qquad t \in [0, \tau[,$$

and hence

$$\psi_\varepsilon(\mathcal{F}_\sigma(t)) \leq \psi_\varepsilon(C_3) \leq C_4 \qquad \text{if } \varepsilon \leq \min\left\{ \frac{1}{2C_3}, \frac{1}{2 r_0 M_0} \right\},$$

where C_3, C_4 are constants depending on the solution u and on T, ϱ_0, C_0, L_0, M_0, r_0, M_1.

Thus, if we choose $r(t) = r_0 e^{C_4 t}$, we obtain for our solution the estimate

$$\mathcal{F}_\sigma(t, u) \leq C_3 , \qquad t \in [0, \tau[, \tag{3.32}$$

for all $1 < \sigma < s$, with C_3 independent of σ.

E) Conclusion of the proof.

The Cauchy-Kowalewski theorem and the uniqueness proved in Step B), ensure that the given solution $u(t, x)$ of (3.1) is in fact analytic in some interval $[0, \tau[$, hence Gevrey of any order $\sigma > 1$ with arbitrarily large radius. Thus (3.27) is fulfilled. Hence we are in the position to apply to $u(t, x)$ the a-priori estimate (3.32), and we find the uniform estimate $\mathcal{F}_\sigma(\tau, u) \leq C_3$, for all $\sigma < 1$. This implies

$$|D^\alpha u(\tau, x)| \leq C \Lambda^{|\alpha|} |\alpha|!^\sigma$$

for some constants C, Λ independent of σ. Letting $\sigma \to 1$, we obtain that $u(t, x)$ is analytic also at $t = \tau$. Applying again the Cauchy-Kovalewski theorem, we conclude that $\tau = T$.

4. – Regularity for general systems.

To conclude the proof of Theorem 1, we shall now show that the study of regularity for a generic first order system of the form

$$\partial_t u + A(t, D)\, u = f(t, x, u) \tag{4.1}$$

can be reduced to the study of a block Sylvester system as in Section 3.

Let $L(t, \tau, \xi)$ be the cofactor matrix of $\tau I + A(t, \xi)$, so that $L(t, \tau, \xi)(\tau I + A(t, \xi)) = \delta(t, \tau, \xi) I$ where

$$\delta(t, \tau, \xi) = \tau^N + \sum_{h=0}^{N-1} b_{N-h}(t, \xi)\, \tau^h$$

is the determinant of $\tau I + A$, and b_h are homogeneous polynomials of order h in ξ. We remark that $L = L(t, \partial_t, D)$ is an $N \times N$ matrix of homogeneous differential operators of order $N-1$, while $\delta(t, \partial_t, D)$ is scalar homogeneous of order N. By applying L to (4.1) we obtain $\delta(t, \partial_t, D)\, u = L[f(u)] + l.o.t.$, i.e.,

$$\partial_t^N u + \sum_{j=0}^{N-1} b_{N-h}(t, D)\, \partial_t^h u = g(t, x, \ldots, \partial_t^j D^\alpha u, \ldots), \qquad j + |\alpha| \le N-1 \tag{4.2}$$

for a suitable function $g(t, x, p)$, analytic in x, p and C^1 in t.

We define now the N^2 column vector

$$U = [\partial_t^{j-1} \Lambda^{N-j} u]_{j=1, \ldots, N}, \qquad \Lambda = \langle D \rangle = (1-\Delta)^{1/2}, \tag{4.3}$$

which satisfies the first order system

$$\partial_t U + \mathcal{A}(t, D)\, U = G(t, x, \ldots, D^\alpha \Lambda^{-h} u, \ldots), \qquad |\alpha| \le h \le N-1\,. \tag{4.4}$$

We thus obtain a block Sylvester system of dimension N^2; more precisely, $\mathcal{A}$ is made of N identical blocks $\mathcal{A}_0$ of size N, and $\mathcal{A}_0$ is the Sylvester matrix

$$\mathcal{A}_0 = \Lambda \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ & 0 & 1 & & & \\ \vdots & & & \ddots & & \\ & & & & 0 & 1 \\ b_1 \Lambda^{-1} & & & \cdots & & b_N \Lambda^{-N} \end{bmatrix}.$$

We are in the position to apply Proposition 2. Indeed, let $u \in C^N([0, T], \gamma^s_{L^2}(\boldsymbol{R}^n))$ be a solution of (4.1), and assume that $u(0, \cdot) \in \mathcal{A}_{L^2}(\boldsymbol{R}^n)$. Then U given by (4.3) belongs to $C^1([0, T], \gamma^s_{L^2}(\boldsymbol{R}^n))$ and solves the block Sylvester system (4.4), while $U(0, \cdot) \in \mathcal{A}_{L^2}(\boldsymbol{R}^n)$. Now, if $s < N/(N-1)$, by Proposition 2 we conclude that $U \in C^1([0, T], \mathcal{A}_{L^2}(\boldsymbol{R}^n))$, and this implies that $u \in C^N([0, T], \mathcal{A}_{L^2}(\boldsymbol{R}^n))$.

This concludes the proof of Theorem 1.

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