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The C^*-Algebra of a Hilbert Bimodule (*).

Sergio Doplicher - Claudia Pinzari - Rita Zuccante

Sunto. – *Un C^*-modulo hilbertiano destro X su una C^*-algebra $\mathcal{A}$ dotato di uno *-omomorfismo isometrico $\phi: \mathcal{A} \to \mathcal{L}_{\mathcal{A}}(X)$ viene qui considerato come un oggetto $X_{\mathcal{A}}$ della C^*-categoria degli $\mathcal{A}$-moduli Hilbertiani destri. Come in [11], associamo ad esso una C^*-algebra $\mathcal{O}_{X_{\mathcal{A}}}$ contenente X come un «$\mathcal{A}$-bimodulo hilbertiano in $\mathcal{O}_{X_{\mathcal{A}}}$». Se X è pieno e proiettivo finito $\mathcal{O}_{X_{\mathcal{A}}}$ è la C^*-algebra $C^*(X)$, la generalizzazione delle algebre di Cuntz-Krieger introdotta da Pimsner [27] (e in un caso particolare da Katayama [31]) Più in generale, $C^*(X)$ è canonicamente immersa in $\mathcal{O}_{X_{\mathcal{A}}}$ come la C^*-sottoalgebra generata da X. Reciprocamente, se X è pieno $\mathcal{O}_{X_{\mathcal{A}}}$ è canonicamente immersa in $C^*(X)^{**}$. Inoltre, considerando X come un oggetto ${}_{\mathcal{A}}X_{\mathcal{A}}$ della C^*-categoria degli $\mathcal{A}$-bimoduli hilbertiani, associamo ad esso una C^*-sottoalgebra $\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$ di $\mathcal{O}_{X_{\mathcal{A}}}$ che commuta con $\mathcal{A}$, su cui X induce un endomorfismo canonico ϱ. Discutiamo condizioni sotto le quali $\mathcal{A}$ ed $\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$ sono l'uno il commutante relativo dell'altro ed X è precisamente il sottospazio degli operatori di allacciamento in $\mathcal{O}_{X_{\mathcal{A}}}$ tra l'identità e ϱ su $\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$. Discutiamo anche condizioni che implicano la semplicità di $C^*(X)$ o di $\mathcal{O}_{X_{\mathcal{A}}}$; in particolare, se X è proiettivo finito e pieno, $C^*(X)$ è semplice se $\mathcal{A}$ è X-semplice e lo «spettro di Connes» di X è $\mathbb{T}$.*

1. – Introduction.

Let $\mathcal{C} \subset \mathcal{B}$ be an inclusion of C^*-algebras and denote by $\mathcal{A} = \mathcal{C}' \cap \mathcal{B}$ the relative commutant. If ϱ is an endomorphism of $\mathcal{C}$, the subset X_ϱ of $\mathcal{B}$ defined by

$$X_\varrho = \{\psi \in \mathcal{B} \mid \psi C = \varrho(C)\, \psi,\ C \in \mathcal{C}\} \tag{1.1}$$

is a *Hilbert $\mathcal{A}$-bimodule in $\mathcal{B}$*, in the sense that X_ϱ is a closed subspace, stable under left and right multiplication by elements of $\mathcal{A}$, and equipped with an $\mathcal{A}$-valued right $\mathcal{A}$-linear inner product given by

$$\langle \psi, \psi' \rangle_{\mathcal{A}} = \psi^* \psi', \qquad \psi, \psi' \in X_\varrho$$

such that $\|\langle \psi, \psi \rangle_{\mathcal{A}}\| = \|\psi\|^2_{\mathcal{B}}$. We say that ϱ is *inner* in $\mathcal{B}$ if X_ϱ is finite projective as a right $\mathcal{A}$-module and if its left annihilator in $\mathcal{B}$ is zero.

This notion reduces to that of inner endomorphism when, e.g., $\mathcal{C} = \mathcal{B}$ has

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centre $\mathbb{C}I$; if $\mathcal{C} \neq \mathcal{B}$ but $\mathcal{A} = \mathbb{C}I$, X_ϱ is a Hilbert space in $\mathcal{B}$ and ϱ is the restriction to $\mathcal{C}$ of an inner endomorphism of $\mathcal{B}$ [9, 10, 11], i.e. $\varrho(C) = \sum_1^d \psi_i C \psi_i^*$, with $\{\psi_i, i = 1, \dots, d\}$ an orthonormal basis of X_ϱ.

The crossed product of a unital C^*-algebra $\mathcal{C}$ with trivial centre by the outer action of a discrete group [13, 19, 25] or by the action of a compact group dual [10] has the characteristic property that the objects (automorphisms, resp. endomorphisms of $\mathcal{C}$) become inner in the crossed product $\mathcal{B}$, and that $\mathcal{A}' \cap \mathcal{B} = \mathbb{C}I$.

These notions of crossed products might prove too narrow to provide a scheme for an abstract duality theory of quantum groups in the spirit of [11], or for the related problem of describing the superselection structure of low dimensional QFT by a symmetry principle [12, 15]. In the last case, indeed, no-go theorems indicate that the relative commutant of the observable algebra in the field algebra might have to be nontrivial [23, 29].

It is therefore interesting to study more general crossed products $\mathcal{B}$ associated to the pairs $\{\mathcal{C}, \varrho\}$ and conditions ensuring existence and uniqueness, in particular of the C^*-algebra $\mathcal{A}$ appearing as the relative commutant $\mathcal{C}' \cap \mathcal{B}$.

As a preliminary step towards this problem, that we hope to treat elsewhere, we consider in this paper the situation where X is given as a Hilbert C^*-bimodule with coefficients in $\mathcal{A}$ (i.e. X is a right Hilbert $\mathcal{A}$-module with a monomorphism of $\mathcal{A}$ into the C^*-algebra $\mathcal{L}(X)$ of the adjointable module maps, defining the left action [27]).

With X^r, $r = 0, 1, 2, \dots$ the bimodule tensor powers of X (where $X^0 = \mathcal{A}$ by convention) we can consider the following C^*-categories:

– the strict *tensor* C^*-category $\mathcal{T}_X$ with objects X^r, $r \in \mathbb{N}_0$, and with arrows the adjointable right $\mathcal{A}$-module maps commuting with the left action of $\mathcal{A}$;

– the C^*-category $\mathcal{S}_X$ with the same objects and with arrows all the adjointable right $\mathcal{A}$-module maps. This is a strict *semitensor* C^*-category in the sense that on arrows only the tensor product on the right with the identity arrows of the category itself is defined (cf. Section 2).

A general construction associates functorially to each object ϱ in a strict tensor C^*-category a C^*-algebra $\mathcal{O}_\varrho$ [11]. It is easy to verify that this applies without substantial modifications to objects in a strict *semitensor* C^*-category. We can thus associate to the bimodule X viewed as an object of $\mathcal{S}_X$ (to mean this we will write for short $X_\mathcal{A}$) a C^*-algebra $\mathcal{O}_{X_\mathcal{A}}$, where $\mathcal{A}$ is embedded as a C^*-subalgebra and X is embedded as a Hilbert $\mathcal{A}$-bimodule in $\mathcal{O}_{X_\mathcal{A}}$. The C^*-algebra $C^*(X)$ constructed by Pimsner [27] from the bimodule X, generalizing the Cuntz-Krieger algebras, can be identified with the C^*-subalgebra of $\mathcal{O}_{X_\mathcal{A}}$

generated by X, and will coincide with $\mathcal{O}_{X_\mathcal{A}}$ if X is full and finite projective (Section 3).

The C^*-algebra $\mathcal{O}_{{}_\mathcal{A}X_\mathcal{A}}$ associated with ${}_\mathcal{A}X_\mathcal{A}$, i.e. with X viewed as an object of the *tensor* category $\mathcal{T}_X$, is embedded in the relative commutant $\mathcal{A}' \cap \mathcal{O}_{X_\mathcal{A}}$ and coincides with it if further conditions are fulfilled (Proposition 3.4). X induces a canonical endomorphism on $\mathcal{A}' \cap \mathcal{O}_{X_\mathcal{A}}$ which acts on $\mathcal{O}_{{}_\mathcal{A}X_\mathcal{A}}$ tensoring the arrows in (X^r, X^s) with the identity arrow of (X, X) on the left. We give conditions which guarantee that $\mathcal{A}$ is *normal* in $\mathcal{O}_{X_\mathcal{A}}$, i.e. $\mathcal{A} = (\mathcal{A}' \cap \mathcal{O}_{X_\mathcal{A}})' \cap \mathcal{O}_{X_\mathcal{A}}$; in this case X identifies with the $\mathcal{A}$-bimodule in $\mathcal{B} = \mathcal{O}_{X_\mathcal{A}}$ which induces ϱ on $\mathcal{C} = \mathcal{A}' \cap \mathcal{O}_{X_\mathcal{A}}$ in the sense of eq. (1.1).

If X is full, $C^*(X)$ is the universal C^*-algebra containing $\mathcal{A}$ and X as an $\mathcal{A}$-bimodule and generated by X; $\mathcal{O}_{X_\mathcal{A}}$ can be canonically identified with a C^*-subalgebra of $C^*(X)^{**}$ (Theorem 3.3).

While $\mathcal{O}_{X_\mathcal{A}}$ generalizes the Cuntz algebras $\mathcal{O}_n$, $n < \infty$ when X is finite projective, if X is not it rather generalizes the C^*-algebra $\mathcal{O}_H$ discussed in [6].

If X is finite projective and full and $\mathcal{A}$ has no closed two sided proper ideal J such that $X^* J X \subset J$, then $C^*(X)$ is *simple* if the Connes spectrum of the dual action of $\mathbb{Z}$ on the crossed product of $C^*(X)$ with the canonical action of $\mathbb{T}$ is full, i.e. coincides with $\mathbb{T}$. If furthermore there is a tensor power X^s of X containing an isometry which commutes with $\mathcal{A}$, then $\mathcal{O}_{X_\mathcal{A}}$ is also simple. These and slightly more general results are discussed in Section 4 (cf. Theorem 4.7).

2. – Representations of Hilbert bimodules in C^*-algebras.

A strict *semitensor* C^*-category is a C^*-category $\mathcal{T}$ for which the set of objects is a unital semigroup, with identity ι, and such that for any object $\tau \in \mathcal{T}$ there is a *-functor («right tensoring» with the identity 1_τ of (τ, τ))

$$\Phi_\tau : (\varrho, \sigma) \to (\varrho\tau, \sigma\tau) \tag{2.1}$$

such that

$$\Phi_\iota = id, \qquad \Phi_\omega \circ \Phi_\tau = \Phi_{\tau\omega}.$$

Here and in the following (ϱ, σ) denotes the set of arrows from the object ϱ to the object σ in our category.

The product on the set of objects will be referred to as the tensor product. In other words $\Phi: \tau \to \Phi_\tau$ is a unital antihomomorphism from the semigroup of objects of $\mathcal{T}$ to the semigroup $\mathrm{End}(\mathcal{T})$ of *-endofunctors of $\mathcal{T}$. We will consider only cases where Φ_τ is injective, and hence isometric. Any strict tensor C^*-category is obviously semitensor choosing Φ_τ: $T \to T \times 1_\tau$.

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule is a right Hilbert $\mathcal{B}$-module X (with $\mathcal{B}$-valued inner product denoted by $\langle x, y\rangle_{\mathcal{B}}$) endowed with a faithful *-homomorphism $\phi: \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(X)$.

It was shown in [3] that a refinement of an argument by Dixmier on approximate units shows that if X is countably generated as a right Hilbert module then there exist elements $x_1, x_2, \ldots$ of X such that $\sum_{j=1}^{N} \vartheta_{x_j, x_j}$ is an approximate unit for $\mathcal{K}_{\mathcal{B}}(X)$, the C^*-algebra of compact operators on X. In particular every $x \in X$ is the norm limit

$$x = \sum_j \vartheta_{x_j, x_j}(x) = \sum_j x_j \langle x_j, x\rangle_{\mathcal{B}} .$$

The set $\{x_j\}$ will be called a *basis* of X. The use of a basis will be helpful to simplify our formalism, hence throughout this paper we will only consider countably generated Hilbert bimodules. However, most of our results extend to the more general setting.

Let $\mathcal{B}$ be a C^*-subalgebra of a C^*-algebra $\mathcal{M}$. A right Hilbert $\mathcal{B}$-module contained in $\mathcal{M}$ is a norm closed subspace such that

$$X\mathcal{B} \subseteq X , \qquad X^* X \subseteq \mathcal{B}$$

(for any pair of subspaces $X, Y \subset \mathcal{M}$, XY denotes the closed linear subspace generated by operator products xy, $x \in X$, $y \in Y$). If furthermore $\mathcal{A} \subset \mathcal{M}$ is a C^*-subalgebra satisfying

$$\mathcal{A}X \subseteq X , \qquad ax = 0 , \qquad x \in X \Rightarrow a = 0 ,$$

X will be called a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule contained in $\mathcal{M}$.

If X and Y are respectively a right Hibert $\mathcal{B}$-module and a Hilbert $\mathcal{B}$-$\mathcal{C}$-bimodule in $\mathcal{M}$ then XY is a right Hilbert $\mathcal{C}$-module in $\mathcal{M}$ naturally isomorphic to $X \otimes_{\mathcal{B}} Y$.

If X and Y are right Hilbert $\mathcal{B}$-modules in $\mathcal{M}$ then YX^* is a subspace of $\mathcal{M}$ naturally isomorphic to the space $\mathcal{K}_{\mathcal{B}}(X, Y)$ of compact operators from X to Y. In general this identification does not extend to the space $\mathcal{L}_{\mathcal{B}}(X, Y)$ of $\mathcal{B}$-linear adjointable maps. However, $\mathcal{L}_{\mathcal{B}}(X, Y)$ may be recovered as a subspace of $\mathcal{M}^{**}$, the enveloping von Neumann algebra of $\mathcal{M}$. Let $1_X \in \mathcal{M}^{**}$ denote the identity of $\overline{XX^*}^{uw}$, the closure of XX^* in $\mathcal{M}^{**}$ in the ultraweak topology.

PROPOSITION 2.1. – *Let X and Y be right Hilbert $\mathcal{B}$-modules in $\mathcal{M}$. Then setting*

$$(X, Y)_{\mathcal{B}} := \{T \in \mathcal{M}^{**}: T1_X = 1_Y T = T,\ TX \subseteq Y,\ Y^* T \subseteq X^*\}$$

one defines a subspace of $\mathcal{M}^{**}$, in fact contained in $\overline{YX^*}^{uw}$, which identifies naturally with $\mathcal{L}_B(X, Y)$. If X and Y are Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodules in $\mathcal{M}$

then ${}_{\mathcal{A}}(X, Y)_{\mathcal{B}} := \mathcal{A}' \cap (X, Y)_{\mathcal{B}}$ *corresponds in the above identification to the set of elements of* $\mathcal{L}_{\mathcal{B}}(X, Y)$ *that commute with the left* $\mathcal{A}$*-action.*

PROOF. – Any $T \in (X, Y)_{\mathcal{B}}$ defines by multiplication in $\mathcal{M}^{**}$ an operator $\widehat{T}: X \to Y$ with adjoint $\widehat{T}^*$, hence $\widehat{T} \in \mathcal{L}_{\mathcal{B}}(X, Y)$. Since $TXX^* \subseteq YX^*$ we conclude, approximating 1_X ultra strongly with elements of XX^*, that $T \in \overline{YX^*}^{us}$. Furthermore $TX = 0$ implies $T = 0$, and this shows that $T \to \widehat{T}$ is injective. On the other hand this map is clearly isometric from YX^* to $\mathcal{K}_{\mathcal{B}}(X, Y)$. If now $S \in \mathcal{L}_{\mathcal{B}}(X, Y)$ then for any basis $\{x_j, j = 1, 2, \ldots\}$ of X, $S \sum_{j=1}^{N} \vartheta_{x_j, x_j}$ is a norm bounded sequence of compact operators hence it is of the form $\widehat{T}_N$, with $T_N \in XX^*$ norm bounded and strictly convergent. Let $T \in \overline{YX^*}^{uw}$ be a weak limit point. Clearly $T1_X = 1_Y T = T$. Furthermore for all $x \in X$ $T_N x$ is norm convergent, necessarily to Tx, so $TX \subseteq Y$. We also conclude that $S = \widehat{T}$, hence the map $T \to \widehat{T}$ is surjective and the proof is complete. ■

A representation of a C^*-category $\mathcal{T}$ in some $\mathcal{B}(H)$ is a collection of maps $\mathcal{F}_{\varrho, \sigma}: (\varrho, \sigma) \to \mathcal{B}(H)$, ϱ, $\sigma \in \mathcal{T}$ such that for any pair of arrows $T \in (\varrho, \sigma)$, $S \in (\sigma, \tau)$,

$$\mathcal{F}_{\varrho, \sigma}(T)^* = \mathcal{F}_{\sigma, \varrho}(T^*), \qquad \mathcal{F}_{\varrho, \tau}(ST) = \mathcal{F}_{\sigma, \tau}(S)\mathcal{F}_{\varrho, \sigma}(T).$$

Let $\mathcal{H}_{\mathcal{B}}$ be the C^*-category of right Hilbert $\mathcal{B}$-bimodules: If X and Y are objects of $\mathcal{H}_{\mathcal{B}}$ the set of arrows from X to Y is $\mathcal{L}_{\mathcal{B}}(X, Y)$. Let $\mathcal{T} \subseteq \mathcal{H}_{\mathcal{B}}$ be a full subcategory. Then the previous Proposition shows that if the objects of $\mathcal{T}$ embed in $\mathcal{M}$ as right Hilbert $\mathcal{B}$-modules then there is a representation of $\mathcal{T}$ in the bounded linear operators on the Hilbert space of the universal representation of $\mathcal{M}$.

Note that in place of the universal representation we may consider any faithful representation of $\mathcal{M}$ on some Hilbert space H. Indeed, the subspace $(X, Y)_{\mathcal{B}} := \{T \in \mathcal{B}(H): T1_X = 1_Y T = T, TX \subseteq Y, Y^* T \subseteq X^*\}$ lies in $\overline{YX^*}^{uw}$ and again identifies naturally with $\mathcal{L}_{\mathcal{B}}(X, Y)$ (1_X is as before the identity of $\overline{XX^*}^{uw} \subseteq \mathcal{B}(H)$). It follows that there is still an obvious faithful representation of $\mathcal{T}$ in $\mathcal{B}(H)$.

Our next aim is to extend the formalism of [9] to Hilbert bimodules. We describe natural realizations of categories of Hilbert bimodules faithfully represented in some C^*-algebra as endomorphism categories of a suitable C^*-algebra. Our starting point is the following. We are given a unital semigroup Δ of Hilbert bimodules over a C^*-algebra $\mathcal{A}$ contained in the C^*-algebra $\mathcal{M}$. We assume, for simplicity, that $\mathcal{M}$ is generated by the elements of Δ. We form the subspaces $(X, Y)_{\mathcal{A}}$, $X, Y \in \Delta$, in $\mathcal{M}^{**}$ and the category $\mathcal{S}_\Delta$ with arrows these intertwining spaces. We denote by $\widetilde{\mathcal{M}}$ the C^*-subalgebra of $\mathcal{M}^{**}$ generated by the $(X, Y)_{\mathcal{A}}$'s. It is now clear that $\mathcal{S}_\Delta$ is a strict semitensor C^*-category. If furthermore we define ${}_{\mathcal{A}}(X, Y)_{\mathcal{A}} \subset (X, Y)_{\mathcal{A}}$ as the subspace of $\mathcal{A}$-bimodule maps, namely

$${}_{\mathcal{A}}(X, Y)_{\mathcal{A}} = \{T \in (X, Y)_{\mathcal{A}} \colon aTx = Tax,\ a \in \mathcal{A},\ x \in X\},$$

the subcategory $\mathcal{T}_\Delta \subset \mathcal{S}_\Delta$ with the same objects of $\mathcal{S}_\Delta$ and arrows ${}_{\mathcal{A}}(X, Y)_{\mathcal{A}}$, is a strict tensor C^*-category.

Let $\mathcal{B} \subseteq \mathcal{C}$ be an inclusion of unital C^*-algebras, and let $\mathrm{End}_{\mathcal{C}}(\mathcal{B})$ be the category of endomorphisms of $\mathcal{B}$ with arrows the intertwiners in $\mathcal{C}$:

$$(\varrho, \sigma) = \{c \in \mathcal{C} \colon c\varrho(I) = c,\ c\varrho(b) = \sigma(b)\, c,\ b \in \mathcal{B}\}.$$

REMARK 2.2. – $\mathrm{End}_{\mathcal{C}}(\mathcal{B})$ is a strict semitensor C^*-category by defining the tensor product on the set of objects to be the composition, and $\Phi_\tau \colon c \in (\varrho, \sigma) \to c \in (\varrho\tau, \sigma\tau)$.

$\mathrm{End}_{\mathcal{B}}(\mathcal{B})$ (simply denoted $\mathrm{End}(\mathcal{B})$) is a tensor C^*-category by $b \times b' = b\varrho(b') \in (\varrho\varrho', \sigma\sigma')$, $b \in (\varrho, \sigma)$, $b' \in (\varrho', \sigma')$.

THEOREM 2.3. – *Let Δ be a unital semigroup of Hilbert $\mathcal{A}$-bimodules in a C^*-algebra $\mathcal{M}$. With the above notation, any $X \in \Delta$ induces a unique endomorphism σ_X on $\mathcal{A}' \cap \widetilde{\mathcal{M}}$ such that*

$$\sigma_X(T)\, x = xT, \qquad x \in X, \qquad T \in \mathcal{A}' \cap \widetilde{\mathcal{M}}.$$

The map $X \in \mathcal{S}_\Delta \to \sigma_X \in \mathrm{End}_{\widetilde{\mathcal{M}}}(\mathcal{A}' \cap \widetilde{\mathcal{M}})$ that acts trivially on the arrows is a faithful functor of semitensor C^*-categories that restricts to a functor of tensor C^*-categories $\mathcal{T}_\Delta \to \mathrm{End}(\mathcal{A}' \cap \widetilde{\mathcal{M}})$. If furthermore $\mathcal{A}$ is normal in $\widetilde{\mathcal{M}}$ then the images of these functors are full subcategories.

PROOF. – Let $\{x_1, x_2, \ldots\}$ be a basis of X. If $T \in \mathcal{M}^{**+}$ then the sequence of positive elements $\sum_{j=1}^{N} x_j T x_j^*$ is increasing and bounded in norm by $\|T\|\,\|1_X\|$. Therefore $\sum_{j=1}^{N} x_j T x_j^*$ is strongly convergent to an element $\phi(T) \in \mathcal{M}^{**}$ for any $T \in \mathcal{M}^{**}$ and ϕ is a norm 1 positive map. If $T \in (Y, Z)_{\mathcal{A}}$, for $Y, Z \in \Delta$, then clearly $\phi(T) \in \overline{XZY^*X^*}^{uw}$ and $\phi(T)\, XY \subseteq XZ$ and $\phi(T)^* XZ \subseteq XY$, hence $\phi(T) \in (XY, XZ)_{\mathcal{A}}$. It follows that ϕ leaves $\widetilde{\mathcal{M}}$ globally invariant. Since $X^*X \subseteq \mathcal{A}$, the restriction σ_X of ϕ to $\mathcal{A}' \cap \widetilde{\mathcal{M}}$ is multiplicative. Clearly if $T \in \mathcal{A}' \cap \widetilde{\mathcal{M}}$ then $\sigma_X(T)x = xT$ for any $x \in X$. Now $\sigma_X(T)$ has support contained in 1_X, thus we conclude that $\sigma_X(T)$ is independent on the basis. In particular, if u is a unitary in $\mathcal{A}$ (or in $\widetilde{\mathcal{A}} := \mathcal{A} + \mathbb{C}1_X$ if $\mathcal{A}$ does not have a unit) then the basis $\{ux_1, ux_2, \ldots\}$ induces the same map σ_X, thus u commutes with $\sigma_X(\mathcal{A}' \cap \widetilde{\mathcal{M}})$, i.e. σ_X leaves $\mathcal{A}' \cap \widetilde{\mathcal{M}}$ invariant. Finally, if $\mathcal{A}$ is normal in $\widetilde{\mathcal{M}}$ and $T \in (\sigma_X, \sigma_Y)$ then in particular for any $x \in X$ and $y \in Y$ $y^*Tx \in (\mathcal{A}' \cap$

$\widetilde{\mathcal{M}})' \cap \widetilde{\mathcal{M}} = \mathcal{A}$. For any basis $\{y_j\}$ of Y, $\sum_j y_j y_j^* Tx$ is norm converging to Tx, thus $TX \subset Y$. Similarly, $T^* Y \subset X$, so $T \in (X, Y)_{\mathcal{A}}$. ■

3. – The C^*-algebra $\mathcal{O}_\varrho$.

In this section we discuss the C^*-algebra $\mathcal{O}_\varrho$ associated with an object ϱ of a strict semitensor C^*-category $\mathcal{T}$. When we specialize ϱ to a Hilbert bimodule X, $\mathcal{A}$ will be embedded in $\mathcal{O}_\varrho$ as a subalgebra, and X as a $\mathcal{A}$-bimodule. In view of Theorem 2.3 we will give sufficient conditions on X in order that $\mathcal{A}$ is embedded as a normal subalgebra.

The construction of the C^*-algebras $\mathcal{O}_\varrho$ was given in [11] when ϱ is an object of a strict tensor C^*-category $\mathcal{T}$. We are now interested, among others, in the categories $\mathcal{S}_X$ with objects the tensor powers of a bimodule X and arrows $(X^r, X^s)_{\mathcal{A}}$, $r, s \in \mathbb{N}_0$, so that $\mathcal{S}_X$ is only a strict semitensor C^*-category. However, the construction in [11] goes through without substantial modifications and for the convenience of the reader we sketch it here in the case of a strict semitensor C^*-category.

We first form the Banach space $\mathcal{O}_\varrho^{(k)}$ inductive limit of $(\varrho^r, \varrho^{r+k})$ via the maps $\Phi_\varrho : (\varrho^r, \varrho^{r+k}) \to (\varrho^{r+1}, \varrho^{r+k+1})$. The composition and the *-involution of $\mathcal{T}$ define on $\bigoplus_{k \in \mathbb{Z}} \mathcal{O}_\varrho^{(k)}$ a structure of $\mathbb{Z}$-graded *-algebra. There is a unique C^*-norm on $\bigoplus_{k \in \mathbb{Z}} \mathcal{O}_\varrho^{(k)}$ for which the automorphic action of $\mathbb{T}$ defined by the grading is isometric, and $\mathcal{O}_\varrho$ is the completion in that norm. We denote by ${}^0\mathcal{O}_\varrho$ the canonical dense *-subalgebra generated by images of intertwiners (ϱ^r, ϱ^s).

If $\mathcal{T}$ is a genuine tensor C^*-category, tensoring on the left by 1_ϱ induces a canonical endomorphism, σ_ϱ of $\mathcal{O}_\varrho$.

Any *-functor $\mathcal{F}: \mathcal{T}_1 \to \mathcal{T}_2$ of strict semitensor C^*-categories induces an obvious *-homomorphism $\mathcal{F}_*: \mathcal{O}_\varrho \to \mathcal{O}_{\mathcal{F}(\varrho)}$.

Let X be a Hilbert $\mathcal{A}$-bimodule as in Section 2. We will consider the semitensor C^*-category $\mathcal{S}_X$ with objects the $\mathcal{A}$-bimodule tensor powers X^r of X and arrows the $(X^r, X^s)_{\mathcal{A}}$, the adjointable right $\mathcal{A}$-module maps. We will write $X_{\mathcal{A}}$ when X is viewed as an object of this strict *semitensor* C^*-category. We can also consider the the strict *tensor* C^*-category $\mathcal{T}_X$ with the same objects and arrows the bimodule maps ${}_{\mathcal{A}}(X^r, X^s)_{\mathcal{A}}$. We will write ${}_{\mathcal{A}}X_{\mathcal{A}}$ when X is considered as an object of this strict tensor category.

The construction of $\mathcal{O}_\varrho$ applied to $\varrho = X_{\mathcal{A}}$ yields a C^*-algebra $\mathcal{O}_{X_{\mathcal{A}}}$ that contains a copy of $\mathcal{A}$ as embedded in $(X, X)_{\mathcal{A}}$ and $X = \mathcal{K}_{\mathcal{A}}(\mathcal{A}, X) \subset (\mathcal{A}, X)_{\mathcal{A}}$ as a Hilbert $\mathcal{A}$-bimodule. $\mathcal{O}_{X_{\mathcal{A}}}$ is generated as a Banach space by the $(X^r, X^s)_{\mathcal{A}}$'s and carries the action α of $\mathbb{T}$ defined by the $\mathbb{Z}$-grading $\mathcal{O}_{X_{\mathcal{A}}}^{(k)}$.

REMARK. – The left annihilator of X in $\mathcal{O}_{X_{\mathcal{A}}}$ is zero. For, given $T \in \mathcal{O}_{X_{\mathcal{A}}}$, $Tx = 0$ for all $x \in X$ implies, by Fourier analysis over the action α of $\mathbb{T}$, $T_k x = 0$ for all $x \in X$, $k \in \mathbb{Z}$, where T_k is the projection of T in $\mathcal{O}^{(k)}_{X_{\mathcal{A}}}$. But each $T_k^* T_k$ can be approximated in norm by elements of $(X^r, X^r)_{\mathcal{A}}$ for large r, and the norm on $(X^r, X^r)_{\mathcal{A}}$ is that of the corresponding bounded operators on $X^r X^{r*}$. Thus $T_k = 0$ and $T = 0$.

REMARK. – In the special case where X is a Hilbert $\mathcal{A}$-bimodule in the C^*-algebra $\mathcal{M}$, $(X^r, X^s)_{\mathcal{A}}$ are identified as in Section 2 with the corresponding subspaces of $\widetilde{\mathcal{M}}$, but the closed linear span in $\widetilde{\mathcal{M}}$ does not necessarily identify with $\mathcal{O}_{X_{\mathcal{A}}}$ since the $\mathbb{Z}$-graded $*$-subalgebra of $\widetilde{\mathcal{M}}$ generated by the $(X^r, X^s)_{\mathcal{A}}$ does not necessarily carry an automorphic action of $\mathcal{T}$ defined by the grading and continuous for the norm of $\widetilde{\mathcal{M}}$.

The following result is an easy consequence of the definition of $\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$ and of functoriality of the construction.

PROPOSITION 3.1. – *Let X and Y be Hilbert C^*-bimodules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and let ${}_{\mathcal{A}}\gamma_{\mathcal{B}}$ be a strong Morita equivalence such that X and $\gamma Y \gamma^{-1}$ are isomorphic as Hilbert C^*-bimodules. Then $\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$ and $\mathcal{O}_{{}_{\mathcal{B}}Y_{\mathcal{B}}}$ are isomorphic according to an isomorphism that transforms ${}_{\mathcal{A}}(X^r, X^s)_{\mathcal{A}}$ into ${}_{\mathcal{B}}(Y^r, Y^s)_{\mathcal{B}}$.*

Pimsner defined in [27] the universal C^*-algebra generated by a Hilbert bimodule X (cf. also [31] for a special case). These C^*-algebras are generalizations of the Cuntz-Krieger algebras and we shall refer to them as CKP-algebras. In the following Proposition we relate the algebras $\mathcal{O}_{X_{\mathcal{A}}}$ to the CKP-algebras.

PROPOSITION 3.2. – *Let X be a Hilbert $\mathcal{A}$-bimodule and $C^*(X)$ the associated CKP-algebra. The identity map on X extends to a $*$-isomorphism of $C^*(X)$ onto the C^*-subalgebra of $\mathcal{O}_{X_{\mathcal{A}}}$ generated by X, which is onto $\mathcal{O}_{X_{\mathcal{A}}}$ if X is full and projective.*

PROOF. – Following Pimsner [27], we consider $\mathcal{F}(X)$, the full Fock space of X, and $J(\mathcal{F}(X))$, the C^*-subalgebra of $\mathcal{L}_{\mathcal{A}}(\mathcal{F}(X))$ generated by $\mathcal{L}_{\mathcal{A}}(\bigoplus_{n=0}^{p} X^n)$, $p \in \mathbb{N}$. For any $x \in X$, let S_x be the image in $M(J(\mathcal{F}(X)))/J(\mathcal{F}(X))$ of the operator that tensors on the left by x. The CKP-algebra is the C^*-subalgebra generated by S_x, $x \in X$. The automorphic action β of $\mathbb{T}$ on $\mathcal{L}(\mathcal{F}(X))$ implemented by the unitary operators $U(z)$ on $\mathcal{F}(X)$ defined by $U(z)x = z^k x$, $x \in X^k$, $k \in \mathbb{N}_0$, induces an action on the quotient C^*-algebra, that restricts to an action γ on the CKP-algebra such that $\gamma_z(x) = zx$ for $x \in X$. It follows that the *-subalgebra generated by S_X is contained in ${}^0\mathcal{O}_{X_{\mathcal{A}}}$ in a canonical way, and that this is an equality if X is full and finite projective. Clearly, the canonical action α corresponds to γ. ■

THEOREM 3.3. – *Let X be a full Hilbert $\mathcal{A}$-bimodule contained in $\mathcal{M}$ such that $\mathcal{M}$ is generated by X as a C^*-algebra* (*hence $\widetilde{\mathcal{M}}$ is generated by the $(X^r, X^s)_{\mathcal{A}}$'s*) *and X has vanishing left annihilator in $\widetilde{\mathcal{M}}$. The following are equivalent*:

i) *$\mathcal{M}$ is the universal C^*-algebra with the properties above,*

ii) *$\mathcal{O}_{X_{\mathcal{A}}}$ is canonically isomorphic to $\widetilde{\mathcal{M}}$, i.e. there is a *-isomorphism acting as the identity on $(X^r, X^s)_{\mathcal{A}}$, $r, s \in \mathbb{N}_0$,*

iii) *the CKP-algebra $C^*(X) \subset \mathcal{O}_{X_{\mathcal{A}}}$ is canonically isomorphic to $\mathcal{M}$, i.e. there is an isomorphism acting as the identity on X,*

iv) *there is an action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{M})$ such that $\alpha_z(x) = zx$, $z \in \mathbb{T}$, $x \in X$.*

PROOF. – If there is an action α as in iv) then the bitransposed action $\alpha^{**}\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{M}^{**})$ restricts to an action on $\widetilde{\mathcal{M}}$, still denoted by α, such that $\alpha_z(T) = z^{s-r}T$, $T \in (X^r, X^s)_{\mathcal{A}}$, and this shows the equivalence of ii) with iv) and with iii) as well, in view of the previous Proposition. If i) holds then iv) follows from the universality property of $\mathcal{M}$. Finally, iii)⇒i) was proved in [27]. ■

Theorem 3.3 can be easily reformulated without assuming that X is full, but requiring that $\mathcal{M}$ is the C^*-algebra generated by X and $\mathcal{A}$. In this case, condition iii) modifies requiring that there is an isomorphism of $\mathcal{M}$ with the augmented algebra of Pimsner [27] which identifies the embeddings of X, respectively of $\mathcal{A}$, in those algebras. In condition iv) the action α will be further required to be trivial on $\mathcal{A}$.

In view of condition i) the CKP-algebra $C^*(X) \subset \mathcal{O}_{X_{\mathcal{A}}}$ can be thought of as the crossed product of $\mathcal{A}$ by X in the spirit of [1] where, however, only bimodules of a more restricted class were considered.

PROPOSITION 3.4. – *a) The inclusion functor $\iota\colon \mathcal{T}_X \subset \mathcal{S}_X$ induces an inclusion *-monomorphism*

$$\iota_*\colon \mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}} \to \mathcal{O}_{X_{\mathcal{A}}}$$

such that

$$\iota_*(\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}) \subseteq \mathcal{A}' \cap \mathcal{O}_{X_{\mathcal{A}}}\,.$$

We have that $\sigma_X \circ \iota_* = \iota_* \circ \sigma_X$.

b) If for some $s \in \mathbb{N}$, ${}_{\mathcal{A}}(\mathcal{A}, X^s)_{\mathcal{A}}$ *contains an isometry then*

$$\iota_*(\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}) = \mathcal{A}' \cap \mathcal{O}_{X_{\mathcal{A}}} .$$

PROOF. – Part *a*) follows from the fact that the dual action of $\mathbb{T}$ on $\mathcal{O}_{X_{\mathcal{A}}}$ that defines its $\mathbb{Z}$-grading transform ${}_{\mathcal{A}}(X^r, X^s)_{\mathcal{A}}$ according to the character $s - r \in \mathbb{Z}$, hence their linear span coincides with ${}^0\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$. The canonical norm of $\mathcal{O}_{X_{\mathcal{A}}}$, i.e. the one for which the $\mathbb{T}$-action is isometric, restricts to the canonical norm of ${}^0\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$. Since $\iota_*(\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}})$ and $\mathcal{A}' \cap \mathcal{O}_{\iota(X)}$ are globally invariant under the action of $\mathbb{T}$, to prove *b*) it suffices to show that the corresponding $\mathbb{T}$-eigenspaces are equal. Let $R \in {}_{\mathcal{A}}(\mathcal{A}, X^s)_{\mathcal{A}}$ be an isometry. Using $R^{p+p'} = \sigma_X^{ps}(R^{p'})R^p$, one can easily show that for T in some $(X^p, X^{p+k})_{\mathcal{A}}$ the sequence $\sigma_X^{r+k}(R^{p'})^* T \sigma_X^r(R^{p'})$, $p' \in \mathbb{N}$, is eventually equal to a constant element of $(X^r, X^{r+k})_{\mathcal{A}}$. Thus the formula

$$E_r(T) = \lim_p \sigma_X^{r+k}(R^p)^* T \sigma_X^r(R^p)$$

defines a norm one projection E_r from $\mathcal{O}^{(k)}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$, the closure of ${}^0\mathcal{O}^{(k)}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$ in $\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$, onto $(X^r, X^{r+k})_{\mathcal{A}}$ that acts identically on $(X^r, X^{r+k})_{\mathcal{A}}$ and satisfies $E_r(aTa') = aE_r(T)\,a'$, $a, a' \in \mathcal{A}$. It follows that the sequence E_r is pointwise convergent to the identity map, thus if $T \in \mathcal{A}' \cap \mathcal{O}^{(k)}_{X_{\mathcal{A}}}$ then $E_r(T) \in (X^r, X^{r+k})_{\mathcal{A}}$ and approximates T. ■

The functorial properties of the construction of $\mathcal{O}_{X_{\mathcal{A}}}$ imply that to each unitary $U \in {}_{\mathcal{A}}(X, X)_{\mathcal{A}}$ we can associate a canonical automorphism σ_U of $\mathcal{O}_{X_{\mathcal{A}}}$, leaving $\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$ globally stable, such that

$$\sigma_U(x) = Ux , \qquad x \in X .$$

We thus establish an isomorphism between $\mathcal{U}({}_{\mathcal{A}}(X, X)_{\mathcal{A}})$ and the group of all automorphisms of $\mathcal{O}_{X_{\mathcal{A}}}$ leaving $\mathcal{A}$ pointwise fixed and X globally stable.

The restriction to $\mathcal{O}_{{}_{\mathcal{A}}X_{\mathcal{A}}}$ of such an automorphism commutes with σ_X; hence for each subgroup G of $\mathcal{U}({}_{\mathcal{A}}(X, X)_{\mathcal{A}})$ the fixed point subalgebra $\mathcal{O}^G_{{}_{\mathcal{A}}X_{\mathcal{A}}}$ is globally stable under σ_X. Thus σ_X induces an endomorphism σ_G of $\mathcal{O}^G_{{}_{\mathcal{A}}X_{\mathcal{A}}}$.

The systems $(\mathcal{O}^G_{{}_{\mathcal{A}}X_{\mathcal{A}}}, \sigma_G)$ have been extensively studied when $\mathcal{A} = \mathbb{C}$; we hope to turn to the general case where $\mathcal{A} \neq \mathbb{C}$ and G is replaced by a quantum group.

In the remaining part of this section we focus our attention on how $\mathcal{A}$ is embedded in $\mathcal{O}_{X_{\mathcal{A}}}$, more precisely, in view of Theorem 2.3 we look for conditions that X should satisfy so that $\mathcal{A}$ is normal in $\mathcal{O}_{X_{\mathcal{A}}}$.

A Hilbert $\mathcal{A}$-bimodule X with left $\mathcal{A}$-action $\phi: \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(X)$ is called nonsingular if $\vartheta_{x,x} \in \phi(\mathcal{A})$ for some $x \in X$ implies $x = 0$. The trivial bimodule $\mathcal{A}$ is always singular. It is easy to see that if X is nonsingular then

$Y \otimes_{\mathcal{A}} X$ is nonsingular for any Hilbert $\mathcal{A}$-bimodule Y. In particular, powers of nonsingular bimodules are nonsingular.

Let $\mathcal{A}$ be a unital, purely infinite C^*-algebra, and let X be a Hilbert $\mathcal{A}$-bimodule such that the left $\mathcal{A}$-action $\phi: \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(X)$ is unital. Then X is singular if and only if it is singly generated. In fact, if there is $a \in \mathcal{A}$ such that $\vartheta_{x,x} = \phi(a) \neq 0$, then by the pure infiniteness of $\mathcal{A}$ there is $b \in \mathcal{A}$ such that $b^* ab = I$, hence $\vartheta_{bx, bx} = I$, and this is to say that X is singly generated.

PROPOSITION 3.5. – *Let X be a Hilbert $\mathcal{A}$-bimodule in $\mathcal{M}$.*

a) If X is nonsingular and ${}_{\mathcal{A}}(\mathcal{A}, X^s)_{\mathcal{A}}$ contains an isometry S for some $s > 1$ then $C^*(S)' \cap \mathcal{O}_{X_{\mathcal{A}}} = \mathcal{A}$.

b) If there are isometries $S_k \in {}_{\mathcal{A}}(\mathcal{A}, X^{n(k)})_{\mathcal{A}}$ such that

$$S_k^* \sigma_X^k(S_k) = \lambda_k$$

with $\|\lambda_k\| < 1$, then $C^*(S_k, k = 1, 2, \dots)' \cap \mathcal{O}_{X_{\mathcal{A}}} = \mathcal{A}$.

In both cases $\mathcal{A}$ is normal in $\mathcal{O}_{X_{\mathcal{A}}}$.

PROOF. – Let $\mathcal{B}$ denote one of the relative commutants described in *a*) or in *b*), and $S \in {}_{\mathcal{A}}(\mathcal{A}, X^s)_{\mathcal{A}}$ an isometry. We show that $\mathcal{B} \cap \mathcal{O}_{X_{\mathcal{A}}}^{(k)}$ is zero for $k \neq 0$ and that it is contained in $\mathcal{A}$ for $k = 0$. Let Φ be a weak limit point of the sequence Φ_p, $\Phi_p(T) = (S^p)^* T S^p$ in some faithful representation of $\mathcal{O}_{X_{\mathcal{A}}}$ on a Hilbert space. Clearly $\Phi(\mathcal{O}_{X_{\mathcal{A}}}^{(k)}) \subseteq X^k$, $k \in \mathbb{N}$, and $\Phi(T) = T$, $T \in \mathcal{B}$. Hence $\mathcal{B} \cap \mathcal{O}_{X_{\mathcal{A}}}^{(k)}$ is contained in X^k for $k \in \mathbb{N}_0$. Let T be an element of this subspace with $k > 0$. If X is nonsingular then $TT^* \in \mathcal{B} \cap \mathcal{O}_{X_{\mathcal{A}}}^{(0)} \subset \mathcal{A}$, so $T = 0$, and *a*) holds. To prove *b*), we note that

$$T = S_k^* T S_k = S_k^* \sigma_X^k(S_k) T = \lambda_k T ,$$

thus $T = 0$. ■

As a consequence of the previous result we can show normalcy of $\mathcal{A}$ in $\mathcal{O}_{X_{\mathcal{A}}}$ when X is a real or pseudoreal bimodule with dimension > 1 in the sense of [24]. More explicitly, and slightly more generally, we have the following result.

COROLLARY 3.6. – *If there is an isometry $S \in {}_{\mathcal{A}}(\mathcal{A}, X^2)_{\mathcal{A}}$ such that*

$$\|S^* \sigma_X(S)\| < 1$$

then $C^*(\sigma_X^k(S), k = 0, 1, 2, \dots)' \cap \mathcal{O}_{X_{\mathcal{A}}} = \mathcal{A}$, hence $\mathcal{A}$ is normal in $\mathcal{O}_{X_{\mathcal{A}}}$.

PROOF. – The isometries

$$S_k := \sigma_X^{k-1}(S)\ldots\sigma_X(S)\, S \in {}_{\mathcal{A}}(\mathcal{A}, X^{2k})_{\mathcal{A}}$$

satisfy $S_k^* \sigma_X^k(S_k) = \lambda^k$, with $\lambda = S^* \sigma_X(S)$. ■

4. – The ideal structure of $\mathcal{O}_{X_\mathcal{A}}$.

In the first part of this section we introduce a natural class of *-representations $\pi : \mathcal{O}_{X_\mathcal{A}} \to \mathcal{B}(H)$, called locally strictly continuous, and we generalize Pimsner's universality result to the algebras $\mathcal{O}_{X_\mathcal{A}}$. We then associate to X certain Connes spectra which allow us to characterize simplicity of $\mathcal{O}_{X_\mathcal{A}}$ and of the CKP-algebra $C^*(X) \subset \mathcal{O}_{X_\mathcal{A}}$ in terms of a suitable class of ideals of $\mathcal{A}$.

The following is a variant of Pimsner's universality result to the C^*-algebras $\mathcal{O}_{X_\mathcal{A}}$.

THEOREM 4.1. – *Let Y be a Hilbert bimodule over a C^*-algebra $\mathcal{B}$ in $\mathcal{B}(H)$, and $\mathcal{D}$ the C^*-subalgebra of $\mathcal{B}(H)$ generated by the subspaces $(Y^r, Y^s)_\mathcal{B}$, $r, s \geq 0$. Assume that the left annihilator of Y in $\mathcal{D}$ is zero, and let (U, ϕ) be a pair consisting of a *-isomorphism $\phi : \mathcal{A} \to \mathcal{B}$ and a linear surjective map $U : X \to Y$ which satisfies*

$$U(x)^* U(x') = \phi(x^* x'),$$

$$U(xa) = U(x)\,\phi(a), \qquad U(ax) = \phi(a)\, U(x),$$

*for $a, a' \in \mathcal{A}$, $x, x' \in X$. Then there is a unique *-representation $\pi : C^*(X) \to \mathcal{B}(H)$ that maps $x \in X$ to $U(x)$, as in* [27, *Theorem* 3.12], *which furthermore extends to a unique *-representation $\tilde{\pi} : \mathcal{O}_{X_\mathcal{A}} \to \mathcal{B}(H)$ via*

$$\tilde{\pi}(T)\,\pi(A) = \pi(TA), \qquad A \in X^s X^{r*}, \qquad T \in (X^s, X^t)_\mathcal{A},$$

$$\tilde{\pi}(a) = \phi(a), \qquad a \in \mathcal{A}.$$

If $\ker \pi$ *is $\mathbb{T}$-invariant then $\tilde{\pi}$ is faithful.*

PROOF. – It is easy to see that for any $T \in (X, X)_\mathcal{A}$ there is a unique operator $\pi_U(T) \in (Y, Y)_\mathcal{B}$ such that $\pi_U(T)Ux = U(Tx)$, $x \in X$, and that π_U is a *-homomorphism s.t. $\pi_U(xy^*) = U(x)U(y)^*$, $x, y \in X$. Let $\{x_1, x_2, \ldots\}$ be a basis of X. Since U has dense range, $\{U(x_1), U(x_2), \ldots\}$ is a basis of Y. Since the left annihilator of Y in the C^*-subalgebra $C^*(Y, \mathcal{B})$ of $\mathcal{B}(H)$ generated by Y and $\mathcal{B}$ is zero, for any $a \in \mathcal{A} \cap XX^*$, $\sum_i \phi(a)\, U(x_i)\, U(x_i)^* = \sum_i \pi_U(a x_i x_i^*)$ is norm converging to $\phi(a)$, therefore by [27, Theorem 3.12] there is a unique *-representation π of $C^*(X) \subseteq \mathcal{O}_{X_{\mathcal{A}}}$ on H such that $\pi(x) = U(x)$. Now if $\max\{r, s\} > 0$, the restriction of π to $X^s X^{r*}$ extends uniquely to a map $\tilde{\pi}_{r,s} : (X^r, X^s)_{\mathcal{A}} \to (Y^r, Y^s)_{\mathcal{B}} \subset \mathcal{B}(H)$ such that for $T \in (X^s, X^t)_{\mathcal{A}}$, $A \in X^s X^{r*}$, $B \in X^v X^{t*}$,

$$\tilde{\pi}_{s,t}(T)\, \pi(A) = \pi(TA),$$

$$\pi(B)\, \tilde{\pi}_{s,t}(T) = \pi(BT).$$

We set, by convention, $\tilde{\pi}_{0,0} = \phi : \mathcal{A} \to \mathcal{B} \subset \mathcal{B}(H)$. Uniqueness implies $\tilde{\pi}_{s,t}(T)^* = \tilde{\pi}_{t,s}(T^*)$, $\tilde{\pi}_{s,t}\tilde{\pi}_{r,s} = \tilde{\pi}_{r,t}$, and also that the restriction of $\tilde{\pi}_{s+1,t+1}$ to $(X^s, X^t)_{\mathcal{A}}$ coincides with $\tilde{\pi}_{s,t}$ since the left annihilator of Y in $C^*(\{(Y^r, Y^s)_{\mathcal{B}}, r, s \geq 0\}) \subset \mathcal{B}(H)$ is zero. We can thus define a unique *-homomorphism $\tilde{\pi} : {}^0\mathcal{O}_{X_{\mathcal{A}}} \to \mathcal{B}(H)$ extending $\tilde{\pi}_{r,s}$ on $(X^r, X^s)_{\mathcal{A}}$. We show that $\tilde{\pi}$ is norm continuous. Let $T = \sum_{k=-n}^{n} T_k$ be an element of ${}^0\mathcal{O}_{X_{\mathcal{A}}}$, with $T_k \in (X^r, X^{r+k})_{\mathcal{A}}$ for a suitable r and $k = -n, \ldots, n$ and let 1_F be the support of a finitely generated right $\mathcal{A}$-submodule of X^r, so $T1_F \in C^*(X)$. Then

$$\|\pi(T1_F)\| \leq \|T1_F\| \leq \|T\|$$

for all F implies $\|\tilde{\pi}(T)\| \leq \|T\|$.

Assume now that $\ker\pi$ is globally invariant under the action of $\mathbb{T}$. Then $\tilde{\pi}_{r,r}$ is faithful on $(X^r, X^r)_{\mathcal{A}}$ since the left annihilator of X^r in $\mathcal{O}_{X_{\mathcal{A}}}$ is zero, therefore, since $\ker\pi \cap \mathcal{O}^0_{X_{\mathcal{A}}}$ is the inductive limit of $\ker\pi \cap (X^r, X^r)_{\mathcal{A}}$, $\tilde{\pi}$ is faithful on $\mathcal{O}^0_{X_{\mathcal{A}}}$, hence, being $\ker\tilde{\pi}$ $\mathbb{T}$-invariant, $\tilde{\pi}$ is faithful on $\mathcal{O}_{X_{\mathcal{A}}}$. ■

As a consequence of Theorem 4.1 the correspondence between unitaries and endomorphisms of the Cuntz algebras generalizes as follows.

PROPOSITION 4.2. – *Any unitary $U \in \mathcal{A}' \cap \mathcal{O}_{X_{\mathcal{A}}}$ defines an endomorphism λ_U of $\mathcal{O}_{X_{\mathcal{A}}}$ acting trivially on $\mathcal{A}$ by*

$$\lambda_U(x) = Ux, \qquad x \in X.$$

If $U \in \mathcal{A}' \cap \mathcal{O}^0_{X_{\mathcal{A}}}$ then λ_U is a monomorphism.

If X is finite projective, the correspondence $U \to \lambda_U$ is a one to one map of the unitaries in $\mathcal{A}' \cap \mathcal{O}_{X_{\mathcal{A}}}$ onto the endomorphisms of $\mathcal{O}_{X_{\mathcal{A}}}$ leaving $\mathcal{A}$ pointwise fixed, which extends the canonical action of $\mathcal{U}({}_{\mathcal{A}}(X, X)_{\mathcal{A}})$ (cf. Section 3).

PROOF. – We represent $\mathcal{O}_{X_{\mathcal{A}}}$ faithfully on a Hilbert space H. We have already noted that the left annihilator of X in $\mathcal{O}_{X_{\mathcal{A}}}$ is zero (cf. a remark in Section 3), therefore also the left annihilator of $Y := UX$ in $\mathcal{O}_{X_{\mathcal{A}}}$ (regarded as a Hilbert $\mathcal{A}$-bimodule in $\mathcal{O}_{X_{\mathcal{A}}}$) is zero. By Theorem 4.1 there is a unique *-representation λ_U of $\mathcal{O}_{X_{\mathcal{A}}}$ on H such that $\lambda_U(x) = Ux$, $x \in X$ and acting trivially on $\mathcal{A}$ provided we show that the left annihilator of Y in $C^*\{(Y^r, Y^s)_{\mathcal{A}}, r, s \geq 0\} \subset \mathcal{B}(H)$ is zero. Now

$$Y^s Y^{r*} = U\sigma_X(U) \dots \sigma_X^{s-1}(U)\, X^s X^{r*}\, \sigma_X^{r-1}(U^*) \dots \sigma_X(U^*)\, U^* ,$$

therefore

$$(Y^r, Y^s)_{\mathcal{A}} = U\sigma_X(U) \dots \sigma_X^{s-1}(U)(X^r, X^s)_{\mathcal{A}}\, \sigma_X^{r-1}(U^*) \dots \sigma_X(U^*)\, U^* \subset \mathcal{O}_{X_{\mathcal{A}}} ,$$

and the claim follows from the previous remarks. Note also that if $T \in X^s X^{r*}$,

$$\lambda_U(T) = U\sigma_X(U) \dots \sigma_X^{s-1}(U)\, T\sigma_X^{r-1}(U^*) \dots \sigma_X(U^*)\, U^* ,$$

therefore the same formula must hold for $T \in (X^r, X^s)_{\mathcal{A}}$, and we conclude that λ_U is an endomorphism of $\mathcal{O}_{X_{\mathcal{A}}}$. If U is a $\mathbb{T}$-fixed point then λ_U commutes with α, so $\ker \lambda_U$ is $\mathbb{T}$-invariant.

Since the left annihilator of X in $\mathcal{O}_{X_{\mathcal{A}}}$ is zero the map $U \to \lambda_U$ is one to one. If X is finite projective and $x_1, \dots, x_d$ is a basis in X, for each endomorphism λ leaving $\mathcal{A}$ pointwise fixed we can define, following Cuntz,

$$U := \sum_i \lambda(x_i)\, x_i^* ,$$

so that U is unitary. For $a \in \mathcal{A}$, $x \in X$, we have

$$Uax = \lambda(ax) = a\lambda(x) = aUx$$

so that $U \in \mathcal{A}' \cap \mathcal{O}_{X_{\mathcal{A}}}$ and $\lambda = \lambda_U$. If $\lambda(X) = X$ clearly $U \in {}_{\mathcal{A}}(X, X)_{\mathcal{A}}$. ■

Our next aim is to determine the ideal structure of $\mathcal{O}_{X_{\mathcal{A}}}$ in certain cases of interest for our pourposes. We first look at ideals invariant under the canonical action of the circle group. Let $\mathfrak{J}$ be a closed ideal of $\mathcal{O}_{X_{\mathcal{A}}}$. We call $\mathfrak{J}$ locally strictly closed if whenever one of r and s is nonzero $\mathfrak{J}_{r,s} := \mathfrak{J} \cap (X^r, X^s)_{\mathcal{A}}$ is strictly closed in $(X^r, X^s)_{\mathcal{A}}$. Note that in this case, $\mathfrak{J}_{r,s}$ is the strict closure of $X^s \mathfrak{J} \cap \mathcal{A}X^{r*}$ in $(X^r, X^s)_{\mathcal{A}}$. An ideal J of $\mathcal{A}$ is called X-invariant if $X^* JX \subset J$. As in [18] we associate to J the ideal $J_X := \{a \in \mathcal{A} : X^* aX \subset J\}$ which is a closed X-invariant ideal containing J. We call J *X-saturated* if $J_X = J$. Note that the zero ideal is X-saturated, and that, if X is full and nondegenerate (in the sense that $\mathcal{A}X = X$) and if J is proper then J_X is proper.

LEMMA 4.3. – *a) Any $\mathbb{T}$-invariant closed ideal $\mathfrak{J}$ of $\mathcal{O}_{X_{\mathcal{A}}}$ is the closed linear span of $\mathfrak{J}_{r,s}$, $r, s = 0, 1, 2, \dots$. Therefore, if $\mathfrak{J}$ is also l.s.c, it is determined by $\mathfrak{J} \cap \mathcal{A}$.*

b) Let J be an X-invariant, X-saturated ideal of $\mathcal{A}$, and let $\tilde{\mathfrak{J}}$ denote the c.l.s. in $\mathcal{O}_{X_{\mathcal{A}}}$ of the strict closures of $X^s JX^{r*}$ in $(X^r, X^s)_{\mathcal{A}}$. If X is full, let $\mathfrak{J}$ be the c.l.s. of the $X^s JX^{r*}$. Then $\tilde{\mathfrak{J}}$ and $\mathfrak{J}$ are respectively *a locally strictly closed* $\mathbb{T}$*-invariant ideal of* $\mathcal{O}_{X_{\mathcal{A}}}$ *and a closed* $\mathbb{T}$*-invariant ideal of* $C^*(X)$ *such that* $\tilde{\mathfrak{J}} \cap \mathcal{A} = \mathfrak{J} \cap \mathcal{A} = J$.

PROOF. – We first note that if $\mathcal{B}$ is any C^*-algebra andowed with a continuous automorphic action α of $\mathbb{T}$ and $\mathfrak{I}$ and $\mathfrak{J}$ are closed α-invariant ideals of $\mathcal{B}$ such that the fixed point subalgebras coincide: $\mathfrak{I}^\alpha = \mathfrak{J}^\alpha$ then $\mathfrak{I} = \mathfrak{J}$. Indeed, by Fourier analysis $\mathfrak{I}$ is generated by by the subspaces $\mathfrak{I}^{(k)}$ that transform like the character $k \in \mathbb{Z} = \widehat{\mathbb{T}}$. Furthermore by [26, Proposition 1.4.5] any element $T \in \mathfrak{I}^{(k)}$ can be written in the form $T = u(T^*T)^{1/4}$ with $u \in \mathfrak{I}$, hence $T \in \mathfrak{I}\mathfrak{J}^\alpha \subset \mathfrak{J}$, i.e. $\mathfrak{I} \subset \mathfrak{J}$. Exchanging the role of $\mathfrak{I}$ and $\mathfrak{J}$ we deduce that $\mathfrak{J} = \mathfrak{I}$. Let now $\mathfrak{J}$ be a closed $\mathbb{T}$-invariant ideal of $\mathcal{O}_{X_{\mathcal{A}}}$ and let $\mathfrak{I}$ be the closed linear span of $\mathfrak{J}_{r,s}$, which is still a $\mathbb{T}$-invariant ideal. Since the homogeneous part of $\mathcal{O}_{X_{\mathcal{A}}}$ is the inductive limit of $(X^r, X^r)_{\mathcal{A}}$, $\mathfrak{J}$ is generated by the subspaces $\mathfrak{J}_{r,r}$'s, hence $\mathfrak{I}^{(0)} = \mathfrak{J}^{(0)}$, therefore the previous argument shows that $\mathfrak{J}$ is generated by the $\mathfrak{J}_{r,s}$. To prove b) we consider, for any $r \geq 0$, the ideal J_r of $(X^r, X^r)_{\mathcal{A}} \subset \mathcal{O}_{X_{\mathcal{A}}}$ defined by the strict closure of $X^r J X^{r*}$ in $(X^r, X^r)_{\mathcal{A}}$, so that the inductive limit of the J_r's generates $\tilde{\mathfrak{J}} \cap \mathcal{O}_{X_{\mathcal{A}}}^{(0)}$. If $a \in \mathcal{A} \cap \tilde{\mathfrak{J}}$ then clearly $\lim_r \operatorname{dist}(a, J_r \cap \mathcal{A}) = \lim_r \operatorname{dist}(a, \mathcal{A} \cap \tilde{\mathfrak{J}}) = 0$. On the other hand $J_r \cap \mathcal{A} = J$ for all r since J is X-saturated, therefore $a \in J$. It follows easily that $\tilde{\mathfrak{J}} \cap (X^r, X^r)_{\mathcal{A}} = J_r$, hence $\tilde{\mathfrak{J}}$ is locally strictly closed and, clearly, $\mathbb{T}$-invariant. In the second case, we may argue in the same way, replacing $\mathcal{O}_{X_{\mathcal{A}}}$ by $C^*(X)$, $\tilde{\mathfrak{J}}$ by $\mathfrak{J}$, $(X^r, X^r)_{\mathcal{A}}$ by $\mathcal{A} + XX^* + \dots X^r X^{r*} \subset C^*(X)$ and $\mathfrak{J}_r$ by $J + XJX^* + \dots X^r J X^{r*}$. Since $\mathcal{A} \cap J + XJX^* + \dots X^r J X^{r*} \subset J_{X^r} = J$, we deduce as above that if $a \in \mathcal{A} \cap \mathfrak{J}$ then $a \in J$. ■

If $\mathfrak{J}$ is a l.s.c. ideal of $\mathcal{O}_{X_{\mathcal{A}}}$ then $\mathfrak{J} \cap \mathcal{A}$ is always X-saturated. However, this is not necessarily true if $\mathfrak{J}$ is an ideal of $C^*(X)$. Indeed, this condition may be stated equivalently requiring that if $\pi: C^*(X) \to C^*(X)/\mathfrak{J}$ is the quotient map and P is the support of the right $\pi(\mathcal{A})$-module $\pi(X)$ contained in $\pi(C^*(X))$ (hence $P \in \pi(C^*(X))^{**}$) then $\pi(a)P = 0$ with $a \in \mathcal{A}$ implies $\pi(a) = 0$. In certain cases, e.g. $\mathcal{A} \subset XX^*$, then $\mathfrak{J} \cap \mathcal{A}$ is X-saturated for every closed ideal $\mathfrak{J}$ of $C^*(X)$. If some positive power X^s of X contains an isometry commuting with $\mathcal{A}$ then every X-invariant ideal is automatically X-saturated.

PROPOSITION 4.4. – *Let* $J \to \mathfrak{J}$ *and* $J \to \tilde{\mathfrak{J}}$ *be the maps described in the previous Lemma.*

a) $J \to \tilde{\mathfrak{J}}$ *is a bijective correspondence between* X*-invariant,* X*-saturated ideals of* $\mathcal{A}$*, and* $\mathbb{T}$*-invariant l.s.c. ideals of* $\mathcal{O}_{X_{\mathcal{A}}}$ *with inverse* $\tilde{\mathfrak{J}} \to \tilde{\mathfrak{J}} \cap \mathcal{A}$.

b) If X *is full and* $\mathcal{A} \subset XX^*$, $J \to \mathfrak{J}$ *is a bijective correspondence between the class of ideals of* $\mathcal{A}$ *described in a) and the set of closed* $\mathbb{T}$*-invariant ideals of* $C^*(X)$ *with inverse the map* $\mathfrak{J} \to \mathfrak{J} \cap \mathcal{A}$.

PROOF. – By Lemma 4.3 and the above remarks we need only to show that if $\mathcal{A} \subset XX^*$ then every closed $\mathbb{T}$-invariant ideal $\mathcal{J}$ of $C^*(X)$ is the c.l.s. of the subspaces $X^r \mathcal{J} \cap \mathcal{A} X^{r*}$. Now $\mathcal{A} \subset XX^*$ implies that $X^r X^{r*} \subset X^{r+1} X^{r+1*}$ for all $r \in \mathbb{N}$, hence the homogeneous part of $C^*(X)$ is the inductive limit of $X^r X^{r*}$, $r \in \mathbb{N}$, and this implies that the homogeneous part of $\mathcal{J}$ is the inductive limit of $\mathcal{J} \cap X^r X^{r*} = X^r \mathcal{J} \cap \mathcal{A} X^{r*}$, therefore $\mathcal{J}$ is generated by the subspaces $X^s \mathcal{J} X^{r*}$. ■

We call $\mathcal{A}$ *X-simple* if it has no proper X-invariant, X-saturated ideal, and *X-prime* if it has no pair of nonzero orthogonal X-invariant, X-saturated ideals.

COROLLARY 4.5. – *If X is a Hilbert $\mathcal{A}$-bimodule, the following properties are equivalent,*

a) $\mathcal{A}$ is X-simple (resp. $\mathcal{A}$ is X-prime),

b) $\mathcal{O}_{X_\mathcal{A}}$ has no proper locally strictly closed $\mathbb{T}$-invariant ideal (resp. $\mathcal{O}_{X_\mathcal{A}}$ has no pair of nonzero orthogonal, locally strictly closed, $\mathbb{T}$-invariant ideals).

Consider the following conditions:

i) $\mathcal{A} \subset XX^*$,

ii) *for some $s \in \mathbb{N}$, X^s contains an isometry S commuting with $\mathcal{A}$.*

If either i) *or* ii) *holds and X is full, a) and b) are also equivalent to*

c) $C^*(X)$ is $\mathbb{T}$-simple (resp. $C^*(X)$ is $\mathbb{T}$-prime).

If ii) *holds, a) and b) are equivalent to*

d) $\mathcal{O}_{X_\mathcal{A}}$ is $\mathbb{T}$-simple (resp. $\mathcal{O}_{X_\mathcal{A}}$ is $\mathbb{T}$-prime).

PROOF. – We prove only the statements concerning simplicity, those concerning primeness can be proved with similar arguments. The equivalence of *a*) and *b*), and of *a*) and *c*), in the case that i) holds, follow from Proposition 4.4. Note that by Lemma 4.3 $c) \Rightarrow a)$ (even without assuming that ii) holds). Conversely, assume that *a*) and ii) hold. Let $\mathcal{J}$ be a nonzero $\mathbb{T}$-invariant ideal of $C^*(X)$, then $\mathcal{J} \cap \mathcal{A}$ is a nonzero, X-invariant, X-saturated ideal of $\mathcal{A}$, hence $\mathcal{J} \cap \mathcal{A} = \mathcal{A}$, that implies $\mathcal{J} = C^*(X)$. We are left to show that ii) and *b*) imply *d*). Let $\mathcal{J}$ be a proper $\mathbb{T}$-invariant ideal of $\mathcal{O}_{X_\mathcal{A}}$, and define $\tilde{\mathcal{J}}$ as the c.l.s. of the strict closures of $\mathcal{J} \cap (X^r, X^s)$. $\tilde{\mathcal{J}}$ is a $\mathbb{T}$-invariant ideal containing $\mathcal{J}$. We claim that $\tilde{\mathcal{J}}$ is locally strictly closed, or, more precisely, that $\tilde{\mathcal{J}} \cap (X^r, X^s)$ is the strict closure of $\mathcal{J} \cap (X^r, X^s)$ and that $\tilde{\mathcal{J}} \cap \mathcal{A} = \mathcal{J} \cap \mathcal{A}$. It suffices to prove the second assertion. Let a be an element of $\tilde{\mathcal{J}} \cap \mathcal{A}$ and let T be in the strict closure of some $\mathcal{J} \cap$

(X^{rs}, X^{rs}) such that $\|a - T\| < \varepsilon$, then $\|a - S^{r*} T S^r\| < \varepsilon$, hence $a \in \mathcal{J}$. It follows, by b), that $\tilde{\mathcal{J}} = \mathcal{O}_{X_\mathcal{A}}$. Let T_α be a net in some $\mathcal{J} \cap (X^{rs}, X^{rs})$ strictly converging to the identity, then $S^{r*} T_\alpha S^r$ is a norm converging sequence in $\mathcal{J}$ to the identity, so $\mathcal{J} = \mathcal{O}_{X_\mathcal{A}}$ and the proof is complete. ■

We denote by $\Gamma(X)$ and $\widehat{\Gamma}(X)$ the Connes spectra of the dual action $\widehat{\alpha}$ of $\mathbb{Z}$ on $C^*(X) \times_\alpha \mathbb{T}$ and $\mathcal{O}_{X_\mathcal{A}} \times_\alpha \mathbb{T}$ respectively. By [25] (cf. also Lemma 8.11.7 of [26])

$$\Gamma(X) = \{\lambda \in \mathbb{T} \colon \mathfrak{J} \cap \alpha_\lambda(\mathfrak{J}) \neq \{0\},\ \mathfrak{J} \text{ all closed non-zero ideal of } C^*(X)\},$$

$$\widehat{\Gamma}(X) = \{\lambda \in \mathbb{T} \colon \mathfrak{J} \cap \alpha_\lambda(\mathfrak{J}) \neq \{0\},\ \mathfrak{J} \text{ all closed non-zero ideal of } \mathcal{O}_{X_\mathcal{A}}\}.$$

We note that if X is a Hilbert $\mathcal{A}$-bimodule such that $\mathcal{O}_{X_\mathcal{A}}$, (resp. $C^*(X)$) is prime or simple then clearly $\widehat{\Gamma}(X) = \mathbb{T}$ (resp. $\Gamma(X) = \mathbb{T}$). Furthermore, by Lemma 4.3 $\mathcal{A}$ (resp. the C^*-subalgebra of $\mathcal{A}$ generated by the scalar products if X is not full) is necessarily X-prime or X-simple. The following results are a partial converse.

PROPOSITION 4.6. – *Let X be a Hilbert $\mathcal{A}$-bimodule with $\mathcal{A}$ X-prime.*

a) If X is full and one of the conditions i) *or* ii) *of* 4.5 *is satisfied and furthermore and $\Gamma(X) = \mathbb{T}$ then $C^*(X)$ is prime.*

b) If ii) *of* 4.5 *is satisfied and $\widehat{\Gamma}(X) = \mathbb{T}$ then $\mathcal{O}_{X_\mathcal{A}}$ is prime.*

PROOF. – If $C^*(X)$ were not prime then the arguments that prove (ii)⇒(i) of Theorem 8.11.10 in [26] would prove the existence of two non-zero $\mathbb{T}$-invariant orthogonal ideals in $C^*(X)$, but this is impossible because by 4.5 $C^*(X)$ is $\mathbb{T}$-prime. We prove the second part of the Proposition. $\widehat{\Gamma}(X) = \mathbb{T}$ and $\mathcal{O}_{X_\mathcal{A}}$ nonprime imply the existence of two orthogonal $\mathbb{T}$-invariant proper ideals of $\mathcal{O}_{X_\mathcal{A}}$ hence the existence of two proper orthogonal X-invariant ideals of $\mathcal{A}$ again by 4.5. ■

The above Proposition can be used to prove the following result.

THEOREM 4.7. – *Let X be a Hilbert $\mathcal{A}$-bimodule with $\mathcal{A}$ X-simple.*

a) If X is full and one of the conditions i) *or* ii) *of* 4.5 *is satisfied and furthermore $\Gamma(X) = \mathbb{T}$ then $C^*(X)$ is simple.*

b) If ii) *of* 4.5 *is satisfied and $\widehat{\Gamma}(X) = \mathbb{T}$ then $\mathcal{O}_{X_\mathcal{A}}$ is simple.*

PROOF. – By Lemma 8.11.11 of [26] it suffices to check that our assumptions in a) and b) imply primeness and $\mathbb{T}$-simplicity of $C^*(X)$ and $\mathcal{O}_{X_\mathcal{A}}$ respectively, and this follows from Proposition 4.6 and Corollary 4.5. ■

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