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Solvability of Two Stationary Free Boundary Problems for the Navier-Stokes Equations.

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Sunto. – *Si studiano due problemi con frontiera libera per equazioni stazionarie di Navier-Stokes: il problema del movimento di un liquido viscoso incomprimibile generato dalla rotazione di una sbarra rigida immersa nel liquido con velocità angolare assegnata e il problema della fuoriuscita di un liquido da un tubo circolare nello spazio libero. Si assegna l'angolo di contatto tra la frontiera libera e la superficie del tubo e, nel secondo problema, il flusso totale del liquido attraverso l'apertura del tubo. Si dimostra che, nel caso di flusso totale piccolo (oppure della velocità di rotazione piccola), questi problemi possiedono delle soluzioni uniche assi-simmetriche appartenenti ad alcuni spazi di Hölder con peso.*

1. – Introduction.

The present paper is concerned with two stationary free boundary problems for the Navier-Stokes equations: the problem governing the motion of a viscous incompressible liquid generated by a slow rotation of a rod immersed into the liquid and the problem on the effluence of the liquid out of a circular tube. A common feature of these two problems is the fact that their solutions possess an axial symmetry, and the proof of their solvability relies on the analysis of the Stokes problem in the half-space with boundary conditions of the same type as on a free surface.

Here is the formulation of the problems.

1. *Problem* 1.

Let V be a domain of revolution of the curve L about x_3-axis where L is a union of the straight line $L' = \{x_1 = d_0,\, x_2 = 0,\, x_3 > -m_1\}$ and of a smooth bounded curve L'' located in the domain $0 \leqslant x_1 \leqslant d_0$, $x_2 = 0$, $-(m_1 + m_2) < x_3 < -m_1$ with the endpoints $(d_0, 0, -m_1)$ and $(0, 0, -m_1 - m_2)$ (m_1, m_2 are positive numbers). It is assumed that V is rotating about x_3-axis with a small angular velocity ε. A viscous incompressible liquid occupies an infinite domain $\Omega \equiv \Omega_1$ bounded by a free (unknown) surface $\Gamma = \{x_3 = h(|x'|),\, |x'| > d_0\}$,

$h(d_0) > -m_1$, and by a part Σ of ∂V contained in the domain $x_3 < h(d_0)$. The liquid is subject to a constant force of gravity directed along the vector $-\boldsymbol{e}_3 = (0, 0, -1)$. It is required to find the velocity vector field $v(x) = (v_1(x), v_2(x), v_3(x))$ and a scalar pressure $p(x)$ satisfying in Ω_1 the Navier-Stokes equations

$$-\nu\nabla^2\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = -g_0\boldsymbol{e}_3, \qquad \nabla\cdot\boldsymbol{v} = 0\,, \tag{1.1}$$

the boundary conditions

$$\begin{cases} \boldsymbol{v}|_{\Sigma} = \varepsilon\boldsymbol{e}_{\varphi}\,|x'\,| = \varepsilon(-x_2, x_1, 0)\,, \\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma} = 0\,, \qquad T\boldsymbol{n} - \sigma H\boldsymbol{n}|_{\Gamma} = 0\,, \\ \dfrac{\nabla'\,h\cdot\boldsymbol{e}_r}{\sqrt{1+|\nabla'\,h|^2}}\Bigg|_{|x'\,|=d_0} = -\cos\vartheta\,, \end{cases} \tag{1.2}$$

and the conditions at infinity

$$\begin{cases} \boldsymbol{v}(x)\to 0\,, \qquad p(x) + g_0x_3 \to 0 \qquad (|x|\to\infty)\,, \\ h(|x'\,|)\to 0\,, \qquad (|x'\,|\to\infty)\,. \end{cases} \tag{1.3}$$

Here ν, σ, g_0 are positive constants (coefficient of viscosity, of surface tension, and gravitational constant, respectively), $x' = (x_1, x_2)$, $|x'\,| = \sqrt{x_1^2 + x_2^2}$, $\boldsymbol{e}_r = \boldsymbol{x}'/|x'\,|$, $\boldsymbol{e}_{\varphi} = (-x_2, x_1, 0)/|x'\,|$, ϑ is a contact angle, i.e. the angle between Σ and Γ at the points of the contact of these surfaces which assumes an arbitrary value from the interval $(0, \pi)$, H is twice the mean curvature of Γ defined by the formula

$$H = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\frac{h_{x_i}}{\sqrt{1+|\nabla'\,h|^2}} = \frac{1}{r}\frac{d}{dr}\frac{rh_r'}{\sqrt{1+h_r'^2}} \tag{1.4}$$

T is the stress tensor: $T = -pI + \nu S(\boldsymbol{v})$, $S(\boldsymbol{v})$ is twice the strain tensor with the elements $S_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_i$.

2. *Problem* 2.

The domain $\Omega \equiv \Omega_2$ occupied by the liquid is a union of the half-cylinder $\Omega_- = \{|x'\,| < d_0, x_3 < 0\}$, of the aperture $S = \{|x'\,| < d_0, x_3 = 0\}$ at the end of the cylinder and of the jet Ω_+ bounded by S and by a free surface Γ which contacts the lateral surface Σ of Ω_- along the line $M = \partial S = \{|x'\,| = d_0, x_3 = 0\}$ and extends to infinity. The free boundary is a surface of revolution about x_3-axis of the line Γ' given by

$$r = |x'\,| = r(s)\,, \qquad x_3 = x_3(s) \tag{1.5}$$

(s is the arclength of Γ' counted from the point $x_3 = 0$, $r = d_0$). The problem consists in the determination of this surface, i.e. of $r(s)$ and $x_3(s)$, as well as of $\boldsymbol{v}(x) = (v_1, v_2, v_3)$ and $p(x)$ satisfying the relations

$$-\nu\nabla^2\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = 0\,, \qquad \nabla\cdot\boldsymbol{v} = 0\,, \qquad (x\in\Omega_2)\,,$$

$$\boldsymbol{v}\,|_\Sigma = 0\,, \qquad \boldsymbol{v}\cdot\boldsymbol{n}\,|_\Gamma = 0\,, \qquad T\boldsymbol{n} - \sigma H\boldsymbol{n}\,|_\Gamma = 0\,,$$

$$\int_S v_3\,dx' = \varepsilon\,,$$

$$\boldsymbol{v}(x) - \boldsymbol{v}_-(x) \to 0\,, \qquad p(x) - p_-(x) - \overline{p} \to 0\,, \qquad (x_3 \to -\infty)\,,$$

$$\boldsymbol{v}(x) \to 0\,, \qquad p(x) \to 0\,, \qquad (|x| \to \infty\,, x_3 > 0)\,,$$

where $\boldsymbol{v}_-(x) = (2\varepsilon/\pi d_0^2)(1 - |x|^2/d_0^2)$, $p_-(x) = -(8\nu\varepsilon/\pi d_0^4)\,x_3$ is a Poiseuille flow and $\overline{p}$ is a constant which is not given a-priori. In addition, there is prescribed a contact angle ϑ, i.e. the angle between Σ and Γ at the contact line M, or, which is the same, the angle $\alpha(s)$ between the tangential vector to the line Γ' and the plane $x_3 = 0$ for $s = 0$:

$$\alpha(0) = \frac{3\pi}{2} - \vartheta \qquad \left(\alpha(0) \in (0, \pi),\, \vartheta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right).$$

The angle $\alpha(s)$ is related to $x_3(s)$ and $r(s)$ by the formulas

$$r'(s) = \cos\alpha(s)\,, \qquad x_3'(s) = \sin\alpha(s)$$

and the mean curvature of Γ is defined by

$$H = H_0 - \frac{\sin\alpha(s)}{r(s)} = -\alpha'(s) - \frac{\sin\alpha(s)}{r(s)} \tag{1.6}$$

where $H_0 = -\alpha'(s)$ is the curvature of the line Γ'. It is clear that this line and, as a consequence, the surface Γ, is completely determined by $\alpha(s)$, since

$$r(s) = d_0 + \int_0^s \cos\alpha(t)\,dt\,, \qquad x_3(s) = \int_0^s \sin\alpha(t)\,dt\,.$$

In both problems the free boundary Γ is noncompact but the domain Ω_1 has only one, and Ω_2 has two «exits to infinity» (by an «exit to infinity» of the domain Ω a connected unbounded component of $\Omega^R = \Omega\backslash B_R(0)$, $B_R(0) \equiv \{|x| < R\}$ is meant). In the second case, one of the «exits» has a form of a cylinder and another «exit» is close to the half-space.

Let us introduce basic weighted spaces of functions and vector fields in which we are going to work. The choice of weight functions is determined by the behaviour of solutions of the above problems for large $|x|$ in the «exits» of different geometrical structure and, in addition, by possible singularity of solutions at the contact line $M = \overline{\Sigma} \cap \overline{\Gamma}$. Let l be a positive non-integral number, $s \in (0, l]$, $b > 0$. By $C^l_s(\Omega_1, b)$ we mean the space of functions (or vector fields) given in Ω_1 and having a finite norm

$$(1.7) \qquad |u|_{C^l_s(\Omega_1, b)} = \sum_{0 \le |j| < l} \sup_{\Omega_1} \varrho(x, b + |j|, |j| - s)\, |D^j u(x)| +$$

$$\sum_{|j| = [l]} \sup_{x \in \Omega_1} \varrho(x, b + l, l - s) \sup_{y \in K(x)} |x - y|^{[l] - l} |D^j u(x) - D^j u(y)| + |u|_{C^s(\Omega_1)}$$

where $C^s(\Omega_1)$ is a standard Hölder space of functions (or the space of s times continuously differentiable functions in the case of integral s), $K(x) = \{y \in \Omega_1 : |x - y| \le (1/2)\, \varrho(x, 1, 1)\}$ and

$$\varrho(x, b, m) = \begin{cases} |x|^b, & \text{if } |x| > 4 d_0, \\ \operatorname{dist}(x, M)^{\max(m, 0)}, & \text{if } \operatorname{dist}(x, M) \le d_0/2; \end{cases}$$

at all the other points $x \in \Omega_1$ $\varrho(x, b, m)$ assumes strictly positive values. This space will be used also for $s < 0$, in which case the last term in (1.7) should be omitted.

To consider Problem 2, we need the spaces of functions and vector fields whose elements decay as power functions as $|x| \to \infty$, $x_3 > 0$, and exponentially as $|x| \to \infty$, $x_3 < 0$. Let $a \ge 0$ and l, s as above. We define $C^l_{s,a}(\Omega_2, b)$ (in the case $s \in (0, l]$) as the space of functions with finite norm

$$|u|_{C^l_{s,a}(\Omega_2, b)} = \sum_{0 \le |j| < l} \sup_{\Omega_2} \varrho(x, a, b + |j|, |j| - s)\, |D^j u(x)| +$$

$$\sum_{|j| = [l]} \sup_{x \in \Omega_2} \varrho(x, a, b + l, l - s) \sup_{y \in K(x)} |x - y|^{[l] - l} |D^j u(x) - D^j u(y)| + |u|_{C^s(\Omega_2)}$$

where $K(x) = \{y \in \Omega_2 : |x - y| \le \varrho(x, 0, 1, 1)/2\}$ and

$$\varrho(x, a, b, m) = \begin{cases} e^{a|x_3|}, & \text{if } x_3 < -2 d_0, \\ |x|^b, & \text{if } |x| > 4 d_0, \quad x_3 > 0, \\ \operatorname{dist}(x, M)^{\max(m, 0)}, & \text{if } \operatorname{dist}(x, M) \le d_0/2\,; \end{cases}$$

at all the other points $x \in \Omega_2$ the function ϱ is strictly positive: $\varrho(x, a, b, m) \ge \varrho_0 > 0$. If $s < 0$, then the last term in the norm should be omitted.

To characterize regularity properties of the free boundary, we need the spaces of functions $u(r)$ given at an infinite interval $J_0 = (0, \infty)$ or $J_{d_0} = (d_0, \infty)$, decaying like power functions as $r \to \infty$ and, possibly, having certain singularities at the left end of the interval ($r = 0$ or $r = d_0$). Let $J_\xi = (\xi, \infty)$,

$s \in (0, l]$ and let $C^l_s(J_\xi, b)$ be the space of functions given on J_ξ with the norm

$$|u|_{C^l_s(J_\xi, b)} = \sum_{j=0}^{[l]} \sup_{r>\xi} \varrho_1(r, b+|j|, |j|-s)\,|D^j u(r)| +$$

$$\sup_{r>\xi} \varrho_1(r, b+l, l-s) \sup_{|r-r'|<|r-\xi|/2} |r-r'|^{[l]-l}\,|D^{[l]}u(r)-D^{[l]}u(r')|+|u|_{C^s(J_\xi)}$$

where

$$\varrho_1(r, b, m) = \begin{cases} r^b, & \text{if } r > \xi+1\,, \\ (r-\xi)^{\max(m,\,0)}, & \text{if } r \leqslant \xi+1\,. \end{cases}$$

Finally, by $\widetilde{C}^{l+3}_{s+1}(J_{d_0}, 3)$ we mean the space of functions given in J_{d_0} and such that $D^j u \in C^{l+1}_{s-1}(J_\xi, 3)$, $j = 0, 1, 2$. The norm in this space can be defined by the formula

$$|u|_{\widetilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} = |u|_{C^{l+1}_{s-1}(J_{d_0}, 3)} + \sup_{J_{d_0}} \varrho_1(r, [l]+4, 2+[l]-s)\,|D^{[l]+2}u(r)| +$$

$$\sup_{J_{d_0}} \varrho_1(r, [l]+4, 3+[l]-s)\,|D^{[l]+3}u(r)| +$$

$$\sup_{r\in J_{d_0}} \varrho_1(r, 4+l, 3+l-s) \sup_{|r'-r|\leqslant r/2} |r'-r|^{[l]-l}\,|D^{[l]+3}u(r)-D^{[l]+3}u(r')|\,.$$

This space differs from the space $C^{l+3}_{s+1}(J_{d_0}, 3)$ by the behaviour at infinity of the derivatives $D^{[l]+2}u(r)$ and $D^{[l]+3}u(r)$ of its elements: the rate of decay of these derivatives is the same as the rate of decay of $D^{[l]+1}u(r)$.

In this paper we deal mainly with axisymmetrical functions and vector fields. For functions this means that they do not depend on the angle φ of rotation about the axis of symmetry, and for vector fields this means that their cylindrical components are independent of φ.

Before presenting main results of the paper, we say a few words about the rest state ($\varepsilon = 0$). Then $\boldsymbol{v}(x) = 0$, and the free boundary is defined by the equations

$$x_3 = h_0(r) \text{ (Problem 1)},$$

$$r = r_0(s), \quad x_3 = x_{03}(s) \text{ (Problem 2)},$$

where $r_0(s) = d_0 + \int_0^s \cos\alpha_0(t)\,dt$, $x_{03}(s) = \int_0^s \sin\alpha_0(t)\,dt$. The functions $h_0(s)$

and $\alpha_0(s)$ satisfy the relations

$$
\left\{
\begin{aligned}
&\frac{\sigma}{r}\frac{d}{dr}\frac{rh_0'(r)}{\sqrt{1+h_0'^2(r)}} - g_0 h_0(r) = 0\,, \\
&\frac{h_0'(r)}{\sqrt{1+h_0'^2(r)}}\Bigg|_{r=d_0} = -\cos\vartheta\,, \qquad h_0(r)\to 0\ \ (r\to\infty)\,,
\end{aligned}
\right.
\tag{1.8}
$$

$$
\left\{
\begin{aligned}
&\alpha_0'(s) + \frac{\sin\alpha_0}{r_0} = 0\,, \\
&\alpha_0(0) = \frac{3\pi}{2} - \vartheta\,, \qquad \alpha_0\to 0\ \ (s\to\infty)\,.
\end{aligned}
\right.
\tag{1.9}
$$

The unique solvability of problem (1.8) for arbitrary $\vartheta \in (0, \pi]$ was established by W. E. Johnson and L. M. Perko [5] (we observe that $h_0(r)$ tends to zero at infinity exponentially). Problem (1.9) can be solved explicitly. First of all, it is easy to see that

$$\frac{d}{ds} r_0(s)\sin\alpha_0(s) = \cos\alpha_0(s)[\sin\alpha_0(s) + r_0(s)\,\alpha_0'(s)] = 0\,,$$

hence,

$$r_0(s)\sin\alpha_0(s) = d_0\sin\alpha_0(0) \equiv c_0$$

which implies

$$\left|\frac{dr_0}{ds}\right| = |\cos\alpha_0(s)| = \frac{1}{r_0(s)}\sqrt{r_0^2(s) - c_0^2}\,.$$

Integration of this equation gives

$$r_0(s) = \sqrt{(s + d_0\cos\alpha_0(0))^2 + d_0^2\sin^2\alpha_0(0)}\,,$$

hence,

$$x_{03}(s) = d_0\sin\alpha_0(0)\log\Big(s + d_0\cos\alpha_0(0) + \sqrt{(s + d_0\cos\alpha_0(0))^2 + d_0^2\sin^2\alpha_0(0)}\Big)\,.$$

If $\alpha_0(0) \in (0, \pi/2]$, then $r_0(s)$ is an increasing function; in the case $\alpha_0(0) \in (\pi/2, \pi)$ $r_0(s)$ decreases for $s \in (0, -d_0\cos\alpha_0(0))$ up to the value $r_{\min} = d_0\sin\alpha_0(0)$ and increases for $s > -d_0\cos\alpha_0(0)$ without limits.

We can find r_0 as a function of $x_{03} \equiv z_0$. Indeed, the equation

$$\frac{dr_0}{dz_0} = \frac{dr_0}{ds}\left(\frac{dz_0}{ds}\right)^{-1} = \frac{\cos\alpha_0}{\sin\alpha_0} = \pm\frac{1}{c_0}\sqrt{r_0^2 - c_0^2} = \pm\sqrt{\left(\frac{r_0}{c_0}\right)^2 - 1}$$

implies

$$r_0 = c_0 \cosh\frac{z_0 - c_1}{c_0}\ ,$$

where c_1 is determined by $\sin\alpha_0(0)\cosh c_1 c_0^{-1} = 1$. We see that Γ' is a chain line.

Main results of the paper are contained in the following two theorems.

THEOREM 1.1. – *If ε is small enough, then Problem* 1 *has an isolated axisymmetrical solution $(h, \boldsymbol{v}, p)$ possessing the following properties: $h \in \tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)$, $\boldsymbol{v} \in C^{l+2}_s(\Omega_1, 2)$, $p + gx_3 \in C^{l+1}_{s-1}(\Omega, 3)$, and*

(1.10) $$|\boldsymbol{v}|_{C^{l+2}_s(\Omega_1,2)} + |p + gx_3|_{C^{l+1}_{s1}(\Omega_1,3)} + |h - h_0|_{\tilde{C}^{l+3}_{s+1}(J_{d_0},3)} \leqslant c|\varepsilon|\ .$$

Here l is an arbitrary positive non-integer, and s satisfies the conditions $s \leqslant l+2$, $s \neq 1$, $s \in (0, \min(\pi/2\vartheta, s_0))$ where $s_0 = \Re\lambda$ and λ is a root of a transcendental equation

(1.11) $$\sin 2\lambda\vartheta = \lambda \sin 2\vartheta$$

with a minimal positive real part different from 1.

THEOREM 1.2. – *If ε is sufficiently small, then Problem* 2 *has an isolated axisymmetrical solution $\alpha, \boldsymbol{v}, p$ (and the angular component of the velocity vanishes) possessing the following properties: $\alpha \in C^{l+2}_s(J_0, 1)$, $\boldsymbol{v} \in C^{l+2}_{s,0}(\Omega_2, 2)$, $\nabla p \in C^{l}_{s-2,0}(\Omega_2, 4)$, and*

(1.12) $$|\boldsymbol{v}|_{C^{l+2}_{s,0}(\Omega_2,2)} + |\nabla p|_{C^{l}_{s-2,0}(\Omega_2,4)} + |\alpha - \alpha_0|_{C^{l+2}_s(J_0,1)} \leqslant c|\varepsilon|$$

where $s \in (0, s_0)$, s_0 is the same as in Theorem 1.1.

REMARK. – Elementary analysis of equation (1.11) (see, for instance, [13]) shows that s_0 is a decreasing function of ϑ, in particular,

$$s_0 = 1/3 \quad \text{for } \vartheta = 3\pi/2\ ,$$
$$s_0 = 1/2 \quad \text{for } \vartheta = \pi\ ,$$
$$s_0 \to 1\ , \quad \text{as } \vartheta \to \vartheta_1\ .$$

where $\vartheta_1 \in (\pi/2, 3\pi/2)$ is a root of the function $\tan 2\vartheta - 2\vartheta$, and s_0 grows without limits, as $\vartheta \to 0$. In Problem 1, $\vartheta \in (0, \pi)$, and in Problem 2, $\alpha(0) \in (0, \pi)$ and $\vartheta \in (\pi/2, 3\pi/2)$, hence, s_0 is a real number from the interval $(1/3, 2)$. The case of limiting values of the contact angle $\vartheta = 0$ and $\vartheta = \pi$ in Problem 1 is excluded only for technical reasons; it can be treated in the same way as in [19].

Problem 1 was considered by D. Sattinger [11] (in the case $\vartheta = \pi/2$, when the solution has no singularity at the contact line) and by I. Mogilevskii [8] for $\vartheta \in (0, \pi)$. In both papers it was assumed that the liquid was contained in a bounded container, so the domain Ω_1 was bounded. In the case of unbounded Ω_1 it is necessary to analyse the behaviour of the solution at infinity. We make use of the axial symmetry of solution and show that this condition guarantees the decay of the velocity vector field of the order $|x|^{-2}$ (symmetrical solutions of the Stokes and Navier-Stokes equations were considered also in the papers [4,17,18]).

Problem 2 was studied in the paper [17] where, in particular, Theorem 1.2 was formulated (in the case when the domain $\Omega_- = \{x \in \Omega_2 : x_3 < 0\}$ is a half-space but not a cylinder, and $\vartheta \in (3\pi/2, 2\pi)$). Main attention in [17] is given to the two-dimensional case, in particular, it is proved that, in contrast to the three-dimensional case, the jet may have a form close to an infinite sector or to a strip with the aperture dependent of the value of the contact angle at the contact set M consisting of two contact points $x_\pm = (\pm d_0, 0)$. In addition, it is shown in [18] that in the three-dimensional axisymmetrical case there exists a solution of Problem 2 with the jet $\Omega_+ = \{x \in R^3 : x_3 > 0\}$ close to a cylinder; in this case the pressure assumes a certain non-zero value at infinity.

Theorems 1.1 and 1.2 reduce to the contraction mapping principle in weighted Hölder spaces. It is convenient to write Problems 1 and 2 in a slightly different equivalent form. We separate tangential and normal parts in dynamic boundary condition $\boldsymbol{Tn} - \sigma H\boldsymbol{n} = 0$, introduce a new pressure function $p + gx_3$ instead of p in Problem 1, and take account of formulas (1.4), (1.6) for the mean curvature of Γ. This makes it possible to write Problems 1 and 2 as follows:

$$
\begin{cases}
-\nu\nabla^2\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{v} = 0, \qquad x \in \Omega_1, \\
\boldsymbol{v}|_\Sigma = \varepsilon \boldsymbol{e}_\varphi |x'|, \\
\boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0, \qquad \boldsymbol{\tau}\cdot S(\boldsymbol{v})\boldsymbol{n}|_\Gamma = 0, \\
\boldsymbol{v}(x) \to 0, \qquad p(x) \to 0, \qquad (|x| \to \infty);
\end{cases}
\tag{1.13}
$$

$$
\begin{cases}
\dfrac{\sigma}{r}\dfrac{d}{dr}\dfrac{rh'(r)}{\sqrt{1 + h_r'^2}} - g_0 h = \boldsymbol{n}\cdot T(\boldsymbol{v}, p)\boldsymbol{n}|_{x_3 = h(r)}, \\
\dfrac{h'(r)}{\sqrt{1 + h_r'^2}}\Big|_{r = d_0} = -\cos\vartheta, \qquad h(r) \to 0, \qquad (r \to \infty);
\end{cases}
\tag{1.14}
$$

$$(1.15)\quad \begin{cases} -\nu\nabla^2 \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} + \nabla p = 0\,, \qquad \nabla\cdot\boldsymbol{v} = 0\,, \qquad x\in\Omega_2\,, \\ \boldsymbol{v}\,|_{\Sigma} = 0\,, \qquad \boldsymbol{v}\cdot\boldsymbol{n}\,|_{\Gamma} = 0\,, \qquad \boldsymbol{\tau}\cdot S(\boldsymbol{v})\,\boldsymbol{n}\,|_{\Gamma} = 0\,, \\ \int_S v_3(x',0)\,dx' = \varepsilon\,, \\ \boldsymbol{v}(x) - \boldsymbol{v}_-(x) \to 0\,, \qquad p(x) - p_-(x) - \overline{p} \to 0\,, \qquad (x_3 \to -\infty)\,, \\ \boldsymbol{v}(x) \to 0\,, \qquad p(x) \to 0\,, \qquad (|x| \to \infty,\, x_3 > 0)\,; \end{cases}$$

$$(1.16)\quad \begin{cases} \alpha'(s) + \dfrac{\sin\alpha(s)}{r(s)} = -\sigma^{-1}\boldsymbol{n}\cdot T(\boldsymbol{v},\,p)\boldsymbol{n}\,|_{x_3 = x_3(s),\; r = r(s)}\,, \\ \alpha(0) = \dfrac{3\pi}{2} - \vartheta\,, \qquad \alpha(s)\to 0\,, \qquad (s\to\infty)\,; \end{cases}$$

here

$$r(s) = d_0 + \int_0^s \cos\alpha(s')\,ds'\,, \qquad x_3 = \int_0^s \sin\alpha(s')\,ds'\,.$$

By $\boldsymbol{\tau}$ in the condition $\boldsymbol{\tau}\cdot S(\boldsymbol{v})\boldsymbol{n}\,|_{\Gamma} = 0$ we mean arbitrary tangential vector on Γ; in fact, it suffices to satisfy this condition for two linearly independent tangential vectors: for $\boldsymbol{\tau} = \boldsymbol{\tau}^{(1)}$ with the components $\tau_r^{(1)} = n_3$, $\tau_3^{(1)} = -n_r$ (τ_r and n_r are radial components of $\boldsymbol{\tau}$ and $\boldsymbol{n}$), and for $\boldsymbol{\tau} = \boldsymbol{\tau}^{(2)} = \boldsymbol{e}_{\varphi} = (-x_2,\, x_1,\, 0)\,|x'|^{-1}$.

The paper is organized as follows. In § 2 a model problem in half-space is considered. § 3 is devoted to auxiliary linear problems for $\boldsymbol{v}$ and p in the domains Ω_1 and Ω_2 with given Γ. In § 4 linearized equation for the free boundary is analysed. Finally, in § 5 the proof of Theorems 1.1 and 1.2 is presented.

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2. – Model problem in the half-space.

Consider in the half-space R^3_+ $\{x_3 > 0\}$ the boundary value problem

$$(2.1)\quad \begin{cases} -\nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x)\,, \qquad \nabla\cdot\boldsymbol{v} = g(x)\,, \\ v_3\,|_{x_3=0} = b(x')\,, \qquad S_{j3}(\boldsymbol{v}) \equiv \dfrac{\partial v_j}{\partial x_3} + \dfrac{\partial v_3}{\partial x_j}\Big|_{x_3=0} = d_j(x')\,, \qquad j = 1, 2\,, \end{cases}$$

where $x' = (x_1, x_2)$ and $\boldsymbol{f}(x)$, $g(x)$, $b(x')$, $d_j(x')$, $j = 1, 2$, are given functions rapidly decaying at infinity.

The solution of this problem can be written explicitly as the sum of some volume and surface potentials. Let $U(x)$ be the fundamental matrix of solutions of the Stokes system of equations. It is 4×4 matrix with the elements

$$U_{ij}(z) = \frac{1}{8\pi\nu}\left(\frac{\delta_{ij}}{|z|} + \frac{z_i z_j}{|z|^3}\right), \qquad i, j = 1, 2, 3,$$

$$U_{i4}(z) = U_{4i}(z) = \frac{1}{4\pi}\frac{z_i}{|z|^3}, \qquad i = 1, 2, 3,$$

$$U_{44}(z) = \nu\delta(x),$$

where $\delta(x)$ is the Dirac δ-function. The solution of the Stokes system (2.1_1) in the whole space R^3 can be written in the form

$$v_i(x) = \sum_{j=1}^{3}\int_{R^3} U_{ij}(x-y) f_j(y)\,dy + \int_{R^3} U_{i4}(x-y)\,g(y)\,dy, \qquad i = 1, 2, 3,$$

$$p(x) = \sum_{j=1}^{3}\int_{R^3} U_{4j}(x-y) f_j(y)\,dy + \nu g(x).$$

The representation formula for the solution of problem (2.1) involves the Green matrix $\mathcal{G}(x, y) = (G_{st}(x, y))_{s, t=1, 2, 3, 4}$ and the Poisson kernels $P_{sj}(x)$ $(s = 1, 2, 3, 4, j = 1, 2, 3)$.

PROPOSITION 2.1. – *The solution of problem* (2.1) *is given by the formula*

$$v_i(x) = \sum_{j=1}^{3}\int_{R^3_+} G_{ij}(x, y) f_j(y)\,dy + \int_{R^3_+} G_{i4}(x, y)\,g(y)\,dy + \tag{2.2}$$

$$\sum_{\mu=1}^{2}\int_{R^2} P_{i\mu}(x'-y', x_3)\,d_\mu(y')\,dy' + \int_{R^2} P_{i3}(x'-y', x_3)\,b(y')\,dy', \qquad i=1, 2, 3,$$

$$p(x) = \sum_{j=1}^{3}\int_{R^3_+} G_{4j}(x, y) f_j(y)\,dy + \nu g(x) + \tag{2.3}$$

$$\sum_{\mu=1}^{2}\int_{R^2} P_{4\mu}(x'-y', x_3)\,d_\mu(y')\,dy' + \int_{R^2} P_{43}(x'-y', x_3)\,b(y')\,dy'$$

where

$G_{st}(x, y) = U_{st}(x-y) + U^*_{st}(x, y), \qquad s, t = 1, 2, 3, 4,$

$U^*_{s\gamma}(x, y) = U_{s\gamma}(x-y^*), \qquad s = 1, 2, 3, 4 \qquad \gamma = 1, 2, \qquad y^* = (y_1, y_2, -y_3),$

$U^*_{s3}(x, y) = -U_{s3}(x-y^*), \qquad U^*_{j4}(x, y) = U_{j4}(x-y^*), \qquad j = 1, 2, 3,$

$U^*_{44}(x, y) = 0,$

$$P_{s\gamma}(x) = -2\nu U_{s\gamma}(x), \qquad s = 1, 2, 3, 4, \qquad \gamma = 1, 2,$$

$$P_{k3}(x) = \frac{3}{2\pi}\frac{x_3^2 x_k}{|x|^5}, \qquad k = 1, 2, 3,$$

$$P_{43}(x) = -\frac{\nu}{\pi}\left(\frac{1}{|x|} - \frac{3x_3^2}{|x|^5}\right).$$

PROOF. – The Green matrix $\mathcal{G}$ for the problem (2.1) was constructed in [7, 20] where also the justification of the formula (2.2) in the case $d_1 = d_2 = b = 0$ can be found. Consider the case $\boldsymbol{f} = 0$, $g = 0$. Since $P_{s\gamma}$ are proportional to $U_{s\gamma}$, and P_{s3} is a kernel of the K. Odquist double layer potential (see [10]), it is clear that the sum of potentials with Poisson kernels P_{sk} in (2.2), (2.3) satisfies the homogeneous Stokes system, and the integrals

$$u_i(x) = \int_{R^2} P_{i3}(x' - y', x_3)\, b(y')\, dy'$$

satisfy the boundary conditions

$$u_i(x)\,|_{x_3 = 0} = \delta_{i3}\, b(x').$$

In addition, we have

$$\int_{R^2} P_{3\mu}(x' - y', x_3)\, d_\mu(y')\, dy' =$$

$$-\frac{x_3}{8\pi\nu}\int_{R^2}\frac{x_\mu - y_\mu}{(|x' - y'|^2 + x_3^2)^{3/2}}\, d_\mu(y')\, dy' \to 0 \qquad (x_3 \to 0).$$

It remains to show that

$$S_{\mu 3}(\boldsymbol{v})\,|_{x_3 = 0} = d_\mu, \qquad \mu = 1, 2.$$

Making use of the relations

$$U_{i\mu}(x) = \frac{1}{8\pi\nu}\left(\frac{2\delta_{i\mu}}{|x|} - \frac{\partial}{\partial x_i}\frac{x_\mu}{|x|}\right),$$

$$\frac{\partial}{\partial x_3}\frac{x_3^2 x_i}{|x|^5} = -\sum_{\mu=1}^{2}\frac{\partial}{\partial x_\mu}\frac{x_3 x_\mu x_i}{|x|^5},$$

we obtain for $i=1,2$

$$\frac{\partial v_i}{\partial x_3} = \sum_{\mu=1}^{2} \int_{R^2} \left(\frac{x_3 \delta_{i\mu} d_\mu(y')}{2\pi(|x'-y'|^2+x_3^2)^{3/2}} - \frac{x_3(x_\mu - y_\mu)}{4\pi(|x'-y'|^2+x_3^2)^{3/2}} \frac{\partial d_\mu}{\partial y_i} \right) dy' -$$

$$\frac{3}{2\pi} \sum_{\mu=1}^{2} \int_{R^2} \frac{x_3(x_i-y_i)(x_\mu-y_\mu)}{(|x'-y'|^2+x_3^2)^{5/2}} \frac{\partial b}{\partial y_\mu} dy' \to d_i(x') - \frac{\partial b(x')}{\partial x_i}$$

as $x_3 \to 0$, hence,

$$\left. \frac{\partial v_i}{\partial x_3} + \frac{\partial v_3}{\partial x_i} \right|_{x_3=0} = d_i(x'), \qquad i=1,2,$$

q.e.d. The proposition is proved.

PROPOSITION 2.2. – *Assume that* $\boldsymbol{f}$ *and* $\boldsymbol{d}' \equiv (d_1, d_2)$ *are axisymmetrical and that the norms*

$$\|\boldsymbol{f}\|_{4+\beta} = \sup_{R_+^3} (1+|x|)^{4+\beta} |\boldsymbol{f}(x)|, \qquad \|g\|_{3+\beta} = \sup_{R_+^3} (1+|x|)^{3+\beta} |\varrho(x)|,$$

$$\|\boldsymbol{d}'\|_{3+\beta} = \sup_{R^2} (1+|x'|)^{3+\beta} |\boldsymbol{d}'(x)|, \qquad \|b\|_{2+\beta} = \sup_{R^2} (1+|x'|)^{2+\beta} |b(x')|,$$

are finite for some $\beta \in (0,1)$. *Then*

(2.4) $$\|\boldsymbol{v}\|_2 = \sup_{R_+^3} (1+|x|)^2 |\boldsymbol{v}(x)| \leqslant c(\|\boldsymbol{f}\|_{4+\beta} + \|g\|_{3+\beta} + \|\boldsymbol{d}'\|_{3+\beta} + \|b\|_{2+\beta}).$$

PROOF. – Since

$$f_1(x) = f_r \frac{x_1}{|x'|} - f_\varphi \frac{x_2}{|x'|}, \qquad f_2 = f_r \frac{x_2}{|x'|} + f_\varphi \frac{x_1}{|x'|},$$

$$d_1(x') = d_r \frac{x_1}{|x'|} - d_\varphi \frac{x_2}{|x'|}, \qquad d_2 = d_r \frac{x_2}{|x'|} + d_\varphi \frac{x_1}{|x'|},$$

and $\boldsymbol{f}$, $\boldsymbol{d}'$ are axisymmetrical, we have

$$\int_{R_+^3} f_1\, dx = \int_{R_+^3} f_2\, dx = 0, \qquad \int_{R^2} d_1\, dx' = \int_{R^2} d_2\, dx' = 0.$$

It is also clear that

$$\int_{R_+^3} x_i f_3(x)\, dx = \int_{R_+^3} x_3 f_i(x)\, dx = 0, \qquad i=1,2.$$

Taking into account that $G_{s3}(x, 0) = 0$ $(s = 1, 2, 3, 4)$, we can write (2.2) in the form

$$(2.5)\quad v_i(x) = \sum_{k=1}^{3} \int_{R^3_+} [G_{ik}(x, y) - G_{ik}(x, 0)] f_k(y)\, dy + \int_{R^3_+} G_{i4}(x, y)\, g(y)\, dy +$$

$$\sum_{\beta=1}^{2} \int_{R^2} [P_{i\beta}(x' - y', x_3) - P_{i\beta}(x', x_3)]\, d_\beta(y')\, dy' + \int_{R^2} P_{i3}(x' - y', x_3)\, b(y')\, dy' =$$

$$\sum_{j,k=1}^{2} \frac{\partial G_{ik}(x, z)}{\partial z_j}\bigg|_{z=0} \int_{R^3_+} y_j f_k(y)\, dy + \frac{\partial G_{i3}(x, z)}{\partial z_3}\bigg|_{z=0} \int_{R^3_+} y_3 f_3(y)\, dy +$$

$$G_{i4}(x, 0) \int_{R^3_+} g(y)\, dy - \sum_{\beta,\gamma=1}^{2} \frac{\partial P_{i\beta}(x', x_3)}{\partial x_\gamma} \int_{R^2} y_\gamma\, d_\beta\, dy' +$$

$$P_{i3}(x', x_3) \int_{R^2} b(y')\, dy' + v_i'(x)$$

with

$$v_i'(x) = \sum_{k=1}^{3} \int_{R^3_+} \left[G_{ik}(x, y) - G_{ik}(x, 0) - \sum_{m=1}^{3} y_m \frac{\partial G_{ik}(x, z)}{\partial z_m}\bigg|_{z=0} \right] f_k(y)\, dy +$$

$$\int_{R^3_+} [G_{i4}(x, y) - G_{i4}(x, 0)]\, g(y)\, dy +$$

$$\sum_{\beta=1}^{2} \int_{R^2} \left[P_{i\beta}(x' - y', x_3) - P_{i\beta}(x', x_3) + \sum_{\mu=1}^{2} y_\mu \frac{\partial P_{i\beta}(x)}{\partial x_\mu} \right] d_\beta(y')\, dy' +$$

$$\int_{R^2} [P_{i3}(x' - y', x_3) - P_{i3}(x', x_3)]\, b(y')\, dy' \equiv I_1 + I_2 + I_3 + I_4\,, \qquad i = 1, 2, 3\,.$$

Let us estimate the functions $v_i - v_i' \equiv v_i^{(0)}$ and v_i' in the domain $\{|x| > 1, x_3 > 0\}$. It is clear that

$$|v_i^{(0)}(x)| \leqslant c|x|^{-2}(\|\boldsymbol{f}\|_{4+\beta} + \|g\|_{3+\beta} + \|\boldsymbol{d}'\|_{3+\beta} + \|b\|_{2+\beta})\,.$$

To estimate $v_i'(x)$, we split each integral I_k, $k = 1, 2, 3, 4$, into two parts, in which the integration is carried out over the domains $|y| \leqslant |x|/2$ (or $|y'| \leqslant$

$|x|/2)$ and $|y| > |x|/2$ ($|y'| > |x|/2$), respectively, and we make use of the inequalities

$$\left| G_{ik}(x, y) - G_{ik}(x, 0) - \sum_{m=1}^{3} y_m \frac{\partial G_{ik}(x, z)}{\partial z_m}\bigg|_{z=0} \right| \leqslant \frac{c|y|^{\gamma+1}}{|x|^{2+\gamma}},$$

$$|G_{i4}(x, y) - G_{i4}(x, 0)| \leqslant \frac{c|y|^{\gamma}}{|x|^{2+\gamma}}, \qquad |y| < |x|/2,$$

$$\left| P_{i\beta}(x' - y', x_3) - P_{i\beta}(x', x_3) + \sum_{\mu=1}^{2} \frac{\partial P_{i\beta}(x)}{\partial x_\mu} \right| \leqslant \frac{c|y'|^{\gamma+1}}{|x|^{2+\gamma}},$$

$$|P_{i3}(x' - y', x_3) - P_{i3}(x', x_3)| \leqslant \frac{c|y'|^{\gamma}}{|x|^{2+\gamma}}, \qquad |y'| < |x|/2, \qquad \gamma \leqslant 1.$$

We take $\gamma \in (0, \beta)$ and evaluate, for instance, I_1 as follows:

$$|I_1| \leqslant c\left(|x|^{-2-\gamma} \int_{|y| \leqslant |x|/2} |y|^{1+\gamma} |\boldsymbol{f}(y)|\, dy + \right.$$

$$\left. \int_{|y| \geqslant |x|/2} (|x - y|^{-1} + |x|^{-1} + |y||x|^{-2}) |\boldsymbol{f}(y)|\, dy \right) \leqslant$$

$$\leqslant c\|\boldsymbol{f}\|_{4+\beta}\left(|x|^{-2-\gamma} \int_{|y| \leqslant |x|/2} |y|^{1+\gamma}(1 + |y|)^{-4-\beta}\, dy + \right.$$

$$\left. + \int_{|y| \geqslant |x|/2} (|x - y|^{-1} + |x|^{-1} + |y||x|^{-2})(1 + |y|)^{-4-\beta}\, dy \right) \leqslant c|x|^{-2-\gamma}\|\boldsymbol{f}\|_{4+\beta}.$$

Integrals I_2 and I_3 should be treated exactly in the same way; finally,

$$|I_4| \leqslant \int_{|y'| \leqslant |x|/2} |P_{i3}(x' - y', x_3) - P_{i3}(x', x_3)||b(y')|\, dy' +$$

$$\int_{|y'| > |x|/2} (|P_{i3}(x' - y', x_3)| + |P_{i3}(x', x_3)|)|b(y')|\, dy' \leqslant$$

$$\|b\|_{2+\beta}\left(|x|^{-2-\gamma} \int_{|y'| \leqslant |x|/2} |y'|^{\gamma}(1 + |y'|)^{-2-\beta}\, dy' + \right.$$

$$|x|^{-2} \int_{|y'| > |x|/2} (1 + |y'|)^{-2-\beta}\, dy' +$$

$$\left. x_3 \int_{|y'|>|x|/2} \frac{dy'}{(|x'-y'|^2+x_3^2)^{3/2}(1+|y'|)^{2+\beta}} \right) \leqslant c\|b\|_{2+\beta}\, |x|^{-2-\gamma}.$$

Hence,

$$|v_i'(x)| \leqslant c|x|^{-2-\gamma}(\|\boldsymbol{f}\|_{4+\beta} + \|g\|_{3+\beta} + \|\boldsymbol{d}'\|_{3+\beta} + \|b\|_{2+\beta}).$$

It remains to bound $|\boldsymbol{v}(x)|$ for $|x| < 1$. This can be easily done by using formula (2.2). We have

$$|v_i(x)| \leqslant c\Bigg(\|\boldsymbol{f}\|_{4+\beta}\int\limits_{R^3_+}|x-y|^{-1}(1+|y|)^{-4-\beta}dy +$$

$$\|g\|_{3+\beta}\int\limits_{R^3_+}|x-y|^{-2}(1+|y|)^{-3-\beta}dy +$$

$$\|\boldsymbol{d}'\|_{3+\beta}\int\limits_{R^2}(|x'-y'|^2+x_3^2)^{-1/2}(1+|y|)^{-3-\beta}dy' +$$

$$\sup_{R^2}|b(y')|\int\limits_{R^2}|P_{i3}(x'-y', x_3)|\,dy'\Bigg) \leqslant c(\|\boldsymbol{f}\|_{4+\beta} + \|\varrho\|_{3+\beta} + \|\boldsymbol{d}'\|_{3+\beta} + \|b\|_{2+\beta}),$$

which completes the proof of (2.4) and of the proposition.

Now, we pass to the estimates of the solution of (2.1) in weighted Hölder spaces $C^l(R^3_+, b)$ with the norm

$$(2.6)\qquad |u|_{C^l(R^3_+, b)} = \sum_{|j|<l}\sup_{R^3_+}(1+|x|)^{|j|+b}|D^j u(x)| +$$

$$\sum_{|j|=[l]}\sup_{x\in R^3_+}(1+|x|)^{l+b}\sup_{y\in K(x)}|x-y|^{[l]-l}|D^j u(x) - D^j u(y)|$$

where $K(x) = \{y \in R^3_+ : |x-y| \leqslant (1+|x|)/2\}$. The space $C^l(R^2, b)$ is defined in an analogous way.

THEOREM 2.1. – *If* $\boldsymbol{f} \in C^l(R^3_+, 4+\beta)$, $g \in C^{l+1}(R^3_+, 3+\beta)$, $\boldsymbol{d}' \in C^{l+1}(R^2, 3+\beta)$, $b \in C^{l+2}(R^2, 2+\beta)$, $\beta \in (0, 1)$, *and* $\boldsymbol{f}, \boldsymbol{d}'$ *are axisymmetrical, then* $\boldsymbol{v} \in C^{l+2}(R^3_+, 2)$, $p \in C^{l+1}(R^3_+, 3)$, *and*

$$|\boldsymbol{v}|_{C^{l+2}(R^3_+, 2)} + |p|_{C^{l+1}(R^3_+, 3)} \leqslant c(|\boldsymbol{f}|_{C^l(R^3_+, 4+\beta)} + |g|_{C^{l+1}(R^3_+, 3+\beta)} +$$

$$|\boldsymbol{d}'|_{C^{l+1}(R^2, 3+\beta)} + |b|_{C^{l+2}(R^2, 2+\beta)}).$$

PROOF. – Let $A_r = \{x \in R^3_+ : r < 1 + |x| \leqslant 2r\}$, $A_r' = A_r \cap \partial R^3_+ = x' \in R^2 : r < 1+|x| \leqslant 2r\}$, $B_r = A_{r/2} \cup A_r \cup A_{2r}$, $B_r' = B_r \cap \partial R^3_+$. We assume that

$r>1$ and make use of local estimates for problem (2.1),

(2.7) $$[\boldsymbol{v}]^{(l+2)}_{A_r}+[p]^{(l+1)}_{A_r}\leqslant$$

$$c([\boldsymbol{f}]^{(l)}_{B_r}+r^{-l}\max_{B_r}|\boldsymbol{f}(x)|+[g]^{(l+1)}_{B_r}+r^{-l-1}\max_{B_r}|g(x)|+[\boldsymbol{d}']^{(l+1)}_{B'_r}+$$

$$r^{-l-1}\max_{B'_r}|d'(x')|+[b]^{(l+2)}_{B'_r}+r^{-l-2}\max_{B'_r}|b(x')|+r^{-l-2}\max_{B_r}|\boldsymbol{v}(x)|).$$

(the proof of this estimate is given in the Appendix). Multiplying this inequality by r^{l+4}, we easily obtain

$$r^{l+4}([\boldsymbol{v}]^{(l+2)}_{A_r}+[p]^{(l+1)}_{A_r})\leqslant c(|\boldsymbol{f}|_{C^l(R^3_+,\,4)}+|g|_{C^{l+1}(R^3_+,\,3)}+$$
$$|\boldsymbol{d}'|_{C^{l+1}(R^2,\,3)}+|b|_{C^{l+2}(R^2,\,2)}+\|\boldsymbol{v}\|_2)$$

and in virtue of (2.4)

(2.8) $$\sup_{r>1} r^{l+4}([\boldsymbol{v}]^{(l+2)}_{A_r}+[p]^{(l+1)}_{A_r})\leqslant$$

$$c(|\boldsymbol{f}|_{C^l(R^3_+,\,4+\beta)}+|g|_{C^{l+1}(R^3_+,\,3+\beta)}+|\boldsymbol{d}'|_{C^{l+1}(R^2,\,3+\beta)}+|b|_{C^{l+2}(R^2,\,2+\beta)}).$$

Next, we observe that for arbitrary function $q(x)$ with finite norm $\sup\limits_{r>1}(r^{-m-\alpha}[q]^{(\alpha)}_{A_r})$, $\alpha\in(0,1)$, $m>1$, or $\sup\limits_{r>1}(r^{-m-1}\sup\limits_{A_r}|\nabla q(x)|)$ there holds inequality

$$|q(x)|\leqslant|q(x)-q(2x)|+|q(2x)-q(4x)|+\ldots=\sum_{k=1}^{\infty}|q(2^{k-1}x)-q(2^kx)|\leqslant$$

$$\sup_{r>1}(r^{-m-\alpha}[q]^{(\alpha)}_{A_r})\sum_{k=1}^{\infty}\frac{(2^{k-1}|x|)^{\alpha}}{(1+2^{k-1}|x|)^{m+\alpha}}\leqslant c|x|^{-m}\sup_{r>1}(r^{-m-\alpha}[q]^{(\alpha)}_{A_r})$$

or

$$|q(x)|\leqslant\sup_{r>1}(r^{-m-1}\sup_{A_r}|\nabla q(x)|)\sum_{k=1}^{\infty}\frac{2^{k-1}|x|}{(1+2^{k-1}|x|)^{m+1}}\leqslant$$

$$c|x|^{-m}\sup_{r>1}(r^{-m-1}\sup_{A_r}|\nabla q(x)|)$$

for arbitrary $x\in R^3_+$ with $|x|>1$. Applying these inequalities to $\boldsymbol{v}$, p and to their derivatives we evaluate successively $|D^j\boldsymbol{v}(x)|$, $j=[l]+2,\ldots,1$, $|D^kp(x)|$, $|k|=[l],\ldots,0$ and deduce from (2.8), (2.4) the estimate

$$|\boldsymbol{v}|_{C^{l+2}(\widehat{R}^3_+,\,2)}+|p|_{C^{l+1}(\widehat{R}^3_+,\,3)}\leqslant$$

$$c(|\boldsymbol{f}|_{C^l(R^3_+,\,4+\beta)}+|g|_{C^{l+1}(R^3_+,\,3+\beta)}+|\boldsymbol{d}'|_{C^{l+1}(R^2,\,3+\beta)}+|b|_{C^{l+2}(R^2,\,2+\beta)})$$

where $\widehat{R}^3_+ = \{x \in R^3_+ : \ |x| > 1\}$, and the norms in the left-hand side are defined by formulas (2.6) with suprema taken in $\widehat{R}^3_+$ instead of R^3_+. The estimate of the Hölder norms of $\boldsymbol{v}$, p in $R^3_+ \setminus \widehat{R}^3_+$ is a classical result of the theory of elliptic boundary value problems. The theorem is proved.

Formula (2.5) gives asymptotic expansion of the solution of (2.1) for large $|x|$. Since

$$\int_{R^3_+} f_1 y_1\, dy = \int_{R^3_+} f_r y_1^2 \, |y'|^{-1} dy = \pi \int_0^\infty \int_0^\infty f_r(r, y_3)\, r^2\, dr\, dy_3 = F_1 ,$$

$$\int_{R^3_+} f_2 y_1\, dy = \int_{R^3_+} f_\varphi y_1^2 \, |y'|^{-1} dy = \pi \int_0^\infty \int_0^\infty f_\varphi(r, y_3)\, r^2\, dr\, dy_3 = F_2 ,$$

$$\int_{R^3_+} f_2 y_2\, dy = F_1 , \qquad \int_{R^3_+} f_1 y_2\, dy = -F_2 ,$$

and

$$\int_{R^2} d_1 y_1\, dy' = \int_{R^2} d_2 y_2\, dy' = \pi \int_0^\infty d_r(r)\, r^2\, dr = D_1 ,$$

$$\int_{R^2} d_2 y_1\, dy' = -\int_{R^2} d_1 y_2\, dy' = \pi \int_0^\infty d_\varphi(r)\, r^2\, dr = D_2 ,$$

we can write $v_i^{(0)}$ in the form

$$v_i^{(0)} = -F_1 \left(\frac{\partial G_{i1}(x, 0)}{\partial x_1} + \frac{\partial G_{i2}(x, 0)}{\partial x_2} \right) - F_2 \left(\frac{\partial G_{i2}(x, 0)}{\partial x_1} - \frac{\partial G_{i1}(x, 0)}{\partial x_2} \right) +$$

$$\frac{\partial G_{i3}(x, z)}{\partial z_3} \Bigg|_{z=0} \int_{R^3_+} y_3 f_3\, dy + G_{i4}(x, 0) \int_{R^3_+} g\, dy + P_{i3}(x) \int_{R^2} b\, dy' -$$

$$D_1 \left(\frac{\partial P_{i1}(x)}{\partial x_1} + \frac{\partial P_{i2}(x)}{\partial x_2} \right) - D_2 \left(\frac{\partial P_{i2}(x)}{\partial x_1} - \frac{\partial P_{i1}(x)}{\partial x_2} \right) =$$

$$C_1 \boldsymbol{V}_1(x) + C_2 \boldsymbol{V}_2(x) + C_3 \boldsymbol{V}_3(x) ,$$

where

$$V_1 = \frac{x_2 \boldsymbol{e}_1 - x_1 \boldsymbol{e}_2}{|x|^3}, \qquad V_2 = \frac{\boldsymbol{x}}{|x|^3} - \frac{3\boldsymbol{x} x_3^2}{|x|^5}, \qquad V_3 = \frac{3\boldsymbol{x} x_3^2}{|x|^5},$$

$$C_1 = -\frac{F_2 - \nu D_2}{2\pi\nu}, \qquad C_2 = \frac{F_1 - \nu D_1}{4\pi\nu} - \frac{1}{4\pi\nu}\int_{R_+^3} y_3 f_3 \, dy + \frac{1}{2\pi}\int_{R_+^3} g \, dy,$$

$$C_3 = \frac{1}{2\pi}\left(\int_{R^2} b dy' + \int_{R_+^3} g \, dy\right).$$

Hence,

$$\boldsymbol{v}(x) = C_1 V_1(x) + C_2 V_2(x) + C_3 V_3(x) + \boldsymbol{v}'(x). \tag{2.9}$$

The corresponding formula for the pressure has the form

$$p(x) = C_2 P_2(x) + C_3 P_3(x) + p'(x),$$

where $P_2 = -P_3 = 2\nu(1/|x|^3 - 3x_3^2/|x|^5)$.

We have proved that $\boldsymbol{v}'(x) = O(|x|^{-2-\gamma})$, $\gamma \in (0, 1)$. Since $(\boldsymbol{v}', p')$ satisfy relations (2.1) in R_+^3, one can show exactly in the same way as in Theorem 2.1 that $\boldsymbol{v}' \in C^{l+2}(\widehat{R}_+^3, 2+\gamma)$, $p' \in C^{l+1}(\widehat{R}_+^3, 3+\gamma)$.

For the half-space problem with the boundary condition $\boldsymbol{v}|_{x_3=0} = 0$ formula of the type (2.9) was obtained in [2, 3] without any assumption on the symmetry of the solution. It may be shown that this formula does not hold for the solution of problem (2.1), if the data are not axisymmetrical.

3. – Auxiliary linear problems.

In this section we consider linear problems

$$\begin{cases} -\nu\nabla^2 \boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \quad \nabla\cdot\boldsymbol{v} = g(x), \quad x \in \Omega_1, \\ \boldsymbol{v}|_\Sigma = 0, \quad \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0, \quad \boldsymbol{\tau}^{(i)}\cdot S(\boldsymbol{v})\boldsymbol{n}|_\Gamma = 0, \quad i = 1, 2, \\ \boldsymbol{v}(x) \to 0, \quad p(x) \to 0, \quad (x \to \infty); \end{cases} \tag{3.1}$$

$$\begin{cases} -\nu\nabla^2 \boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \quad \nabla\cdot\boldsymbol{v} = g(x), \quad x \in \Omega_2, \\ \boldsymbol{v}|_\Sigma = 0, \quad \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0, \quad \boldsymbol{\tau}^{(i)}\cdot S(\boldsymbol{v})\,\boldsymbol{n}|_\Gamma = 0, \quad i = 1, 2, \\ \boldsymbol{v}(x) \to 0, \quad (|x| \to \infty), \\ p(x) \to 0, \quad (|x| \to \infty, x_3 > 0), \quad p(x) - \overline{p} \to 0, \quad (|x| \to \infty, x_3 < 0), \end{cases} \tag{3.2}$$

in given domains Ω_1 and Ω_2 of the type described in § 1. We recall that $\boldsymbol{\tau}^{(1)}$ is a tangential vector to Γ with the components $\tau_r^{(1)} = n_3$, $\tau_3^{(1)} = -n_r$ and $\boldsymbol{\tau}^{(2)} = \boldsymbol{e}_\varphi$. We prove the following theorems.

THEOREM 3.1. – *Let $\partial\Omega_1 = \Sigma \cup \Gamma$ where Γ is given by equation $x_3 = h(r)$ with $h \in \widetilde{C}_{s+1}^{l+3}(J_{d_0}, 3)$ (l and s are the same as in Theorem 1.1). For arbitrary axisymmetric $\boldsymbol{f} \in C_{s-2}^{l}(\Omega_1, 4+\beta)$, $g \in C_{s-1}^{l+1}(\Omega_1, 3+\beta)$, $\beta \in (0, 1)$, satisfying the condition*

$$\int_{\Omega_1} g(x)\, dx = 0\,,$$

and, in the case $\vartheta < \vartheta_1$, $s > 1$, the compatibility condition $g(x)\,|_{x \in M} = 0$, problem (3.1) has a unique axisymmetric solution $\boldsymbol{v} \in C_s^{l+2}(\Omega_1, 2)$, $p \in C_{s-1}^{l+1}(\Omega_1, 3)$, and

$$(3.3)\quad |\boldsymbol{v}|_{C_s^{l+2}(\Omega_1, 2)} + |p|_{C_{s-1}^{l+1}(\Omega_1, 3)} \leq c(|\boldsymbol{f}|_{C_{s-2}^{l}(\Omega_1, 4+\beta)} + |g|_{C_{s-1}^{l+1}(\Omega_1, 3+\beta)})\,.$$

If the angular component of $\boldsymbol{f}$ vanishes, so does the angular component of $\boldsymbol{v}$.

THEOREM 3.2. – *Let $\partial\Omega_2 = \Sigma \cup \Gamma$ where Γ is a surface of revolution of the line Γ' about the x_3-axis, and Γ' is given by equations (1.5) with $x_3(s) = \int_0^s \sin\alpha(s')\, ds'$, $r = d_0 + \int_0^s \cos\alpha(s')\, ds'$, $\alpha \in C_s^{l+2}(J_0, 1)$. For arbitrary axisymmetric $\boldsymbol{f} \in C_{s-2,\,a}^{l}(\Omega_2, 4+\beta)$, $g \in C_{s-1,\,a}^{l+1}(\Omega_2, 3+\beta)$, $\beta \in (0, 1)$, satisfying the conditions*

$$f_\varphi = 0\,, \qquad \int_{\Omega_2} g(x)\, dx = 0\,,$$

and, in the case $\vartheta < \vartheta_1$, $s > 1$, the compatibility condition $g(x)\,|_{x \in M} = 0$, problem (3.2) has a unique axisymmetric solution $\boldsymbol{v} \in C_{s,\,a}^{l+2}(\Omega_2, 2)$, $\nabla p \in C_{s-2,\,a}^{l}(\Omega_2, 4)$ such that $p \in C_{s-1}^{l+1}(\Omega_+, 3)$, $v_\varphi = 0$ and

$$(3.4)\quad |\boldsymbol{v}|_{C_{s,a}^{l+2}(\Omega_2, 2)} + |\nabla p|_{C_{s-2,a}^{l}(\Omega_2, 4)} + |p|_{C_{s-1}^{l+1}(\Omega_+, 3)} \leq$$
$$c(|\boldsymbol{f}|_{C_{s-2,a}^{l}(\Omega_2, 4+\beta)} + |g|_{C_{s-1,a}^{l+1}(\Omega_2, 3+\beta)}).$$

The parameters l, s are defined in the same way as in Theorem 1.2, $a \in (0, a_0)$.

Both theorems are proved in several steps, the first of which is the analysis of generalized solutions of Problems 1 and 2. Let $\mathcal{H}(\Omega_i)$, $i = 1, 2$, be the space of vector fields with a bounded Dirichlet integral

$$\|D\boldsymbol{v}\|_{L_2(\Omega_i)} = \left(\int_{\Omega_i} |D\boldsymbol{v}|^2 \, dx \right)^{1/2} \equiv \left(\sum_{i,j=1}^{3} \int_{\Omega_i} \left| \frac{\partial v_j}{\partial x_i} \right|^2 dx \right)^{1/2}$$

satisfying the boundary conditions

$$\boldsymbol{v}|_{\Sigma} = 0 , \qquad \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma} = 0 .$$

By a generalised solution of Problem 1 or 2 we mean a vector field $\boldsymbol{v} \in \mathcal{H}(\Omega_i)$, $i = 1, 2$, and a function $p \in L_{2,\,\mathrm{loc}}(\Omega_i)$ satisfying the equation

$$\nabla\cdot\boldsymbol{v} = g(x) , \qquad x \in \Omega_i ,$$

and the integral identity

$$\frac{\nu}{2} \int_{\Omega_i} S(\boldsymbol{v}) : S(\boldsymbol{\eta}) \, dx - \int_{\Omega_i} p \nabla\cdot\boldsymbol{\eta} \, dx = \int_{\Omega_i} (\boldsymbol{f}\cdot\boldsymbol{\eta} + \nu g \nabla\cdot\boldsymbol{\eta}) \, dx \tag{3.5}$$

for arbitrary $\boldsymbol{\eta} \in \mathcal{H}(\Omega_i)$ with a compact support.

Before proving the existence of a generalized solutions of Problems 1 and 2, we formulate some auxiliary propositions whose proofs can be found in [6, 1, 15, 18].

PROPOSITION 3.1. – *Let $G \subset R^3$ be a bounded domain possessing the cone property. Arbitrary $f \in L_2(G)$ such that $\int_G f(x)\, dx = 0$ can be represented in the form*

$$f(x) = \nabla\cdot\boldsymbol{u}(x) , \qquad \boldsymbol{u}(x)|_{x \in \partial G} = 0 , \tag{3.6}$$

where $\boldsymbol{u}$ is a vector field with a bounded Dirichlet integral, and

$$\|D\boldsymbol{u}\|^2_{L_2(G)} \equiv \sum_{i,k=1}^{3} \int_G \left| \frac{\partial u_i}{\partial x_k} \right|^2 dx \leqslant c \|f\|^2_{L_2(G)} \tag{3.7}$$

with the constant c independent of f. The correspondence between f and $\boldsymbol{u}$ is linear.

If G is a special Lipschitz domain, i.e. the domain of the form

$$x_3 < F(x_1, x_2) \equiv F(x') \tag{3.8}$$

where $F(x')$ is a function satisfying the Lipschitz condition

$$|F(x') - F(y')| \leqslant c|x' - y'| , \qquad \forall x', y' \in R^2 ,$$

then (3.6) *and* (3.7) *hold for arbitrary $f \in L_2(G)$.*

PROPOSITION 3.2. – *If $G \subset R^3$ is a bounded domain satisfying the cone condition, then for arbitrary $\boldsymbol{u} \in W_2^1(G)$ such that $\boldsymbol{u}|_\Sigma = 0$, where $\Sigma \in \partial G$,* mes $\Sigma >$ 0, *the Korn inequality*

$$\|D\boldsymbol{u}\|^2_{L_2(G)} \leq c\|S(\boldsymbol{u})\|^2_{L_2(G)} \tag{3.9}$$

holds with the constant c independent of $\boldsymbol{u}$.

In special Lipschitz domains (3.8) *Korn's inequality* (3.9) *is satisfied for arbitrary vector field* $\boldsymbol{u}(x)$ *with a bounded Dirichlet integral.*

COROLLARY. – *For arbitrary $\boldsymbol{u} \in \mathcal{H}(\Omega_i)$, $i = 1, 2$, there holds the inequality*

$$\|D\boldsymbol{u}\|^2_{L_2(\Omega_i)} \leq c\|S(\boldsymbol{u})\|^2_{L_2(\Omega_i)} . \tag{3.10}$$

Indeed, Ω_1 is a union of a bounded domain $\Omega_{11} = \{x \in \Omega_1 : |x - x_R| < R\}$, $x_R = (0, 0, R/2)$, $R/2 > m_1 + m_2$, satisfying the cone condition, and of a special Lipschitz domain Ω_{12} (3.8) with the Lipschitz continuous function $F(x) = \min(h(|x'|), R/2 - \sqrt{R^2 - |x'|^2})$, so (3.10) follows from the estimates (3.9) for $G = \Omega_{11}$ and $G = \Omega_{12}$.

Further, $\Omega_2 = \Omega_{21} \cup \Omega_{22} \cup \Omega_{23}$ where $\Omega_{21} = \{x \in \Omega_2 : |x| < R, x_3 > 0\}$ is a bounded domain satisfying the cone condition, $\Omega_{22} = \{x \in \Omega_2 : |x| > R, x_3 > 0\}$ is an unbounded domain which can be defined by inequality $x_3 > F(x')$ with a certain Lipschitz continuous $F(x')$, if R is large enough, and $\Omega_{23} = \{x \in \Omega_2 : x_3 < 0\}$ is an infinite cylinder on whose lateral surface Σ the condition $\boldsymbol{u}|_\Sigma = 0$ holds for arbitrary $\boldsymbol{u} \in \mathcal{H}(\Omega_2)$. It is evident that (3.9) is satisfied in $G = \Omega_{2i}$ for arbitrary $\boldsymbol{u} \in \mathcal{H}(\Omega_2)$, hence, (3.10) holds in Ω_2.

Now, we turn to the proof of the solvability of Problems 1 and 2.

PROPOSITION. – 3.3. – *Assume that* $\boldsymbol{f}$ *and g are axisymmetrical and have finite norms*

$$\sup_{\Omega_1} \varrho(x, 4 + \beta, 2 - s)|\boldsymbol{f}(x)|$$

and

$$\sup_{\Omega_1} \varrho(x, 3 + \beta, 1 - s)|g(x)| .$$

Then Problem 1 *has a unique axisymmetrical generalized solution* $\boldsymbol{v} \in \mathcal{H}(\Omega_1)$, $p \in L_2(\Omega_1)$, *and*

$$\|D\boldsymbol{v}\|_{L_2(\Omega_1)} + \|p\|_{L_2(\Omega_1)} \leq c\big(\sup_{\Omega_1} \varrho(x, 4 + \beta, 2 - s)|\boldsymbol{f}(x)| + \sup_{\Omega_1} \varrho(x, 3 + \beta, 1 - s)|g(x)|\big) . \tag{3.11}$$

If the angular component of $\boldsymbol{f}$ *vanishes, so does the angular component of* $\boldsymbol{v}$.

PROPOSITION 3.4. – *If* $\boldsymbol{f}$ *and* g *are axisymmetrical, have finite norms*

$$\sup_{\Omega_2} \varrho(x, a, 4+\beta, 2-s)\,|\boldsymbol{f}(x)|, \qquad \sup_{\Omega_2} \varrho(x, a, 3+\beta, 1-s)\,|g(x)|,$$

and $f_\varphi = 0$, *then Problem 2 has a unique axisymmetrical generalized solution with the following properties:* $\boldsymbol{v} \in \mathcal{H}(\Omega_2)$, $p \in L_2(\Omega_2 \backslash \Omega(z))$, $\forall z < 0$, *where* $\Omega(z) = \{x \in \Omega_2 \colon x_3 < z\}$, *and* $v_\varphi = 0$. *The solution satisfies the inequalities*

$$\|D\boldsymbol{v}\|_{L_2(\Omega_2)} + \|p\|_{L_2(\Omega_+)} \leqslant c\big(\sup_{\Omega_2} \varrho(x, a, 4+\beta, 2-s)\,|\boldsymbol{f}(x)| + \sup_{\Omega_2} \varrho(x, a, 3+\beta, 1-s)\,|g(x)|\big) \tag{3.12}$$

with $a \in (0, a_0)$, *and*

$$\|D\boldsymbol{v}\|^2_{L_2(\omega(t))} \leqslant c e^{-2at}\big(\sup_{\Omega_2} \varrho(x, a, 4+\beta, 2-s)\,|\boldsymbol{f}(x)| + \sup_{\Omega_2} \varrho(x, a, 3+\beta, 1-s)\,|g(x)|\big)^2 \tag{3.13}$$

where $\omega(t) = \{x \in \Omega_2 \colon -t-1 < x_3 < -t\}$, t *is an arbitrary number greater than* 2.

REMARK. – It may be shown that in the domain $\Omega_- = \{x \in \Omega_2 \colon x_3 < 0\}$ the pressure p has the form $p(x) = p'(x) + \int_{x_3}^{0} P(t)\,dt$ with $p' \in L_2(\Omega_-)$, $P \in L_2(-\infty, 0)$ (see [12]).

We restrict ourselves to the proof of somewhat more complicated Proposition 3.4.

PROOF OF PROPOSITION 3.4. – We reduce problem (3.2) to a similar problem with $g(x) = 0$ by the construction of an auxiliary vector field $\boldsymbol{w}(x)$ satisfying the equation $\nabla \cdot \boldsymbol{w}(x) = g(x)$. We define a smooth monotone function $\chi(x_3)$ equal to 1 for $x_3 < -1$ and to 0 for $x_3 > 0$ and a smooth function $A(x') \in C_0^\infty(|x'| < d_0)$ possessing the property $\int_{|x'|<d_0} A(x')\,dx' = 1$. The vector field

$$\boldsymbol{w}_1(x) = \chi(x_3) A(x')\,\boldsymbol{e}_3 \int_{\Omega(x_3)} g(y)\,dy$$

satisfies the equation

$$\nabla \cdot \boldsymbol{w}_1 = \chi(x_3)\,A(x') \int_{|x'|<d_0} g(x)\,dx' + \chi'(x_3)\,A(x') \int_{\Omega(x_3)} g(y)\,dy \equiv g_1(x)$$

and the estimate

$$\|\boldsymbol{w}_1\|_{W_2^1(\Omega_-)} \leqslant c \sup_{\Omega_2} \varrho(x, a, 3+\beta, 1-s)\,|g(x)|\,.$$

Further, we construct a vector field $\boldsymbol{w}_2$ also satisfying this estimate and the equation $\nabla \cdot \boldsymbol{w}_2 = g(x) - g_1(x) \equiv g_2(x)$. We observe that

$$\int_{|x'|<d_0} g_2(x)\, dx' = \int_{|x'|<d_0} g(x)\, dx' - \int_{|x'|<d_0} g_1(x)\, dx' = 0$$

for all $x_3 < -1$, and we decompose the domain $\Omega(-1)$ into an infinite number of bounded subdomains $\omega_k \equiv \omega(k)$, $k = 1, 2, \dots$. By Proposition 3.1, there exist vector fields $\boldsymbol{w}_{2,k} \in \overset{\circ}{W}{}_2^1(\omega_k)$ such that $\nabla \cdot \boldsymbol{w}_{2,k} = g_2$ in ω_k and

$$\|D\boldsymbol{w}_{2,k}\|^2_{L_2(\omega_k)} \leqslant c\|g_2\|^2_{L_2(\omega_k)}\,.$$

In the domain $\Omega'_{21} = \{x \in \Omega_2 \colon x_3 > -1,\ |x| < 2R\}$ there exists $\boldsymbol{w}_2^{(1)} \in \overset{\circ}{W}{}_2^1(\Omega'_{21})$ satisfying the equation

$$\nabla \cdot \boldsymbol{w}_2^{(1)} = g_2(x) - \phi(x) \int_{\Omega'_{21}} g_2(z)\, dz \equiv \tilde{g}_2(x), \qquad x \in \Omega'_{21}\,,$$

where $\phi(x) \in C_0^\infty(\Omega'_{21})$, $\operatorname{supp}\phi \in \Omega'_{21} \backslash \Omega_{21}$, $\int_{\Omega'_{21}\backslash\Omega_{21}} \phi(z)\, dz = 1$, and the inequality

$$\|D\boldsymbol{w}_2^{(1)}\|^2_{L_2(\Omega'_{21})} \leqslant c\|\tilde{g}_2\|^2_{L_2(\Omega'_{21})}\,.$$

Finally, in Ω_{22} there exists a vector field $\boldsymbol{w}_2^{(2)}$ such that

$$\nabla \cdot \boldsymbol{w}_2^{(2)} = \begin{cases} \phi(x) \int_{\Omega'_{21}} g_2(z)\, dz\,, & \text{if } x \in \Omega_{22} \cap \Omega'_{21}\,, \\ g_2(x), & \text{if } x \in \Omega_{22} \backslash \Omega'_{21}\,, \end{cases}$$

and

$$\|D\boldsymbol{w}_2^{(2)}\|^2_{L_2(\Omega_{22})} \leqslant c(\|g_2\|^2_{L_2(\Omega_{22})} + \|g_2\|^2_{L_2(\Omega'_{21})})\,.$$

We extend $\boldsymbol{w}_{2,k}(x)$ into Ω_2 setting $\boldsymbol{w}_{2,k}(x) = 0$ for $x \in \Omega_2 \backslash \omega_k$, extend $\boldsymbol{w}_2^{(1)}(x)$ and $\boldsymbol{w}_2^{(2)}(x)$ in the same manner, and we set

$$\boldsymbol{w}_2(x) = \sum_{k=1}^\infty \boldsymbol{w}_{2,k}(x) + \boldsymbol{w}_2^{(1)}(x) + \boldsymbol{w}_2^{(2)}(x)\,.$$

It is clear that $\nabla \cdot \boldsymbol{w}_2 = g_2$ and

$$\|D\boldsymbol{w}_2\|^2_{L_2(\Omega_2)} = \sum_{k=1}^\infty \|\boldsymbol{w}_{2,k}\|^2_{L_2(\omega_k)} + \|\boldsymbol{w}_2^{(1)} + \boldsymbol{w}_2^{(2)}\|^2_{L_2(\Omega'_{21} \cup \Omega_{22})} \leqslant$$

$$c\big(\sup_{\Omega_2} \varrho(x, a, 3+\beta, 1-s)\,|g(x)|\big)^2\,.$$

Hence, $\boldsymbol{w}(x)=\boldsymbol{w}_1(x)+\boldsymbol{w}_2(x)$ satisfies the equation $\nabla\cdot\boldsymbol{w}=g$ and the inequality

$$\|D\boldsymbol{w}\|^2_{L_2(\Omega_2)}\leqslant c\big(\sup_{\Omega_2}\varrho(x,\,a,\,3+\beta,\,1-s)\,|g(x)|\big)^2\,.$$

A new unknown vector field $\boldsymbol{u}(x)=\boldsymbol{v}(x)-\boldsymbol{w}(x)$ should be solenoidal: $\nabla\cdot\boldsymbol{u}=0$ and satisfy the integral identity

$$\frac{\nu}{2}\int_{\Omega_2}S(\boldsymbol{u}):\;S(\boldsymbol{\eta})\,dx=\int_{\Omega_2}\boldsymbol{f}\cdot\boldsymbol{\eta}\,dx-\frac{\nu}{2}\int_{\Omega_2}S(\boldsymbol{w}):\;S(\boldsymbol{\eta})\,dx \tag{3.14}$$

for arbitrary divergence free $\boldsymbol{\eta}\in\mathcal{H}(\Omega_2)$ (we denote by $\mathcal{J}(\Omega_2)$ the space of such vector fields). By virtue of Korn's inequality, the bilinear form in the left-hand side can be considered as a new scalar product in $\mathcal{J}(\Omega_2)$. We observe also that

$$\Big|\int_{\Omega_2}\boldsymbol{f}\cdot\boldsymbol{\eta}\,dx\Big|\leqslant c\sup_{\Omega_2}\varrho(x,\,a,\,4+\beta,\,2-s)\,|\boldsymbol{f}(x)|\Big(\int_{\Omega_{23}}e^{ax_3}\,|\boldsymbol{\eta}(x)|\,dx+$$

$$\int_{\Omega_{21}}|\operatorname{dist}(x,\,M)|^{\min(0,\,s-2)}\,|\boldsymbol{\eta}(x)|\,dx+\int_{\Omega_{22}}|x|^{-4-\beta}\,|\boldsymbol{\eta}(x)|\,dx\Big)\leqslant$$

$$c\|D\boldsymbol{\eta}\|_{L_2(\Omega_2)}\sup_{\Omega_2}\varrho(x,\,a,\,4+\beta,\,2-s)\,|\boldsymbol{f}(x)|\;,$$

hence, $\boldsymbol{u}\in\mathcal{J}(\Omega_2)$ is uniquely determined by (3.14). Setting in (3.14) $\boldsymbol{\eta}=\boldsymbol{u}$ we obtain

$$\|D\boldsymbol{u}\|_{L_2(\Omega_2)}\leqslant c\big(\sup_{\Omega_2}\varrho(x,\,a,\,4+\beta,\,2-s)\,|\boldsymbol{f}(x)|\;+\|D\boldsymbol{w}\|_{L_2(\Omega_2)}\big)\leqslant$$

$$c\big(\sup_{\Omega_2}\varrho(x,\,a,\,4+\beta,\,2-s)\,|\boldsymbol{f}(x)|\,+\sup_{\Omega_2}\varrho(x,\,a,\,3+\beta,\,1-s)\,|g(x)|\big)\,.$$

The vector field $\boldsymbol{v}=\boldsymbol{w}+\boldsymbol{u}$ satisfies the equation $\nabla\cdot\boldsymbol{v}=g$, the estimate

$$\|D\boldsymbol{v}\|_{L_2(\Omega_2)}\leqslant\|D\boldsymbol{w}\|_{L_2(\Omega_2)}+\|D\boldsymbol{u}\|_{L_2(\Omega_2)}\leqslant$$

$$\leqslant c\big(\sup_{\Omega_2}\varrho(x,\,a,\,4+\beta,\,2-s)\,|\boldsymbol{f}(x)|\,+\sup_{\Omega_2}\varrho(x,\,a,\,3+\beta,\,1-s)\,|g(x)|\big)$$

and the integral identity

$$\frac{\nu}{2}\int_{\Omega_2}S(\boldsymbol{v}):\;S(\boldsymbol{\eta})dx-\int_{\Omega_2}(\boldsymbol{f}\cdot\boldsymbol{\eta}+\nu g\nabla\cdot\boldsymbol{\eta})\,dx=0$$

for arbitrary $\boldsymbol{\eta}\in\mathcal{J}(\Omega_2)$. It follows from Proposition 3.1 (see [6, 12, 13] in this connection) that for arbitrary $z<0$ there exists the function $p_z\in L_2(\Omega_2\backslash\Omega(z))$

such that

$$\frac{\nu}{2}\int_{\Omega_2} S(\boldsymbol{v}):\ S(\boldsymbol{\eta})\,dx-\int_{\Omega_2}(\boldsymbol{f}\cdot\boldsymbol{\eta}+\nu g\nabla\cdot\boldsymbol{\eta})\,dx=\int_{\Omega_2} p_z\nabla\cdot\boldsymbol{\eta}\,dx$$

where $\boldsymbol{\eta}\in\mathcal{H}(\Omega_2)$ and $\boldsymbol{\eta}(x)=0$ for $x_3\leqslant z$. Clearly, $p_{z_1}(x)=p_{z_2}(x)$ for $x_3>\max(z_1, z_2)$, hence, we have constructed a generalized solution $(\boldsymbol{v}, p)$, $\boldsymbol{v}\in\mathcal{H}(\Omega_2)$, $p\in L_{2,\,\mathrm{loc}}(\Omega_2)$ of Problem (3.1) with homogeneous boundary conditions such that $p\in L_2(\Omega_2\setminus\Omega(z))$, $\forall z<0$. Taking in (3.5) as a test function the vector field $\boldsymbol{\eta}\in\overset{\circ}{W}{}^1_2(\Omega_+)$ extended by zero into Ω_- and such that $\nabla\cdot\boldsymbol{\eta}=p$ and

$$\|D\boldsymbol{\eta}\|_{L_2(\Omega_+)}\leqslant c\|p\|_{L_2(\Omega_+)}$$

we easily obtain

$$\|p\|_{L_2(\Omega_+)}\leqslant$$

$$c\big(\|S(\boldsymbol{v})\|_{L_2(\Omega_+)}+\sup_{\Omega_2}\varrho(x,\,a,\,4+\beta,\,2-s)\,|\boldsymbol{f}(x)|+\sup_{\Omega_2}\varrho(x,\,a,\,3+\beta,\,1-s)\,|g(x)|\big)\leqslant$$

$$c\big(\sup_{\Omega_2}\varrho(x,\,a,\,4+\beta,\,2-s)\,|\boldsymbol{f}(x)|\,+\sup_{\Omega_2}\varrho(x,\,a,\,3+\beta,\,1-s)\,|g(x)|\big)\,.$$

The uniqueness of the solution is obvious: the difference $\boldsymbol{v}=\boldsymbol{v}_1-\boldsymbol{v}_2$ of two solutions is an element of $\mathcal{J}(\Omega_2)$, and

$$\int_{\Omega_2} S(\boldsymbol{v}):\ S(\boldsymbol{\eta})\,dx=0\,,\qquad \forall\eta\in\mathcal{J}(\Omega_2)\,,$$

hence, $\boldsymbol{v}_1=\boldsymbol{v}_2$ and $p_1-p_2=\mathrm{Const}=0$.

Axial symmetry of the solution is a consequence of the symmetry of the problem: if the data are symmetric, then it is easy to verify that $U\boldsymbol{v}(U^{-1}x)$, $p(U^{-1}x)$, where U is an orthogonal matrix corresponding to rotation about the x_3-axis, is a generalized solution of (3.2) together with $\boldsymbol{v}(x)$, $p(x)$. Hence,

$$U\boldsymbol{v}(U^{-1}x)=\boldsymbol{v}(x)\,,\qquad p(U^{-1}x)=p(x)\,,$$

q.e.d. Further, if $\boldsymbol{f}_\varphi=0$, then, taking in (3.5) $\boldsymbol{\eta}=\zeta(|x|/R)v_\varphi\boldsymbol{e}_\varphi$ where $\zeta\in C_0^\infty(R^3)$ is a cut-off function equal to 1 for $|x|<1/2$ and to zero for $|x|>1$, and letting R tend to infinity, one obtains:

$$0=\frac{\nu}{2}\int_\Omega S(\boldsymbol{v}):S(\zeta v_\varphi\boldsymbol{e}_\varphi)dx\to\nu\int_\Omega\left[\left(\frac{\partial v_\varphi}{\partial x_3}\right)^2+\left(\frac{\partial v_\varphi}{\partial r}-\frac{1}{r}v_\varphi\right)^2\right]r\,dr\,d\varphi\,dx_3$$

which implies $v_\varphi=Cr$, and this is possible only in the case $C=0$, i.e., $v_\varphi=0$.

It remains to prove inequality (3.13). For the vector field $\boldsymbol{u}$ satisfying (3.14) this inequality follows from Theorem 3.3 in [14] (with ω_k defined as above and with $\kappa_k = e^{ak}$). By virtue of this theorem, there exists $a_0 > 0$ such that

$$\|D\boldsymbol{u}\|^2_{L_2(\omega(t))} \leqslant ce^{-2at}\big(\sup_{\tau>2} e^{2a\tau}\|\boldsymbol{f}\|^2_{L_2(\omega(\tau))} +$$

$$\big(\sup_{\omega(1)} (\operatorname{dist}(x, M))^{\max(2-s,0)}|\boldsymbol{f}(x)|\big)^2 + \sup_{\tau>1} e^{2a\tau}\|S(\boldsymbol{w})\|^2_{L_2(\omega(\tau))} + \|D\boldsymbol{u}\|^2_{L_2(\omega(0))}\big) \leqslant$$

$$ce^{-2at}\big(\sup_{\Omega_2} \varrho(x, a, 4+\beta, 2-s)|\boldsymbol{f}(x)| + \sup_{\Omega_2} \varrho(x, a, 3+\beta, 1-s)|g(x)|\big)^2.$$

In addition,

$$\|D\boldsymbol{w}_1\|^2_{L_2(\omega(t))} \leqslant c\left(\sup_{\Omega(t)} e^{-ax_3}|g(x)|\int_{-\infty}^{-t} e^{ay}\,dy\right)^2 \leqslant ce^{-2at}\big(\sup_{\Omega(t)} e^{-ax_3}|g(x)|\big)^2,$$

$$\|D\boldsymbol{w}_2\|^2_{L_2(\omega(t))} \leqslant c\|g_2\|^2_{L_2(\omega_k \cap \omega_{k-1})} \leqslant ce^{-2at}\big(\sup_{\Omega(t)} e^{-ax_3}|g(x)|\big)^2,$$

with $t \leqslant k < t+1$, $a \in (0, a_0)$. These estimates of $D\boldsymbol{u}$, $D\boldsymbol{w}_1$, $D\boldsymbol{w}_2$ imply (3.13). The proposition is proved.

PROOF OF THEOREM 3.2. – The proof consists in the investigation of regularity properties of a generalized solution. Consider the solution in the domain $\tilde{\omega}(\xi) = \omega(\xi-1) \cup \omega(\xi) \cup \omega(\xi+1)$, $\xi > 3$. It is easily seen that $\boldsymbol{v}$ and $\tilde{p} = p - |\tilde{\omega}(\xi)|^{-1}\int_{\tilde{\omega}(\xi)} p(y)\,dy$ satisfy (3.5) for arbitrary $\boldsymbol{\eta} \in \mathcal{H}(\Omega_2)$ with $\operatorname{supp}\boldsymbol{\eta} \subset \tilde{\omega}(\xi)$, and

$$\|\tilde{p}\|_{L_2(\tilde{\omega}(\xi))} \leqslant c(\|\boldsymbol{f}\|_{L_2(\tilde{\omega}(\xi))} + \|S(\boldsymbol{w})\|_{L_2(\tilde{\omega}(\xi))} + \|S(\boldsymbol{v})\|_{L_2(\tilde{\omega}(\xi))}) \leqslant$$

$$ce^{-a\xi}\big(\sup_{\Omega_2} \varrho(x, a, 4+\beta, 2-s)|\boldsymbol{f}(x)| + \sup_{\Omega_2} \varrho(x, a, 3+\beta, 1-s)|g(x)|\big).$$

Further, by virtue of local Schauder estimate for the solution of the problem under consideration,

$$|\boldsymbol{v}|_{C^{l+2}(\omega(\xi))} + |\nabla p|_{C^l(\omega(\xi))} \leqslant c(|\boldsymbol{f}|_{C^l(\tilde{\omega}(\xi))} + |g|_{C^{l+1}(\tilde{\omega}(\xi))} + \|\boldsymbol{v}\|_{L_2(\tilde{\omega}(\xi))} + \|\tilde{p}\|_{L_2(\tilde{\omega}(\xi))})$$

which implies

$$(3.15)\qquad |\boldsymbol{v}|_{C^{l+2}((\omega(\xi))} + |\nabla p|_{C^l(\omega(\xi))} \leqslant ce^{-a\xi}(|\boldsymbol{f}|_{C^l_{s-2,a}(\Omega_2, 4+\beta)} + |g|_{C^{l+1}_{s-1,a}(\Omega_2, 3+\beta)}).$$

It follows that $p(x) \to \bar{p}$ as $x \to -\infty$, and

$$|\bar{p}| \leqslant c(|\boldsymbol{f}|_{C^l_{s-2,a}(\Omega_2, 4+\beta)} + |g|_{C^{l+1}_{s-1,a}(\Omega_2, 3+\beta)}).$$

Similar estimates hold also in the domain $\omega(\xi)$, $\xi \in (0, 3)$, and in arbitrary compact subdomain of Ω_2 which is bounded away from M. The simplest way of the investigation of the regularity properties of the solution near M is to use its axial symmetry. In cylindrical coordinates the Stokes equations and the boundary conditions take the form

$$(3.16)\qquad \begin{cases} -\nu\left(\dfrac{\partial^2 v_r}{\partial r^2}+\dfrac{\partial^2 v_r}{\partial x_3^2}\right)+\dfrac{\partial p}{\partial r}=-\dfrac{\nu}{r}\dfrac{\partial v_r}{\partial r}+\dfrac{\nu}{r^2}v_r+f_r\,,\\ -\nu\left(\dfrac{\partial^2 v_3}{\partial r^2}+\dfrac{\partial^2 v_3}{\partial x_3^2}\right)+\dfrac{\partial p}{\partial x_3}=-\dfrac{\nu}{r}\dfrac{\partial v_3}{\partial r}+f_3\,,\\ \dfrac{\partial v_r}{\partial r}+\dfrac{\partial v_3}{\partial x_3}=-\dfrac{v_r}{r}+g\,,\qquad (r,x_3)\in G\,, \end{cases}$$

$$(3.17)\qquad \begin{cases} v_r n_r+v_3 n_3=0\,,\qquad \boldsymbol{\tau}\cdot\dfrac{\partial \boldsymbol{v}}{\partial n}+\boldsymbol{n}\cdot\dfrac{\partial \boldsymbol{v}}{\partial \tau}=0\,,\qquad (r,x_3)\in\Gamma'\,,\\ v_r=v_3=0\,,\qquad (r,x_3\in\Sigma')\,. \end{cases}$$

where $G\subset R^2$ is the domain bounded by the straight lines $\{r=0,\,x_3\in R\}$, $\Sigma'=\{r=d_0,\,x_3<0\}$ and by the curve Γ', $\boldsymbol{n}$ is the normal to the line Γ', $\tau_r=n_3$, $\tau_3=-n_r$. Near the contact point $r=d_0$, $x_3=0$ system (3.16) can be considered as the two-dimensional Stokes system for the vector field $\boldsymbol{V}(r,x_3)$ with $V_1=v_r$, $V_2=v_3$ perturbed by lower order terms, and boundary conditions (3.17) can be written in the form

$$\boldsymbol{V}\cdot\boldsymbol{n}|_{\Gamma'}=0\,,\qquad \boldsymbol{\tau}\cdot\frac{\partial \boldsymbol{V}}{\partial n}+\boldsymbol{n}\cdot\frac{\partial \boldsymbol{V}}{\partial \tau}\bigg|_{\Gamma'}=0\,,\qquad \boldsymbol{V}|_{\Sigma'}=0\,.$$

Hence, $(\boldsymbol{V},p)$ have exactly the same regularity properties near the contact point as the solution of two-dimensional Stokes equations satisfying the same boundary conditions (see [13]). In the δ-neighbourhood $B_\delta=\{(r,x_3)\in G:\,\varrho^2(r,x_3)\equiv(r-d_0)^2+x_3^2\leqslant\delta^2\}$ of the contact point there holds the estimate

$$(3.18)\qquad |\boldsymbol{V}|_{C_s^{l+2}(B_{\delta/2})}+|p|_{C_{s-1}^{l+1}(B_{\delta/2})}\leqslant$$

$$c(|\boldsymbol{F}|_{C_{s-2}^{l}(B_\delta)}+|g|_{C_{s-1}^{l+1}(B_\delta)}+\|\boldsymbol{V}\|_{W_2^1(B_\delta)}+\|p\|_{L_2(B_\delta)})\leqslant$$

$$c(|\boldsymbol{f}|_{C_{s-2,a}^{l}(\Omega_2,4+\beta)}+|g|_{C_{s-1,a}^{l+1}(\Omega_2,3+\beta)})$$

where $\boldsymbol{F}=(f_r,f_3)$, $s\in(0,s_0)$, $s\leqslant l+2$, $s\neq 1$, and

$$|u|_{C_\sigma^l(B_\delta)}=\sum_{\sigma<|j|<l}\sup_{B_\delta}\varrho^{|j|-\sigma}(r,x_3)\,|D^j u(r,x_3)|+|u|_{C^\sigma(B_\delta)}+$$

$$+\sum_{|j|=l}\sup_{B_\delta}\varrho^{l-\sigma}(r,x_3)\sup_{B'_\varrho}((r'-r)^2+(x_3'-x_3)^2)^{([l]-l)/2}\,|D^j u(r,x_3)-D^j u(r',x_3')|\,,$$

if $\sigma \in (0, l]$ (in the case $\sigma < 0$, as usual, the term $|u|_{C^\sigma(B_\delta)}$ should be excluded from the norm; $B'_\varrho = \{(r', x'_3) \in B_\delta :\ \sqrt{(r-r')^2 + (x_3 - x'_3)^2} \leqslant (1/2)\,\varrho(r, x_3)\}$).

Finally, we should clarify the behaviour of a generalized solution for large $|x|$, when $x_3 > 0$. For large s we have $\cos\alpha(s) > 0$, the equation

$$r(s) = d_0 + \int_0^s \cos\alpha(t)\,dt$$

is invertible, and the line Γ' can be given in the form

$$x_3 = x_3(s(r)) \equiv \Phi(r), \qquad r > r_0 .$$

The derivatives of Φ are expressed in terms of $\alpha(s)$:

$$\frac{d\Phi}{dr} = \frac{dx_3}{ds}\left(\frac{dr}{ds}\right)^{-1} = \tan\alpha(s)\,\Big|_{s=s(r)}, \qquad \frac{d^2\Phi}{dr^2} = \frac{\alpha'(s)}{\cos^3\alpha(s)}\,\Bigg|_{s=s(r)} \tag{3.19}$$

etc., hence, $\Phi' \in C^{l+2}(J_{r_0}, 1)$.

For arbitrary $R > 2r_0$ we construct the extension of $\Phi(r)$, $\Phi_R(r)$, from the interval $r > R$ into the interval $(0, R)$, according to the formula

$$\Phi_R(r) = \Phi(r) \qquad \text{for } r > R ,$$

$$\Phi_R(r) = \Phi(R) - \int_r^R \Phi'(t)\,\psi_R(t)\,dt \qquad \text{for } r < R ,$$

where $\psi_a(t) = 1 - \zeta(t/a)$. Clearly, $\Phi_R(r) = \Phi_R(R/2)$ for $r \leqslant R/2$, $\Phi'_R = \Phi'\,\psi_R \in C^{l+2}(J_0, 1)$, and

$$|\Phi'_R|_{C^{l+2}(J_0,1)} \leqslant c\,|\Phi'|_{C^{l+2}(J_{R/2},1)} . \tag{3.20}$$

The mapping $x = \mathcal{Z}_R(y)$:

$$x_i = y_i , \qquad i = 1, 2 , \qquad x_3 = y_3 + \Phi_R(|y|) \tag{3.21}$$

which is invertible, if $\Phi'_R(t) < 1$ (and this is the case for large R, by virtue of (3.19)), transforms the half-space R^3_+ into the domain $\widetilde{\Omega}^{(R)} = \{x_3 > \Phi_R(|x'|)\}$.

Let $R_1 \gg R$. The functions $\boldsymbol{u} = \psi_{R_1}\boldsymbol{v}$, $q = \psi_{R_1} p$ satisfy the relations

$$-\nu\nabla^2\boldsymbol{u} + \nabla q = \psi_{R_1}\boldsymbol{f} + \boldsymbol{f}', \qquad \nabla\cdot\boldsymbol{u} = \psi_{R_1} g + g', \qquad x \in \Omega_2, \quad |x| \geqslant R_1 , \tag{3.22}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 , \qquad \boldsymbol{\tau}\cdot S(\boldsymbol{u})\boldsymbol{n} = d' \qquad x \in \Gamma,\ |x| \geqslant R_1 , \tag{3.23}$$

where

$$f' = -2\nu\nabla \boldsymbol{v}\nabla\psi_{R_1} - \boldsymbol{v}\nabla^2\psi_{R_1} + p\nabla\psi_{R_1}, \qquad g' = \nabla\psi_{R_1}\cdot\boldsymbol{v},$$

$$d' = (\boldsymbol{v}\cdot\boldsymbol{\tau})\frac{\partial\psi_{R_1}}{\partial n} + (\boldsymbol{v}\cdot\boldsymbol{n})\frac{\partial\psi_{R_1}}{\partial\tau}.$$

Extending all the functions in (3.22), (3.23) by zero we may consider (3.22) and (3.23) as equations in $\widetilde{\Omega}^{(R)}$ and on $\partial\widetilde{\Omega}^{(R)}$, respectively. Under transformation (3.21) they take the form

$$(3.24)\quad \begin{cases} -\nu\nabla_R'^2\boldsymbol{u} + \nabla_R' p = \psi_{R_1}\boldsymbol{f} + \boldsymbol{f}', \qquad \nabla_R'\cdot\boldsymbol{u} = \psi_{R_1} g + g', \qquad y\in R^3_+, \\ u_3 - \Phi_{Ry_1}(|y|)\,u_1 - \Phi_{Ry_2}(|y|)\,u_2\,|_{y_3=0} = 0, \\ S'(\boldsymbol{u})\,\boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n}\cdot S'(\boldsymbol{u})\,\boldsymbol{n})|_{y_3=0} = d'\,\boldsymbol{\tau}, \end{cases}$$

where

$$\nabla_R' = J^T\nabla = \left(\sum_{m,k=1}^{3} J_{mk}\frac{\partial}{\partial y_m}\right)_{k=1,2,3},$$

$$S'(\boldsymbol{v}) = \left(\sum_{m=1}^{3}\left(J_{mk}\frac{\partial v_j}{\partial y_m} + J_{mj}\frac{\partial v_k}{\partial y_m}\right)\right)_{j,k=1,2,3},$$

J_{mk} are elements of the Jacobi matrix

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{\Phi_{Ry_1}}{1+\Phi_{Ry_3}} & -\dfrac{\Phi_{Ry_2}}{1+\Phi_{Ry_3}} & \dfrac{1}{1+\Phi_{Ry_3}} \end{bmatrix},$$

$\boldsymbol{n}$ is a vector with the components

$$n_r = -\frac{\Phi_R'(|y'|)}{\sqrt{1+\Phi_R'^2(|y'|)}}, \qquad n_3 = \frac{1}{\sqrt{1+\Phi_R'^2(|y'|}}),$$

$\tau_r = n_3$, $\tau_3 = -n_r$. We can also write (3.24) in an equivalent form

$$(3.25)\quad \begin{cases} -\nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{f}_1 + l_1(\boldsymbol{u}, q) \qquad \nabla\cdot\boldsymbol{u} = g_1 + l_2(\boldsymbol{u}), \\ u_3\,|_{y_3=0} = l_3(\boldsymbol{u}), \\ S_{i3}(\boldsymbol{u})\,|_{y_3=0} = d_{i1} + l_{4i}(\boldsymbol{u}), \qquad i=1,2, \end{cases}$$

where $\boldsymbol{f}_1 = \psi_{R_1}\boldsymbol{f} + \boldsymbol{f}'$, $g_1 = \psi_{R_1} g + g'$, $d_{i1} = d'\tau_i = d' n_3 y_i/|y'|$,

$$l_1(\boldsymbol{u}, q) = \nu(\nabla_R'^2 - \nabla^2)\,\boldsymbol{u} + (\nabla - \nabla_R')\,q\,,$$

$$l_2(\boldsymbol{u}) = (\nabla - \nabla_R')\cdot\boldsymbol{u}\,,$$

$$l_3(\boldsymbol{u}) = \Phi_{Ry_1}(|y|)\,u_1 + \Phi_{Ry_2}(|y|)\,u_2\,|_{y_3=0}\,,$$

$$l_{4i}(\boldsymbol{u}) =$$

$$(S_{i3}(\boldsymbol{u}) - S_{i3}'(\boldsymbol{u})) + S_{i3}'(\boldsymbol{u})(1-n_3) - S_{i1}'(\boldsymbol{u})\,n_1 - S_{i2}'(\boldsymbol{u})\,n_2 + n_i(\boldsymbol{n}\cdot S'(\boldsymbol{u})\,\boldsymbol{n})\,|_{y_3=0}\,.$$

The coefficients of these operators are proportional to the derivatives of $\Phi_R(|y|)$, and it follows from (3.20) that for arbitrary $\boldsymbol{u} \in C^{l+2}(R_+^3, 2)$, $q \in C^{l+1}(R_+^3, 3)$ there holds the estimate

$$(3.26)\qquad |l_1(\boldsymbol{u}, q)|_{C^l(R_+^3, 4+\beta)} + |l_2(\boldsymbol{u})|_{C^{l+1}(R_+^3, 3+\beta)} + |l_3(\boldsymbol{u})|_{C^{l+2}(R^2, 2+\beta)} + \sum_{i=1}^{2} |l_{4i}(\boldsymbol{u})|_{C^{l+1}(R^2, 3+\beta)} \leqslant cR^{\beta-1}(|\boldsymbol{u}|_{C^{l+2}(R_+^3, 2)} + |\nabla q|_{C^l(R_+^3, 4)})\,,\quad \beta \in (0, 1).$$

Hence, by the contraction mapping principle, problem (3.25) with $\boldsymbol{f}_1 \in C^l(R_+^3, 4+\beta)$, $g_1 \in C^{l+1}(R_+^3, 3+\beta)$, $d_{i1} \in C^{l+1}(R^2, 3+\beta)$ defined above has a unique solution $\boldsymbol{U} \in C^{l+2}(R_+^3, 2)$, $Q \in C^{l+1}(R_+^3, 3)$, and

$$(3.27)\qquad |\boldsymbol{U}|_{C^{l+2}(R_+^3, 2)} + |Q|_{C^{l+1}(R_+^3, 3)} \leqslant c\left(|\boldsymbol{f}_1|_{C^l(R_+^3, 4+\beta)} + |g_1|_{C^{l+1}(R_+^3, 3+\beta)} + \sum_{i=1}^{2} |d_{i1}|_{C^{l+1}(R^2, 3+\beta)}\right) \leqslant (|\boldsymbol{f}|_{C^l_{s-2,a}(\Omega_2, 4+\beta)} + |g|_{C^{l+1}_{s-1,a}(\Omega_2, 3+\beta)})\,.$$

The difference $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{U}$, $\kappa = q - Q$ has a finite Dirichlet integral in R_+^3 and satisfies the equations

$$-\nu\nabla_R'^2\boldsymbol{w} + \nabla_R'\kappa = 0\,,\qquad \nabla_R'\cdot\boldsymbol{w} = 0\,,\qquad y \in R_+^3\,,$$

or

$$(3.28)\qquad -\nu\nabla_R' S'(\boldsymbol{w}) + \nabla_R'\kappa = 0\,,\qquad \nabla_R'\cdot\boldsymbol{w} = 0\,,\qquad y \in R_+^3\,,$$

and boundary conditions

$$\boldsymbol{w}\cdot\boldsymbol{n} = 0\,,\qquad S'(\boldsymbol{w})\boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n}\cdot S'(\boldsymbol{w})\boldsymbol{n}) = 0\,,\qquad y_3 = 0\,.$$

Multiplying (3.28) by $\mathscr{D}(y)\boldsymbol{w}(y)$, $\mathscr{D}(y) = \det \mathfrak{J}^{-1} = 1 + \Phi_{Ry_3}$, and integrating over R^3_+, one easily obtains

$$\int_{R^3_+} |S'(\boldsymbol{w})|^2 \mathcal{D}(y)\, dy = 0 ,$$

i.e., $\boldsymbol{u} = \boldsymbol{U}$, $q = Q$. Estimates (3.15), (3.18), (3.27) imply (3.4). Theorem 3.2 is proved.

Theorem 3.1 is proved in a similar way but it should be observed that the angular component of $\boldsymbol{v}$, v_φ, in general does not vanish. Therefore problem (3.1) with homogeneous boundary conditions in cylindrical coordinates takes the form

$$-\nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{\partial^2 v_r}{\partial x_3^2}\right) + \frac{\partial p}{\partial r} = -\frac{\nu}{r}\frac{\partial v_r}{\partial r} + \frac{\nu}{r^2} v_r + f_r ,$$

$$-\nu\left(\frac{\partial^2 v_3}{\partial r^2} + \frac{\partial^2 v_3}{\partial x_3^2}\right) + \frac{\partial p}{\partial x_3} = -\frac{\nu}{r}\frac{\partial v_3}{\partial r} + f_3 ,$$

$$\frac{\partial v_r}{\partial r} + \frac{\partial v_3}{\partial x_3} = -\frac{v_r}{r} + g , \qquad (r, x_3) \in G ,$$

$$v_r n_r + v_3 n_3 = 0 , \qquad \boldsymbol{\tau}\cdot\frac{\partial \boldsymbol{v}}{\partial n} + \boldsymbol{n}\cdot\frac{\partial \boldsymbol{v}}{\partial \tau} = 0 , \qquad x_3 = h(r) ,$$

$$v_r = v_3 = 0 , \qquad (r, x_3) \in \Sigma' .$$

$$-\nu\left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{\partial^2 v_\varphi}{\partial x_3^2}\right) = f_\varphi - \frac{\nu}{r}\frac{\partial v_\varphi}{\partial r} - \frac{\nu v_\varphi}{r^2} , \qquad (r, x_3) \in G ,$$

$$v_\varphi|_{\Sigma'} = 0 , \quad \frac{\partial v_\varphi}{\partial n}\Big|_L = 0 ,$$

where $G \subset R^2$ is a domain bounded by the straight line $\{r = 0,\, x_3 > -m_1 - m_2\}$ and by the curves $\Sigma' = \{(r, x_3) \in L\colon x_3 < h(r)\}$ and $x_3 = h(r)$; $\boldsymbol{n}$ is a vector with the components

$$n_r = -\frac{h_r}{\sqrt{1 + h'^2}} , \qquad n_3 = \frac{1}{\sqrt{1 + h'^2}} ,$$

$\tau_r = n_3$, $\tau_3 = -n_r$. In comparison with (3.16), (3.17), we have here additionally a mixed Dirichlet-Neumann problem for v_φ satisfying the Laplace equation with lower order terms. Therefore we should subject the parameter s to an additional constraint $s < \pi/2\vartheta$ (see [9]). Inequality (3.18) and all the subsequent estimates hold with $s < \min(s_0, \pi/2\vartheta)$.

Now, let us consider problems (3.1), (3.2) with general non-homogeneous boundary conditions on Γ:

$$(3.29)\qquad \begin{cases} -\nu\nabla^2 \boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \quad \nabla\cdot\boldsymbol{v} = g(x), \quad x\in\Omega_1, \\ \boldsymbol{v}|_\Sigma = 0, \quad \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = b(x), \quad \boldsymbol{\tau}^{(i)}\cdot S(\boldsymbol{v})\boldsymbol{n}|_\Gamma = d_i(x), \quad i=1,2, \\ \boldsymbol{v}(x)\to 0, \quad p(x)\to 0, \quad (|x|\to\infty); \end{cases}$$

$$(3.30)\qquad \begin{cases} -\nu\nabla^2 \boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \quad \nabla\cdot\boldsymbol{v} = g(x), \quad x\in\Omega_2, \\ \boldsymbol{v}|_\Sigma = 0, \quad \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = b(x), \quad \boldsymbol{\tau}^{(1)}\cdot S(\boldsymbol{v})\boldsymbol{n}|_\Gamma = d_1(x), \\ \boldsymbol{v}(x)\to 0, \quad (|x|\to\infty), \\ p(x)\to 0, \quad (|x|\to\infty,\ x_3>0), \quad p(x)-\overline{p}\to 0, \quad (|x|\to\infty,\ x_3<0). \end{cases}$$

The domains Ω_i, as well as the data ($\boldsymbol{f}$, g, b, $\boldsymbol{d} = d_1\boldsymbol{\tau}^{(1)} + d_2\boldsymbol{\tau}^{(2)}$) in these problems are axisymmetrical. We say that an axisymmetrical function $f(x)$ given on Γ belongs to the space $C_b^l(\Gamma, b)$, if $f(r(\cdot), x_3(\cdot))\in C_s^l(J_0, b)$, and we set

$$|f|_{C_s^l(\Gamma, b)} \equiv |f(r(\cdot), x_3(\cdot))|_{C_s^l(J, b)}\,.$$

THEOREM 3.3. – *Let $\partial\Omega_1 = \Sigma\cup\Gamma$ where Γ is given by equation $x_3 = h(r)$ with $h\in\widetilde{C}_{s+1}^{l+3}(J_{d_0}, 3)$ (l and s are the same as in Theorem* 1.1). *For arbitrary axisymmetric $\boldsymbol{f}\in C_{s-2}^l(\Omega_1, 4+\beta)$, $g\in C_{s-1}^{l+1}(\Omega_1, 3+\beta)$, $b\in C_s^{l+2}(\Omega_1, 2+\beta)$, $d_i\in C_{s-1}^{l+1}(\Omega_1, 3+\beta)$, $\beta\in(0,1)$, satisfying the condition*

$$\int_{\Omega_1} g(x)\,dx - \int_\Gamma b(x)\,dS = 0\,,$$

and, in the case $\vartheta<\vartheta_1$, $s>1$, the compatibility condition

$$(3.31)\qquad \sin\vartheta\cos\vartheta g(x) = d_1(x)\sin^2\vartheta + \frac{\partial b}{\partial\tau}(\cos^2\vartheta - \sin^2\vartheta), \qquad x\in M\,,$$

problem (3.29) *has a unique axisymmetric solution $\boldsymbol{v}\in C_s^{l+2}(\Omega_1, 2)$, $p\in C_{s-1}^{l+1}(\Omega_1, 3)$, and*

$$(3.32)\qquad |\boldsymbol{v}|_{C_s^{l+2}(\Omega_1,2)} + |p|_{C_{s-1}^{l+1}(\Omega_1,3)} \leqslant$$

$$c\big(|\boldsymbol{f}|_{C_{s-2}^l(\Omega_1,4+\beta)} + |g|_{C_{s-1}^{l+1}(\Omega_1,3+\beta)} + |b|_{C_s^{l+2}(\Gamma,2+\beta)} + \sum_{i=1}^2 |d_i|_{C_{s-1}^{l+1}(\Gamma,3+\beta)}\big)\,.$$

If $f_\varphi = 0$ and $d_2 = 0$, then $v_\varphi = 0$.

THEOREM 3.4. – *Let $\partial\Omega_2 = \Sigma\cup\Gamma$ where Γ is a surface of revolution of the line Γ' about the x_3-axis, and Γ' is given by equations* (1.5) *with $x_3(s) =$*

$\int_0^s \sin \alpha(s')\, ds'$, $r = d_0 + \int_0^s \cos \alpha(s')\, ds'$, $\alpha \in C_s^{l+2}(J_0, 1)$. *For arbitrary axisymmetric* $\boldsymbol{f} \in C^l_{s-2,a}(\Omega_2, 4+\beta)$, $g \in C^{l+1}_{s-1,a}(\Omega_2, 3+\beta)$, $b \in C_s^{l+2}(\Omega_1, 2+\beta)$, $d_1 \in C^{l+1}_{s-1}(\Omega_1, 3+\beta)$, $\beta \in (0, 1)$, *satisfying the condition* $f_\varphi = 0$,

$$\int_{\Omega_1} g(x)\, dx - \int_\Gamma b(x)\, dS = 0 ,$$

and, in the case $\vartheta < \vartheta_1$, $s > 1$, *the compatibility condition* (3.31), *problem* (3.30) *has a unique axisymmetric solution* $\boldsymbol{v} \in C^{l+2}_{s,a}(\Omega_2, 2)$, $\nabla p \in C^l_{s-2,a}(\Omega_2, 4)$ *such that* $p \in C^{l+1}_{s-1}(\Omega_+, 3)$, $v_\varphi = 0$, *and*

$$(3.33)\qquad |\boldsymbol{v}|_{C^{l+2}_{s,a}(\Omega_2, 2)} + |\nabla p|_{C^l_{s-2,a}(\Omega_2,4)} + |p|_{C^{l+1}_{s-1}(\Omega_+, 3)} \leqslant$$
$$c(|\boldsymbol{f}|_{C^l_{s-2,a}(\Omega_2, 4+\beta)} + |g|_{C^{l+1}_{s-1,a}(\Omega_2, 3+\beta)} + |b|_{C^{l+2}_s(\Gamma, 2+\beta)} + |d_1|_{C^{l+1}_{s-1}(\Gamma, 3+\beta)}).$$

These theorems reduce to Theorems 3.1 and 3.2 by the construction of an axiliary axisymmetrical vector field $\boldsymbol{w}(x)$, $x \in \Omega_i$, $i = 1, 2$, such that

$$\boldsymbol{w}|_\Sigma = 0 , \qquad \boldsymbol{w}\cdot\boldsymbol{n}|_\Gamma = b , \qquad \boldsymbol{\tau}^{(i)}\cdot S(\boldsymbol{w})\boldsymbol{n}|_\Gamma = d_i , \qquad i = 1, 2 .$$

We restrict ourselves with the proof of Theorem 3.4.

PROOF OF THEOREM 3.4. – It is easy to verify that the boundary conditions on Γ are satisfied, if

$$\boldsymbol{w}|_\Gamma = b\boldsymbol{n} , \qquad \frac{\partial \boldsymbol{w}}{\partial n}\bigg|_\Gamma = d_1 \boldsymbol{\tau}^{(1)} - \sum_{j=1}^3 n_j \nabla_\Gamma(bn_j)$$

where ∇_Γ is the gradient on the surface Γ. The construction of a vector field with given values of this field and of its normal derivative on a surface is a standard problem (also in weighted Hölder spaces, see ([16], Theorem 4.1). In the axisymmetrical case it is convenient to use cylindrical coordinates. It can be shown that there exists an axisymmetrical vector field $\boldsymbol{w}' \in C^{l+2}_{s,a}(\Omega_2, 2)$ satisfying the conditions

$$\boldsymbol{w}'|_\Gamma = b\boldsymbol{n}\zeta' , \qquad \frac{\partial \boldsymbol{w}}{\partial n}\bigg|_\Gamma = d_1 \zeta' \tau^{(1)} - \sum_{j=1}^3 n_j \nabla_\Gamma(b\zeta' n_j)$$

where ζ' is a smooth axisymmetrical cut-off function equal to zero, if dist$(x, M) < 1$ and to 1, if dist$(x, M) > 2$, and the inequality

$$|\boldsymbol{w}'|_{C^{l+2}_{s,a}(\Omega_2, 2+\beta)} \leqslant (|b|_{C^{l+2}_s(\Gamma, 2+\beta)} + |d_1|_{C^{l+1}_{s-1}(\Gamma, 3+\beta)}) .$$

To construct a vector field $\boldsymbol{w}''$ satisfying the necessary boundary conditions both on Σ and on Γ near M and possessing the necessary regularity properties, one has to use Theorems 4.4 and 4.5 in [13], according to which there exists $\boldsymbol{w}'' \in C^{l+2}_{s,\,a}(\Omega_2, 2+\beta)$ with a compact support, such that

$$\boldsymbol{w}''|_{\Sigma} = 0\,, \qquad \boldsymbol{w}''\cdot\boldsymbol{n}|_{\Gamma} = b(1-\zeta')\,, \qquad \boldsymbol{\tau}^{(i)}\cdot S(\boldsymbol{w}'')\,\boldsymbol{n}|_{\Gamma} = d_i(1-\zeta')\,, \qquad i=1,2\,,$$

and

$$|\boldsymbol{w}''|_{C^{l+2}_s(\Omega_2,\,2+\beta)} \leqslant$$

$$(|b|_{C^{l+2}_s(\Gamma,\,2+\beta)} + |d_1|_{C^{l+1}_{s-1}(\Gamma,\,3+\beta)} + |\boldsymbol{f}|_{C^{l}_{s-2,\,a}(\Omega_2,\,4+\beta)} + |g|_{C^{l+1}_{s-1,\,a}(\Omega_2,\,3+\beta)})\,.$$

Clearly, one can set

$$\boldsymbol{w} = \zeta_1\boldsymbol{w}' + \zeta_2\boldsymbol{w}''$$

where ζ_i are appropriate smooth cut-off functions. The theorem is proved.

Theorem 3.3 is proved in a similar way.

At the conclusion of this section, we consider the problems

$$(3.34)\qquad \begin{cases} -\nu\nabla^2\boldsymbol{v} + \nabla p = 0\,, \qquad \nabla\cdot\boldsymbol{v} = 0\,, \qquad x\in\Omega_1\,,\\ \boldsymbol{v}|_{\Sigma} = \boldsymbol{a}(x) = (-x_2, x_1, 0)\,,\\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma} = 0\,, \qquad \boldsymbol{\tau}^{(i)}\cdot S(\boldsymbol{v})\boldsymbol{n}|_{\Gamma} = 0\,, \qquad i=1,2\,,\\ \boldsymbol{v}(x)\to 0\,, \qquad p(x)\to 0\,, \qquad (x\to\infty)\,,\end{cases}$$

and

$$(3.35)\qquad \begin{cases} -\nu\nabla^2\boldsymbol{v} + \nabla p = 0\,, \qquad \nabla\cdot\boldsymbol{v} = 0\,, \qquad x\in\Omega_2\,,\\ \boldsymbol{v}|_{\Sigma} = 0\,, \qquad \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma} = 0\,, \qquad \boldsymbol{\tau}^{(i)}\cdot S(\boldsymbol{v})\boldsymbol{n}|_{\Gamma} = 0\,, \qquad i=1,2\,,\\ \int_S v_3(x',0)\,dx' = 1\,,\\ \boldsymbol{v}(x) - \boldsymbol{v}_-(x)\to 0\,, \qquad p(x) - p_-(x) - \overline{p}\to 0\,, \qquad (x_3\to-\infty)\,,\\ \boldsymbol{v}(x)\to 0\,, \qquad p(x)\to 0\,, \qquad (|x|\to\infty,\, x_3>0)\,.\end{cases}$$

THEOREM 3.5. – *If* $\boldsymbol{a}$ *satisfies the hypotheses of Theorem* 1.1, *then problem* (3.34) *has a unique axisymmetrical solution* $\boldsymbol{v}\in C^{l+2}_s(\Omega_1, 2))$, $p\in C^{l+1}_{s-1}(\Omega_1, 3)$ (Γ, l, s *are the same as in Theorem* 3.1), *and*

$$(3.36)\qquad |\boldsymbol{v}|_{C^{l+2}_s(\Omega_1,\,2)} + |p|_{C^{l+1}_{s-1}(\Omega_1,\,3)} \leqslant c|\boldsymbol{a}|_{C^{l+2}(\gamma)}\,.$$

PROOF. – We construct the solution in the form $\boldsymbol{v}(x) = A(x) + \boldsymbol{u}(x)$, $\boldsymbol{A}(x) = \boldsymbol{a}(x)\,\omega(x)$ where $\omega(x)$ is an axisymmetrical smooth cut-off function equal to one in the neighbourhood of ∂V and to zero for large $|x'|$ and $|x_3|$. It is easily verified that $\nabla\cdot\boldsymbol{A}=0$, $\boldsymbol{A}\cdot\boldsymbol{n}|_\Gamma=0$, $S(\boldsymbol{A})\boldsymbol{n}|_\Gamma=\boldsymbol{a}(\partial\omega/\partial n)|_\Gamma$, so, for $(\boldsymbol{u}=\boldsymbol{v}-\boldsymbol{A},\,p)$ we obtain problem (3.29) where the data $\boldsymbol{f}, g, b, d_1$ are smooth and have compact supports bounded away from M. By Theorem 3.3, this problem is uniquely solvable, and (3.35) follows from (3.31) (moreover, it can be shown that $\boldsymbol{v}=v_\varphi(r, x_3)\, e_\varphi$ and $p=0$). The theorem is proved.

THEOREM 3.6. – *Problem* (3.35) *has a unique axisymmetrical solution with the following properties:* $\boldsymbol{v}\in C^{l+2}_{s,0}(\Omega_2, 2)$, $\nabla p\in C^l_{s-2,0}(\Omega, 4)$, $v_\varphi=0$ *(*Γ, l, s *are the same as in Theorem* 3.2*), and*

$$|\boldsymbol{v}|_{C^{l+2}_{s,0}(\Omega_2,2)}+|\nabla p|_{C^l_{s-1,0}(\Omega_2,4)}\leqslant c\,. \tag{3.37}$$

PROOF. – We construct the solution in the form

$$\boldsymbol{v}(x)=\zeta_-(x)\,\boldsymbol{v}_-(x)+\zeta_+(x)\,\boldsymbol{v}_+(x)+\boldsymbol{u}(x)\,,$$

$$p(x)=\zeta_-(x)\,p_-(x)+\zeta_+(x)\,p_+(x)+q(x)\,,$$

where $(\boldsymbol{v}_-, p_-)$ is a Poiseuille flow with a unit net flux,

$$\boldsymbol{v}_+(x)=\frac{3x_3^2\boldsymbol{x}}{2\pi|x|^5}\,,\qquad p_+(x)=\frac{\nu}{\pi}\left(\frac{1}{|x|^3}-\frac{3x_3^2}{|x|^5}\right),$$

$\zeta_-(x)=\zeta_-(x_3)$ and $\zeta_+(x)=\zeta_+(|x|, x_3)$ are smooth cut-off functions such that $\zeta_-(x_3)=1$ for $x_3>-2$, $\zeta_-(x_3)=0$ for $x_3>-1$, $\zeta_+(x)=1$ for $|x|>4d_0$, $x_3>0$, $\zeta_+(x)=0$ for $|x|<2d_0$ and for $x_3<0$, and, finally, $(\boldsymbol{u}, q)$ is a solution of problem (3.30) where $\boldsymbol{f}(x)$, $g(x)$ are smooth functions with compact supports,

$$b(x)=-\zeta_+\boldsymbol{v}_+\cdot\boldsymbol{n}=-\frac{3\zeta_+x_3^2}{2\pi|x|^5}(\boldsymbol{n}\cdot\boldsymbol{x})|_\Gamma\,,$$

$$d_1(x)=\frac{3\zeta_+}{2\pi}\left(-\frac{2x_3n_3(\boldsymbol{x}\cdot\boldsymbol{\tau})+2x_3\tau_3(\boldsymbol{x}\cdot\boldsymbol{n})}{|x|^5}+\frac{10x_3^2(\boldsymbol{x}\cdot\boldsymbol{n})(\boldsymbol{x}\cdot\boldsymbol{\tau})}{|x|^7}\right)-$$

$$\frac{3x_3^2}{2\pi|x|^5}\left((\boldsymbol{x}\cdot\boldsymbol{\tau})\frac{\partial\zeta_+}{\partial n}+(\boldsymbol{x}\cdot\boldsymbol{n})\frac{\partial\zeta_+}{\partial\tau}\right)\Bigg|_\Gamma$$

$d_2=0$. Since $\boldsymbol{x}\cdot\boldsymbol{n}=-r\sin\alpha+x_3\cos\alpha$, $\alpha\in C^{l+2}_s(J_0, 1)$ and $x_3'\in C^{l+2}_s(J_0, 1)$ (which means that $x_3(t)$ may have only logarithmic growth at infinity), it is clear that $b\in C^{l+2}_s(\Gamma, 2+\beta)$, $d_1\in C^{l+1}_{s-1}(\Gamma, 3+\beta)$, so, the existence of axisymmetrical $\boldsymbol{u}\in C^{l+1}_{s,a}\Omega_2, 2)$, $\nabla q\in C^l_{s-2,a}(\Omega_2, 4)$, such that $q\in C^{l+1}_{s-1}(\Omega_2, 3)$, $u_\varphi=0$ follows from Theorem 3.4.

The uniqueness of the solution in the class indicated in the formulation of the theorem follows from the fact that the solution of a homogeneous problem decays exponentially as $x_3 \to -\infty$; as a consequence, it vanishes, which follows from the energy estimate. The theorem is proved.

4. – Second auxiliary problem.

In this section we consider Cauchy and boundary value problem for ordinary differential equations arising after linearization of (1.14) and (1.16), i.e.,

$$(4.1)\qquad \begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\dfrac{r\psi'(r)}{(1+h_0'^2(r))^{3/2}} - b_0^2\,\psi(r) = f(r), & r > d_0\,, \\ \psi'_r\,|_{r=d_0} = 0\,, \quad \psi'(r) \to 0\,, & r \to \infty\,, \end{cases}$$

where $b_0^2 = g_0/\sigma$, and

$$(4.2)\quad \xi'(s) + \frac{\cos\alpha_0(s)}{r_0(s)}\xi(s) + \frac{\sin\alpha_0(s)}{r_0^2(s)}\int_0^s \sin\alpha_0(t)\,\xi(t)\,dt = g(s)\,, \qquad \xi(0) = 0\,.$$

($h_0(r)$, $\alpha_0(s)$ are defined in § 1). We prove the following theorems.

THEOREM. – 4.1. – *For arbitrary* $f \in C_{s-1}^{l+1}(J_{d_0}, 3)$, $s \in (0, l+2]$, *problem* (4.1) *has a unique solution* $\psi \in \tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)$, *and*

$$(4.3)\qquad |\psi|_{\tilde{C}_{s+1}^{l+3}(J_{d_0},3)} \le c|f|_{C_{s-1}^{l+1}(J_{d_0},3)}\,.$$

THEOREM 4.2. – *For arbitrary* $g \in C_{s-1}^{l+1}(J_0, 3)$, $s \in (0, l+2]$, *problem* (4.2) *has a unique solution* $\xi \in C_{s+1}^{l+3}(J_0, 1)$, *and*

$$(4.4)\qquad |\xi|_{C_s^{l+2}(J_0,1)} \le c|g|_{C_{s-1}^{l+1}(J_0,3)}\,.$$

We start with a less complicated second theorem.

PROOF OF THEOREM 4.2. – We make use of the relation $\sin\alpha_0(s) = c_0 r_0(s)^{-1}$ (see § 1), write (4.2) in the form

$$(4.5)\qquad \frac{d}{ds}r_0(s)\xi(s) + \frac{c_0^2}{r_0^2(s)}\int_0^s \frac{\xi(t)}{r_0(t)}dt = r_0(s)\,g(s)\,, \qquad \xi(0) = 0\,,$$

and introduce new dependent and independent variables

$$\zeta(s) = r_0(s)\, \xi(s),$$

$$u = -\int_s^\infty \frac{dt}{r_0^2(t)} \in [-u_0, 0], \quad u_0 = \int_0^\infty \frac{dt}{r_0^2(t)} .$$

Then (4.5) is transformed into

$$\frac{d\hat{\zeta}}{du} + c_0^2 \int_{-u_0}^{u} \hat{\zeta}(v)\, dv = \hat{r}_0^3 \hat{g}(u), \qquad u \in [-u_0, 0],$$

$$\hat{\zeta}(-u_0) = 0$$

where $\hat{\zeta}(u(s)) = \zeta(s)$ etc. Hence,

$$\hat{\zeta}(u) = \int_{-u_0}^{u} \cos c_0(u-v)\, \hat{g}(v)\, \hat{r}_0^3(v)\, dv, \tag{4.6}$$

This formula gives a bound for $|\hat{\zeta}|$. We have

$$|\hat{\zeta}(u)| \leqslant \int_{-u_0}^{0} |\hat{g}(v)|\hat{r}_0^3(v) dv \leqslant \sup_{t>0} |g(t)| \varrho(t, 3, 1-s) \int_0^\infty \frac{r_0(t)\, dt}{\varrho(t, 3, 1-s)} \leqslant$$

$$\leqslant c \sup_{t>0} |g(t)| \varrho(t, 3, 1-s).$$

Hence,

$$|\xi(t)| \leqslant c r_0^{-1}(t) \sup_{\tau>0} |g(\tau)| \varrho(\tau, 3, 1-s) \leqslant \frac{c}{1+t} \sup_{\tau>0} |g(\tau)| \varrho(\tau, 3, 1-s).$$

Further estimates can be deduced from (4.2). Assume that $s \in (0, 1)$. For arbitrary $t' \in (0, t)$ we have

$$\xi(t+t') - \xi(t) = \int_t^{t+t'} \xi'(\tau)\, d\tau = \int_t^{t+t'} \left(g(\tau) - \frac{\cos \alpha_0(\tau)}{r_0(\tau)} \xi(\tau) + \frac{c_0^2}{r_0^3} \int_0^\tau \frac{\xi(\tau')}{r_0(\tau')} d\tau' \right) d\tau$$

which gives

$$|\xi(t+t')-\xi(t)|\leq c\sup_{r>0}|g(r)|\varrho(r,3,1-s)\int_t^{t+t'}\frac{d\tau}{\varrho(\tau,3,1-s)}+$$

$$\sup_{r>0}|\xi(r)|(1+r)\left(\int_t^{t+t'}\frac{d\tau}{r_0(\tau)(1+\tau)}+\int_t^{t+t'}\frac{d\tau}{r_0^3(\tau)}\int_0^\tau\frac{d\tau'}{r_0(\tau')(1+\tau')}\right)\leq$$

$$ct'^s\big(\sup_{r>0}|g(r)|\varrho(r,3,1-s)+\sup_{r>0}|\xi(r)|(1+r)\big)\leq ct'^s\sup_{r>0}|g(r)|\varrho(r,3,1-s);$$

hence,

$$|\xi|_{C^s(J_0)}\leq c\sup_{r>0}|g(r)|\varrho(r,3,1-s).$$

Further, for arbitrary $s\in(0,l+2]$ we have

$$|\xi'(t)|\leq\frac{|\xi(t)|}{r_0(t)}+\sup_{\tau>0}(1+\tau)|\xi(\tau)|\frac{c_0^2}{r_0^3(t)}\int_0^t\frac{d\tau'}{1+\tau'}+|g(t)|\leq$$

$$c(\varrho(t,2,1-s))^{-1}\sup_{\tau>0}|g(\tau)|\varrho(\tau,3,1-s).$$

In the same way, differentiating (4.2), we can estimate higher order derivatives of ξ and their Hölder constants which leads to (4.4). Theorem 4.2 is proved.

Theorem 4.1 is proved in several steps. First of all, it is convenient to introduce a generalized solution of problem (4.1) as a function $\psi(r)$, $r>d_0$, with a finite norm

$$\left(\int_{d_0}^\infty r(\psi'^2+\psi^2)\,dr\right)^{1/2}\tag{4.7}$$

satisfying the integral identity

$$\int_{d_0}^\infty\left(\frac{\psi'(r)\eta'(r)}{(1+h_0'^2(r))^{3/2}}+b_0^2\psi\eta\right)r\,dr=-\int_{d_0}^\infty f\eta r\,dr\tag{4.8}$$

where $\eta(r)$ is an arbitrary function also with a finite norm (4.7).

PROPOSITION 4.1. – *For arbitrary* $f(r)$ *with*

$$\sup_{r>d_0}\varrho(r,3,1-s)|f(r)|\equiv|||f|||<\infty$$

problem (4.1) *has a unique generalized solution, and it satisfies the inequality*

$$\int_{d_0}^{\infty} (\psi'^2(r) + \psi^2(r))\, r\, dr \leq c\, |||f|||^2 . \tag{4.9}$$

Moreover, for arbitrary $t > d_0 + 2$

$$\int_{t}^{t+1} (\psi'^2 + \psi^2)\, r\, dr \leq c t^{-5}\, |||f|||^2 \tag{4.10}$$

with the constant c *independent of* t.

PROOF. – The existence of a unique generalized solution follows from the Riesz representation theorem, since $\int_{d_0}^{\infty} f\eta r\, dr$ is a linear functional in the space $\mathcal{H}$ of functions with a finite norm (4.7):

$$\Big|\int_0^{\infty} f\eta r\, dr\Big| \leq |||f||| \int_0^{\infty} \varrho^{-1}(r, 3, 1-s)\, |\eta(r)|\, r\, dr \leq$$

$$\leq c\, |||f||| \left(\int_{d_0}^{d_0+1} (r-d_0)^{s-1}\, |\eta(r)|\, dr + \int_{d_0+1}^{\infty} r^{-2}\, |\eta(r)|\, dr \right) \leq$$

$$\leq c\, |||f||| \left[\left(\int_{d_0}^{d_0+1} (\eta_r^2 + \eta^2)\, dr \right)^{1/2} + \left(\int_{d_0+1}^{\infty} |\eta(r)|^2\, r\, dr \right)^{1/2} \right] \leq c\, |||f|||\, \|\eta\|_{\mathcal{H}} .$$

Taking $\eta = \psi$ in (4.8) we easily obtain (4.9). Inequality (4.10) can be proved by the «technics of the Saint-Venant principle» in the form presented in [14]. We assume that $t > d_0 + 2$ and put in (4.8) $\eta(r) = \psi(r)\chi_k(r)$ with

$$\chi_k(r) = \begin{cases} 1, & \text{if } t-k < r < t+k+1, \\ 0, & \text{if } r > t+k+2 \text{ or } r < t-k-1, \\ r-(t-k-1), & \text{if } t-k-1 < r \leq t-k, \\ t+k+2-r, & t+k+1 \leq r < t+k+2, \end{cases}$$

$k=0, 1, 2, \ldots$, to obtain

$$\int_{t-k-1}^{t+k+2}\left(\frac{r\psi_r^2}{(1+h_0'^2(r))^{3/2}}+b_0^2 r\psi^2\right)\chi_k(r)\,dr+\int_{t-k-1}^{t+k+2}\frac{r\psi_r\psi}{(1+h_0'^2)^{3/2}}\,\chi_k'(r)\,dr=$$

$$-\int_{t-k-1}^{t+k+2} rf(r)\,\psi(r)\,\chi(r)\,dr\leqslant |||f|||\left(\int_{t-k-1}^{t+k+2} r\psi^2\,dr\right)^{1/2}\left(\int_{t-k-1}^{t+k+2}\frac{dr}{r^5}\right)^{1/2}\leqslant$$

$$|||f|||\left(\int_{t-k-1}^{t+k+2} r\psi^2(r)\,dr\right)^{1/2}\left(\frac{2k+3}{(t-k-1)^5}\right)^{1/2}.$$

After easy calculations we show that $y_k=\int_{t-k}^{t+k+1}(\psi_r^2+\psi^2)\,r\,dr$ satisfies the inequality

$$y_k\leqslant c_0(y_{k+1}-y_k)+c_1F_k\,,$$

with $F_k=|||f|||^2\,\frac{2k+3}{(t-k-1)^5}$. Hence,

$$y_k\leqslant\frac{c_0}{c_0+1}y_{k+1}+\frac{c_1}{c_0+1}F_k$$

and

$$y_k\leqslant\frac{c_1}{c_0+1}(F_k+\delta F_{k+1}+\ldots+\delta^m F_{m+k})+\delta^{m+1}y_{k+m+1}$$

where $\delta=c_0/(c_0+1)\in(0,1)$, $m>0$, $k+m<[t]-1$. In particular, taking $k=0$, $m=\beta[t]-1$, $\beta\in(0,1)$ and making use of the inequality

$$t^5F_j\delta^j=|||f|||^2\,\frac{\delta^j(2j+3)}{(1-(j+1)/t)^5}\leqslant|||f|||^2\,\frac{\delta^j(2j+3)}{(1-\beta)^5}\,,$$

we obtain

$$y_0\leqslant\frac{c_1}{c_0+1}\,|||f|||^2t^{-5}(1-\beta)^{-5}\sum_{j=0}^{m}\delta^j(2j+3)+c\delta^{\beta[t]}\,|||f|||^2\leqslant ct^{-5}\,|||f|||^2\,,$$

q.e.d. The proposition is proved.

We need one more lemma concerning the estimate of the convolution integral

$$v(r)=\int_{-\infty}^{\infty}K(r-s)\,F(s)\,ds \tag{4.11}$$

with $K(z) = e^{-b_0|z|}$ or $K(z) = e^{-b_0|z|}\operatorname{sign} z$ in weighted Hölder norms

$$\|F\|_m = \sup_{r > d_0 + 1} r^m |F(r)|, \qquad m > 0,$$

$$\|F\|_{m,\alpha} = \|F\|_m + \sup_{r > d_0+1} r^{m+\alpha} \sup_{0 < t < r/2} t^{-\alpha} |F(z+t) - F(z)|, \qquad \alpha \in (0, 1).$$

PROPOSITION 4.2. – *Let $F(t)$ be a function with $\|F\|_{m,\alpha} < \infty$ vanishing for $t < d_0 + 1$. Then the convolution* (4.11) *satisfies the inequalities*

$$\|v\|_m \leqslant c\|F\|_m, \tag{4.12}$$

$$\|v\|_{m,\alpha} \leqslant c\|F\|_{m,\alpha}. \tag{4.13}$$

PROOF. – Inequality (4.12) follows immediately from the elementary estimate

$$\int_{-\infty}^{\infty} e^{-b_1|r-s|}(1 + |s|)^{-k}\, ds \leqslant c(1 + |r|)^{-k}, \qquad k, b_1 > 0; \tag{4.14}$$

indeed,

$$|v(r)| \leqslant \|F\|_m \int_{d_0+1}^{\infty} e^{-b_0|r-s|}\, |s|^{-m}\, ds \leqslant cr^{-m}\|F\|_m .$$

Now, we evaluate the difference $v(r+\varrho) - v(r)$ with $r > d_0$, $\varrho \in (0, r/2)$. We have

$$v(r+\varrho) - v(r) = \int_{3\varrho/2}^{\infty} K(r-s)[F(s+\varrho) - F(s)]\, ds +$$

$$\int_0^{3\varrho/2} K(r-s)[F(s+\varrho) - F(s)]\, ds + \int_{-\infty}^0 K(r-s)\, F(s+\varrho)\, ds \equiv I_1 + I_2 + I_3,$$

Using again (4.14), we obtain

$$|I_1| \leqslant c\|F\|_{m,\alpha}\varrho^{\alpha} \int_{\max(d_0+1,\, 3\varrho/2)}^{\infty} e^{-b_0|r-s|}\, s^{-m-\alpha}\, ds \leqslant c\varrho^{\alpha} r^{-m-\alpha}\|F\|_{m,\alpha};$$

$$|I_2| \leqslant c\|F\|_m \int_0^{3\varrho/2} e^{-b_0|r-s|}[(1 + |s+\varrho|)^{-m} + (1+s)^{-m}]\, ds \leqslant$$

$$\|F\|_m \int_0^{3\varrho/2} e^{-b_0|r-s|}\left[(1+|s+\varrho|)^{-m}\left(\frac{5\varrho}{2(s+\varrho)}\right)^\alpha + (1+s)^{-m}\left(\frac{3\varrho}{2s}\right)^\alpha\right] ds \leqslant$$

$$c\|F\|_m \varrho^\alpha r^{-m-\alpha};$$

finally, we observe that $I_3 = 0$ in the case $\varrho < d_0 + 1$; hence,

$$|I_3| \leqslant e^{-(b_0 r)/2}\|F\|_m \int_{d_0+1-\varrho}^{0} e^{-\frac{b_0}{2}(r-s)} |s+\varrho|^{-m} ds \left(\frac{\varrho}{d_0+1}\right)^\alpha \leqslant$$

$$c\left(\frac{\varrho}{d_0+1}\right)^\alpha r^{-m} e^{-(b_0 r)/2}\|F\|_m \leqslant c\varrho^\alpha r^{-m-\alpha}\|F\|_m,$$

so, (4.13) is proved.

PROOF OF THEOREM 4.1. – We estimate weighted Hölder norm of a weak solution of problem (4.1). First of all, since $W_2^1(t, t+1)$ is continuously imbedded into $C(t, t+1)$, we have

$$\sup_{t<r<t+1} |\psi(r)| \leqslant c\left(\int_t^{t+1} (\psi_r'^2 + \psi^2)\, dr\right)^{1/2} \leqslant$$

$$ct^{-1/2}\left(\int_t^{t+1} (\psi_r'^2 + \psi^2)\, r\, dr\right)^{1/2} \leqslant ct^{-3}\, |||f|||, \qquad \text{if } t \geqslant d_0 + 2.$$

From this inequality and from (4.9) we obtain

$$\sup_{r>d_0} r^3 |\psi(r)| \leqslant c|||f||| . \tag{4.15}$$

Further, we consider $\tilde\psi(r) = \psi(r) - \psi(d_0+2)$ as a solution of the Sturm-Liouville problem

$$\begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\dfrac{r\tilde\psi_r'(r)}{(1+h_0'^2(r))^{3/2}} - b_0^2\tilde\psi = f(r) + b^2\psi(d_0+2) \equiv \tilde f(r), \\ r \in (d_0, d_0+2) \equiv I, \qquad \tilde\psi'|_{r=d_0} = 0, \qquad \tilde\psi|_{r=d_0+2} = 0. \end{cases} \tag{4.16}$$

From the representation formula of the solution of this problem in terms of the Green function and from the equation (4.16) we obtain (see [13])

$$|\psi|_{C^{l+3}_{s+1}(I)} \leqslant (|\psi(d_0+2)| + |\tilde{\psi}|_{C^{l+3}_{s+1}(I)}) \leqslant \tag{4.17}$$

$$\leqslant c|\tilde{f}|_{C^{l+1}_{s-1}(I)} + |\psi(d_0+2)| \leqslant c|f|_{C^{l+1}_{s-1}(I)}$$

where

$$|\psi|_{C^l_\sigma(I)} = \sum_{\sigma<j<l} \sup_I (r-d_0)^{j-\sigma)} \left| \frac{d^j\psi(r)}{dr^j} \right| +$$

$$\sup_I (r-d_0)^{(l-\sigma)} \sup_{|r'-r|<(r-d_0)/2} |r'-r|^{[l]-l} \left| \frac{d^{[l]}\psi(r')}{dr'^{[l]}} - \frac{d^{[l]}\psi(r)}{dr^{[l]}} \right| + |\psi|_{C^\sigma(I)},$$

$0<\sigma\leqslant l$ (in the case $\sigma<0$, as usual, the term $|\psi|_{C^\sigma(I)}$ is omitted).

Finally, we consider $\psi(r)$ in the interval $r>d_0+1$. From the equation (4.1) for ψ and from (4.15) we deduce

$$\sup_{r>d_0+1} r^3|\psi'(r)| + \sup_{r_0>d_0+1} r^3|\psi''(r)| \leqslant c|||f|||\,,$$

and, after the differentiation of this equation,

$$\sum_{i=0}^{[l]+3} \sup_{r>d_0+1} r^3 \left| \frac{d^i\psi(r)}{dr^i} \right| \leqslant c\left(\sum_{i=0}^{[l]+1} \sup_{r>d_0+1} r^3 \left| \frac{d^if(r)}{dr^i} \right| + |||f||| \right). \tag{4.18}$$

To get a sharper estimate of the derivatives of ψ,we introduce the functions $\omega(r)=r\psi(r)$ and $\Sigma(r)=\omega(r)\,\mu(r)$ where $\mu\in C_0^\infty(R)$, $\mu(t)=1$ for $t\geqslant d_0+2$, $\mu(t)=0$ for $t\leqslant d_0+1$, and we set $\Sigma(r)=0$ for $r\leqslant d_0+1$. These function satisfy the equations

$$\frac{d}{dr}\,p(r)\frac{d\omega(r)}{dr} - b_0^2\,\omega = rf + \frac{d}{dr}\left(\frac{p(r)}{r}\omega\right), \qquad r>d_0\,, \tag{4.19}$$

where $p(r)=(1+h_0'^2(r))^{-3/2}$, and

$$\frac{d^2\Sigma}{dr^2} - b_0^2\Sigma = \frac{dF_1}{dr} + F_0\,, \qquad r\in R\,,$$

$$F_1(r) = \mu(r)\left[(1-p(r))\frac{d\omega}{dr} + \frac{p(r)}{r}\omega\right],$$

$$F_0(r) = r\mu f + \mu'(r)\left[(1-p(r))\frac{d\omega}{dr} + \frac{p(r)}{r}\omega\right] + 2\mu'\omega' + \mu''\omega$$

(we assume that $\Sigma, F_1, F_0 = 0$ for $r < d_0 + 1$). Hence,

$$\Sigma(r) = -\frac{1}{2b_0}\int_{-\infty}^{\infty} e^{-b_0|r-s|}\left(\frac{dF_1(s)}{ds} + F_0(s)\right) ds =$$

$$-\frac{1}{2b_0}\int_{-\infty}^{\infty} e^{-b_0|r-s|}(b_0 \operatorname{sign}(r-s)\, F_1(s) + F_0(s))\, ds\,.$$

When we differentiate this formula and make use of Proposition 4.2, we obtain

$$\|D_r^j \Sigma\|_{2+j,\,\alpha} \le c(\|D_r^j F_1\|_{2+j,\,\alpha} + \|D_r^j F_0\|_{2+j,\,\alpha}) \le \tag{4.20}$$

$$c\left(\sum_{i=0}^{j}\|D_r^i \omega\|_{1+i,\,\alpha} + \sum_{i=0}^{j}\|D_r^i f\|_{3+i,\,\alpha} + \sum_{i=0}^{j}|D_r^i \omega(r)|_{C^\alpha(J_{d_0+1})}\right)$$

where $\alpha = l - [l]$ (we have used the fact that the function $1 - p(r)$ and the derivatives of $p(r)$ decay exponentially at infinity). Now, using (4.18) and the elementary interpolation inequality

$$\|D_r^i \omega\|_{1+i,\,\alpha} \le \varepsilon\|D_r^i \omega\|_{2+i,\,\alpha} + c(\varepsilon)\,|D_r^i \omega(r)|_{C^\alpha(J_{d_0+1})} \qquad \forall \varepsilon \in (0,\,1)\,,$$

we obtain from (4.18) and (4.20) the following estimate of $D_r^j \omega(r)$:

$$\|D_r^j \omega\|_{2+j,\,\alpha} \le \varepsilon \sum_{i=0}^{j}\|D_r^i \omega\|_{2+i,\,\alpha} + c'(\varepsilon)\left(\sum_{i=0}^{j}\|D_r^i f\|_{3+i,\,\alpha} + |||f|||\right).$$

Hence,

$$\sum_{i=0}^{[l]+1}\|D_r^i \omega\|_{2+i,\,\alpha} \le c\left(\sum_{i=0}^{[l]+1}\|D_r^i f\|_{3+i,\,\alpha} + |||f|||\right). \tag{4.21}$$

The derivatives $D_r^i \omega$, $i = [l]+2, [l]+3$, may be expressed in terms of lower order derivatives with the help of equation (4.19), and it can be shown that

$$\|D_r^{[l]+2}\omega\|_{3+[l]} + \|D_r^{[l]+3}\omega\|_{3+[l],\,\alpha}$$

also can be estimated by the right-hand side of (4.21). Clearly, these two estimates, together with (4.18), imply (4.3). The theorem is proved.

5. – Proof of Theorems 1.1 and 1.2.

We begin this section with the construction of special mappings of the domains Ω_{0i}, $i = 1, 2$, corresponding to the rest state in both problems (see § 1) onto Ω_i.

PROPOSITION 5.1. – *Let the surface Γ be given by equation $x_3 = h(r)$, $r = |x'| > d_0$, and*

$$|h - h_0|_{\tilde{C}^{l+3}_{s+1}(J_{d_0},\,3)} \leq \delta_1 \ll 1\,. \tag{5.1}$$

There exists an invertible axisymmetrical mapping $x = \mathcal{Z}_1(y)$ of the domain Ω_{01} onto Ω_1 with the following properties:

1) $\mathcal{Z}_1(y', h_0(|y'|)) = h(|y'|)$.

2) *The elements J_{km} of the Jacobi matrix of $\mathcal{Z}_1$, as well as the elements J^{km} of the Jacobi matrix of the inverse transformation $\mathcal{Z}_1^{-1}$ satisfy the inequality*

$$|J_{km} - \delta_{km}|_{\tilde{C}^{l+2}_s(\Omega_{01},\,4)} + |J^{km} - \delta_{km}|_{\tilde{C}^{l+2}_s(\Omega_{01},\,4)} \leq c|h - h_0|_{\tilde{C}^{l+3}_{s+1}(J_{d_0},\,3)} \tag{5.2}$$

where $s \in (0, s_0)$,

$$|u|_{\tilde{C}^{l+2}_s(\Omega_{01},\,b)} = |u|_{C^s(\Omega_{01})} + \sum_{0 \leq |j| < l+2} \sup_{\Omega_{01}} \varrho(y, q(|j|) + b, |j| - s)|D^j u(y)| +$$

$$\sum_{|j| = [l]+2} \sup_{\Omega_{01}} \varrho(y, l+b, l+2-s) \sup_{K(y)} |z - y|^{[l]-l} |D^j u(y) - D^j u(z)|\,,$$

$q(|j|) = |j|$ for $|j| \leq [l]$, $q(|j|) = [l]$ for $|j| = [l]+1, [l]+2$, $K(y) = \{z \in \Omega_1$: $|z - y| \leq \varrho(y, 1, 1)/2\}$,

$$\varrho(y, k, m) = \begin{cases} |y|^k, & \text{if } |y| > 2d_0, \\ (\operatorname{dist}(y, M))^{\max(m,\,0)}, & \text{if } \operatorname{dist}(y, M) < d_0/2\,. \end{cases}$$

and if the functions $h_1(|y'|)$ and $h_2(|y'|)$ satisfy (5.1), *then the elements of corresponding Jacobi matrices $J^{(i)}_{km}$ and $J^{(i)km}$, $i = 1, 2$, satisfy the inequalities*

$$|J^{(1)}_{km} - J^{(2)}_{km}|_{\tilde{C}^{l+2}_s(\Omega_{01},\,4)} + |J^{(1)km} - J^{(2)km}|_{\tilde{C}^{l+2}_s(\Omega_{01},\,4)} \leq c|h_1 - h_2|_{\tilde{C}^{l+3}_{s+1}(J_{d_0},\,3)} \tag{5.3}$$

PROPOSITION 5.2. – *Let Γ' be given by* (1.5) *with $r(t) = d_0 + \int_0^t \cos\alpha(t')\,dt'$, $x_3(t) = \int_0^t \sin\alpha(t')\,dt'$, $\alpha(0) = 3\pi/2 - \vartheta$ and*

$$|\alpha - \alpha_0|_{C^{l+2}_s(J_0,\,1)} \leq \delta_2 \ll 1\,, \qquad s \in (0, \min(s_0, \pi/2\vartheta))\,. \tag{5.4}$$

There exists an invertible axisymmetrical mapping $x = \mathcal{Z}_2(y)$ of the domain Ω_{02} onto Ω_2 with the following properties:

1) *If $|y'| = r_0(t)$, $y_3 = x_{03}(t)$, then $|x'| = r(t)$, $x_3 = x_3(t)$*

2) $\mathcal{Z}_2(y) = y$ *for $y \in \Sigma$ and for $y_3 < -1$.*

3) *The elements J_{km} of the Jacobi matrix of $\mathcal{Z}_2$, as well as the elements*

J^{km} *of the Jacobi matrix of the inverse transformation* $\mathcal{Z}_2^{-1}$, *satisfy the inequality*

$$(5.5)\quad |J_{km}-\delta_{km}|_{\overset{\circ}{C}{}^{l+2}_{s_1}(\Omega_{02},1)}+|J^{km}-\delta_{km}|_{\overset{\circ}{C}{}^{l+2}_{s_1}(\Omega_{02},1)}\leqslant c|\alpha-\alpha_0|_{C^{l+2}_s(J_0,1)}$$

where $s_1=\min(s,1)$,

$$|u|_{\overset{\circ}{C}{}^{l+2}_s(\Omega_{02},b)}=\sum_{0\leqslant|j|<l+2}\sup_{\Omega_{02}}\varrho(y,|j|+b,|j|-s)|D^ju(y)|+$$

$$\sum_{|j|=[l]+2}\sup_{\Omega_{01}}\varrho(y,l+2+b,l+2-s)|\sup_{K(y)}|z-y|^{[l]-l}|D^ju(y)-D^ju(z)|\,,$$

$$K(y)=\{z\in\Omega_1:\ |z-y|\leqslant\varrho(y,1,1)/2\},$$

$$\varrho(y,k,m)=\begin{cases}|y|^k, & \text{if } |y|>2d_0\,,\\ (\operatorname{dist}(y,M))^m, & \text{if } \operatorname{dist}(y,M)<d_0/2\,.\end{cases}$$

(*in particular,* (5.2) *shows that* $J_{km}(y)=\delta_{km}$ *for* $y\in M$). *Moreover, if the curves* Γ_1' *and* Γ_2' *satisfy the above conditions, then the elements of the corresponding Jacobi matrices satisfy the inequalities*

$$(5.6)\ |J^{(1)}_{km}-J^{(2)}_{km}|_{\overset{\circ}{C}{}^{l+2}_{s_1}(\Omega_{02},1)}+|J^{(1)km}-J^{(2)km}|_{\overset{\circ}{C}{}^{l+2}_{s_1}(\Omega_{02},1)}\leqslant c|\alpha_1-\alpha_2|_{C^{l+2}_s(J_0,1)}\,.$$

PROOF OF PROPOSITION 5.1. – Using standard methods (see [13], § 2, and [16], Theorem 4.1), we can construct an extension $\Phi\in\tilde{C}^{l+3}_{s+1}(\Omega_{01},3)$ of $h(|y'|)-h_0(|y'|)$ from Γ_0 into Ω_{01} satisfying the inequality

$$(5.7)\qquad |\Phi|_{\tilde{C}^{l+3}_{s+1}(\Omega_{01},3)}\leqslant c|h-h_0|_{\tilde{C}^{l+3}_{s+1}(J_{d_0},3)}$$

and possessing the property of axial symmetry (this can be easily achieved if we pass to cylindrical coordinates and make all the constructions on the plane (r,x_3)). Moreover, multiplying our extension by an appropriate cut-off function we can satisfy the condition

$$\Phi(|y'|,y_3)=0\qquad\text{for } y_3<-m_1\,.$$

The mapping $\mathcal{Z}_1$ can be defined by the formula

$$x'=y',\qquad x_3=y_3+\Phi(y);$$

it is easily seen that (5.2), (5.3) follow from (5.7). The proposition is proved.

PROOF OF PROPOSITION 5.2. – We pass to cylindrical coordinates, set $\xi_1=r$, $\xi_2=x_3$ and define a mapping of the domain G_0 onto G (see the notation in § 3) as follows:

(5.8) $$\eta = \xi + \Phi(\xi), \qquad \xi \in G_0 ,$$

Here $\Phi(\xi) = (\Phi_1(\xi), \Phi_2(\xi))$ is a vector field satisfying the boundary conditions

$$\Phi(\xi)\,|_{\Sigma'} = 0 , \qquad \Phi(\xi_0(s)) = \xi(s) - \xi_0(s) ,$$

$$\xi_0(s) = (r_0(s), x_{03}(s)) , \qquad \xi(s) = (r(s), x_3(s)) .$$

From the relations

$$r(t) - r_0(t) = \int_0^t [\cos \alpha(\tau) - \cos \alpha_0(\tau)]\, d\tau =$$

$$- (\alpha(t) - \alpha_0(t)) \int_0^1 du \int_0^t \sin (\alpha_0 + u(\alpha - \alpha_0))\, d\tau ,$$

$$x_0(t) - x_{03}(t) = (\alpha(t) - \alpha_0(t)) \int_0^1 du \int_0^t \cos (\alpha_0 + u(\alpha - \alpha_0))\, d\tau ,$$

it follows that $\xi - \xi_0 \equiv (r - r_0, x_3 - x_{03}) \in C^{l+3}_{s+1}(J_0, 0)$, and

$$|r - r_0|_{C^{l+3}_{s+1}(J_0, 0)} + |x_3 - x_{03}|_{C^{l+3}_{s+1}(J_0, 0)} \leq c|\alpha - \alpha_0|_{C^{l+2}_s(J_0, 1)} .$$

Therefore, using standard methods, we can construct an extension of $\xi(t) - \xi_0(t)$, $\Phi_0 \in C^{l+3}_{s+1}(G, 0)$, from Γ' into G, such that

$$|\Phi_0|_{C^{l+3}_{s+1}(G, 0)} \leq c|\alpha - \alpha_0|_{C^{l+2}_s(J_0, 1)} .$$

We can also assume that

$$\Phi_0(\xi) = 0 \qquad \text{for } \xi_1 \equiv r < d_0/3 \text{ and for } \xi_2 \equiv x_3 < -1 .$$

Now, we set

$$\Phi(\xi) = \Phi_0(\xi)\, \chi(\xi_1 - d_0, \xi_2)$$

where $\chi(\eta) = \chi_0(\eta/|\eta|)$, χ_0 is a smooth function given on the unit circle and satisfying the conditions $\chi_0(0, -1) = 0$, $\chi_0(\eta) = 1$ for $\eta_2 > 0$. Clearly, $\Phi(\xi)$ satisfies the necessary boundary conditions both on Γ' and on Σ', moreover, as $\xi(0) = \xi_0(0)$ and $\xi'(0) = \xi_0'(0)$, $\Phi_0(\xi)$ vanishes at the contact point $(d_0, 0)$ together with its first derivatives, and

(5.9) $$|\Phi|_{\mathring{C}^{l+3}_{s1+1}(\Omega_{02}, 0)} \leq c|\alpha - \alpha_0|_{C^{l+2}_s(J_0, 1)} .$$

We define the mapping $\mathcal{Z}_2$ by formula (5.8) or, in the Cartesian coordinates

$x = (x_1, x_2, x_3)$, by

$$x_i = y_i + \frac{y_i}{|y'|}\Phi_1(|y'|, y_3), \qquad i = 1, 2, \quad x_3 = y_3 + \Phi_2(|y'|, y_3).$$

It is easily seen that inequalities (5.2) follow from (5.9). The proposition is proved.

Let us proceed to the proof of Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1. – We map Ω onto Ω_0 by means of the transformation $\mathcal{Z}_1^{-1}$ and we observe that the operators ∇ and $S(\boldsymbol{v})$ are transformed into $\nabla' = \sum_{m=1}^{3} (J^{mk}(\partial/\partial y_m))_{k=1,2,3}$ and

$$S'(\boldsymbol{v}) = \left(\sum_{m=1}^{3} J^{mk}\frac{\partial v_i}{\partial y_m} + \sum_{m=1}^{3} J^{mi}\frac{\partial v_k}{\partial y_m}\right)_{i,k=1,2,3},$$

respectively. Hence, in the coordinates $y = \mathcal{Z}_1^{-1}(x) \in \Omega_0$, (1.13) take the form

$$(5.10)\quad \begin{cases} -\nu\nabla'^2\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla')\,\boldsymbol{v} + \nabla' p = 0\,, \qquad \nabla'\cdot\boldsymbol{v}(y) = 0\,, \\ \boldsymbol{v}|_{\Sigma_0} = \varepsilon\boldsymbol{a}\,, \qquad \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma_0} = 0\,, \qquad \boldsymbol{\tau}^{(i)}\cdot S'(\boldsymbol{v})\,\boldsymbol{n}|_{\Gamma_0} = 0\,, \qquad i = 1, 2\,, \\ \boldsymbol{v}(y) \to 0\,, \qquad p(y) \to 0\,, \qquad (|y| \to \infty) \end{cases}$$

where $\boldsymbol{a}(y) = (-y_2, y_1, 0)$ and $\boldsymbol{\tau}^{(i)}$, $\boldsymbol{n}$ are tangential and normal vectors to Γ, respectively. Since

$$\boldsymbol{n} = \left(-\frac{h_{y_1}}{\sqrt{1+\nabla' h'^2}}, -\frac{h_{y_2}}{\sqrt{1+\nabla' h'^2}}, \frac{1}{\sqrt{1+\nabla' h'^2}}\right)$$

and

$$\boldsymbol{n}_0 = \left(-\frac{h_{0y_1}}{\sqrt{1+\nabla' h_0'^2}}, -\frac{h_{0y_2}}{\sqrt{1+\nabla' h_0'^2}}, \frac{1}{\sqrt{1+\nabla' h_0'^2}}\right)$$

are related to each other by the formula

$$\boldsymbol{n} = \frac{(J^{-1})^\top \boldsymbol{n}_0}{|(J^{-1})^\top \boldsymbol{n}_0|}$$

(J is the Jacobi matrix of the mapping $\mathcal{Z}_1$), the condition $\boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0$ is equivalent to $J^{-1}\boldsymbol{v}\cdot\boldsymbol{n}_0|_{\Gamma_0} = 0$. We multiply this condition and equation $\nabla'\cdot\boldsymbol{v} = 0$ by $\det J$

and write (5.10) in the form

$$(5.11)\quad \begin{cases} -\nu\nabla^2 \boldsymbol{v} + \nabla p = \nu(\nabla'^2 - \nabla^2)\,\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla')\,\boldsymbol{v} + (\nabla - \nabla')\,p \equiv \boldsymbol{f}[\boldsymbol{v}, p, \psi], \\ \nabla\cdot\boldsymbol{v} = (\nabla - \det J(J^{-1})^\top\,\nabla)\cdot\boldsymbol{v} \equiv g[\boldsymbol{v}, \psi], \\ \boldsymbol{v}|_{\Sigma_0} = \varepsilon\boldsymbol{a}, \qquad \boldsymbol{v}\cdot n_0|_{\Gamma_0} = (I - \det JJ^{-1})\,\boldsymbol{v}\cdot\boldsymbol{n}_0|_{\Gamma_0} \equiv b[\boldsymbol{v}, \psi], \\ \boldsymbol{\tau}_0^{(i)}\cdot S(\boldsymbol{v})\,\boldsymbol{n}_0|_{\Gamma_0} = (\boldsymbol{\tau}_0^{(i)}\cdot S(\boldsymbol{v})\,\boldsymbol{n}_0 - \boldsymbol{\tau}^{(i)}\cdot S'(\boldsymbol{v})\,\boldsymbol{n})|_{\Gamma_0} \equiv d_i[\boldsymbol{v}, \psi], \quad i=1, 2, \\ \boldsymbol{v}(y)\to 0, \qquad p(y)\to 0, \qquad (|y|\to\infty), \end{cases}$$

where $\psi = h - h_0$ (this function determines completely the transformation $\mathcal{Z}_1$ and the vectors $\boldsymbol{n}$ and $\boldsymbol{\tau}^{(1)}$, whereas $\boldsymbol{\tau}^{(2)} = \boldsymbol{\tau}_0^{(2)} = \boldsymbol{e}_\varphi$).

As

$$\det J\nabla'\cdot\boldsymbol{v} = \nabla\cdot((\det J)J^{-1}\boldsymbol{v}),$$

the expressions $g[\boldsymbol{v}, \psi]$ and $b[\boldsymbol{v}, \psi]$ are related to each other by the formula

$$\int_{\Omega_{01}} g[\boldsymbol{v}, \psi]\,dx = \int_{\Gamma_0} b[\boldsymbol{v}, \psi]\,dS.$$

In addition, the following proposition holds.

PROPOSITION 5.3. – 1) *For arbitrary* $\boldsymbol{v}\in C_s^{l+2}(\Omega_{01}, 2)$, $p\in C_{s-1}^{l+1}(\Omega_{01}, 3)$, $s\in(0, s_0)$, *and arbitrary small* $\psi\in\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)$ (*so small that* (5.1) *is satisfied*) *there hold the estimates*

$$(5.12)\quad \begin{cases} |g[\boldsymbol{v}, \psi]|_{C_{s-1}^{l+1}(\Omega_{01}, 3+\beta)} + |b[\boldsymbol{v}, \psi]|_{C_s^{l+2}(\Gamma_0, 2+\beta)} + \\ \qquad \sum_{i=1}^{2} |d_1[\boldsymbol{v}, \psi]|_{C_{s-1}^{l+1}(\Gamma_0, 3+\beta)} \leq c|\psi|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}|\boldsymbol{v}|_{C^{l+2}(\Omega_{01}, 2)}, \\ |\boldsymbol{f}[\boldsymbol{v}, p, \psi]|_{C_{s-2}^{l}(\Omega_{01}, 4+\beta)} \leq \\ \qquad c|\psi|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}\big(|\boldsymbol{v}|_{C_s^{l+2}(\Omega_{01}, 2)} + |\nabla p|_{C_{s-2}^{l}(\Omega_{01}, 4)}\big) + c|\boldsymbol{v}|^2_{C_s^{l+2}(\Omega_{01}, 2)}. \end{cases}$$

2) *If* $(\boldsymbol{v}_1, p_1, \psi_1)$ *and* $(\boldsymbol{v}_2, p_2, \psi_2)$ *satisfy the above hypotheses, then*

$$(5.13)\quad |\boldsymbol{f}[\boldsymbol{v}_1, p_1, \psi_1] - \boldsymbol{f}[\boldsymbol{v}_2, p_2, \psi_2]|_{C_{s-2}^{l}(\Omega_{01}, 4+\beta)} +$$

$$|g[\boldsymbol{v}_1, \psi_1] - g[\boldsymbol{v}_2, \psi_2]_{C_{s-1}^{l+1}(\Omega_{01}, 3+\beta)} +$$

$$|b[\boldsymbol{v}_1, \psi_1] - b[\boldsymbol{v}_2, \psi_2]|_{C_s^{l+2}(\Gamma_0, 2+\beta)} + \sum_{i=1}^{2} |d_i[\boldsymbol{v}_1, \psi_1] - d_i[\boldsymbol{v}_2, \psi_2]|_{C_{s-1}^{l+1}(\Gamma_0, 3+\beta)} +$$

$$c|\psi_1 - \psi_2|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}\big(|\boldsymbol{v}_2|_{C_s^{l+2}(\Omega_{01}, 2)} + |\boldsymbol{v}_2|^2_{C_s^{l+2}(\Omega_{01}, 2)} + |\nabla p_2|_{C_{s-2}^{l}(\Omega_{01}, 4)}\big) +$$

$$c|\boldsymbol{v}_1 - \boldsymbol{v}_2|_{C_s^{l+2}(\Omega_{0,1}, 2)}\big(|\psi_1|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)} + |\boldsymbol{v}_1|_{C_s^{l+2}(\Omega_{01}, 2)} + |\boldsymbol{v}_2|_{C_s^{l+2}(\Omega_{01}, 2)}\big) +$$

$$c|\nabla(p_1 - p_2)|_{C_{s-2}^{l}(\Omega_{01}, 4)}|\psi_1|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}.$$

Estimates (5.12), (5.13) are obtained as a result of standard and somewhat lengthy calculations based on the inequality

$$(5.14)\qquad |fu|_{C^{l+i}_{s_2}(\Omega_{01},\, b_1)} \leqslant c|f|_{\tilde{C}^{l+2}_s(J_{d_0},\, 4)}\, |u|_{C^{l+i}_{s_2}(\Omega_{01},\, b_2)}\,,$$

$$0 \leqslant i \leqslant 2\,, \quad s_2 \leqslant s\,, \quad b_1 \leqslant 2 + b_2\,.$$

In virtue of (5.14) and (5.1),

$$|g[\boldsymbol{v},\, \psi]|_{C^{l+1}_{s-1}(\Omega_{01},\, 3+\beta)} \leqslant c \sum_{k,\, m=1}^{3} |J^{mk} - \delta_{mk}|_{\tilde{C}^{l+2}_s(\Omega_{01},\, 4)}\, |D\boldsymbol{v}|_{C^{l+1}_{s-1}(\Omega_{01},\, 3)} \leqslant$$

$$c|\psi|_{\tilde{C}^{l+3}_{s+1}(J_{d_0},\, 3)}\, |\boldsymbol{v}|_{C^{l+2}_s(\Omega_{01},\, 2)}\,.$$

Further, since

$$g[\boldsymbol{v}_1,\, \psi_1] - g[\boldsymbol{v}_2,\, \psi_2] = (I - \det J_1 (J_1^{-1})^{\top})\nabla \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) +$$

$$(\det J_1 (J_1^{-1})^{\top} - (\det J_2 (J_2^{-1})^{\top})\nabla \cdot \boldsymbol{v}_2\,,$$

we have

$$|g[\boldsymbol{v}_1,\, \psi_1] - g[\boldsymbol{v}_2,\, \psi_2]|_{C^{l+1}_{s-1}(\Omega_{01},\, 3+\beta)} \leqslant c\big(|\psi_1|_{\tilde{C}^{l+3}_{s+1}(J_{d_0},\, 3)}\, |\boldsymbol{v}_1 - \boldsymbol{v}_2|_{C^{l+2}_s(\Omega_{01},\, 2)} +$$

$$|\psi_1 - \psi_2|_{\tilde{C}^{l+3}_{s+1}(J_{d_0},\, 3)}\, |\boldsymbol{v}_2|_{C^{l+2}_s(\Omega_{01},\, 2)}\big)\,.$$

Similar estimates hold for $b[\boldsymbol{v},\, \psi]$, $d_i[\boldsymbol{v},\, \psi]$ and for $\boldsymbol{f}[\boldsymbol{v},\, p,\, \psi]$ which is a linear combination of the terms $(J^{mk} - \delta_{mk})D^2 v_i$, $DJ^{mk} Dv_i$, $(J^{mk} - \delta_{mk})(\partial p/\partial y_i)$ and $(\boldsymbol{v} \cdot \nabla')\, v_i$.

Let $\mathcal{L} = (\mathcal{L}_1,\, \mathcal{L}_2)$ be a linear operator assigning the solution of problem (3.29) in the domain Ω_{01} to the data $R = (\boldsymbol{f},\, g,\, b,\, d_1,\, d_2)$, according to the formulas $\boldsymbol{v} = \mathcal{L}_1 R$, $p = \mathcal{L}_2 R$. It is clear that (5.10) is equivalent to the equations

$$\boldsymbol{v} = \varepsilon \boldsymbol{v}_0 + \mathcal{L}_1 R[\boldsymbol{v},\, p,\, \psi] \equiv \varepsilon \boldsymbol{v}_0 + \mathcal{A}_1[\boldsymbol{v},\, p,\, \psi]\,,$$

$$p = \varepsilon p_0 + \mathcal{L}_2 R[\boldsymbol{v},\, p,\, \psi] \equiv \varepsilon p_0 + \mathcal{A}_2[\boldsymbol{v},\, p,\, \psi]\,,$$

where $R[\boldsymbol{v}, p, \psi] = (\boldsymbol{f}[\boldsymbol{v}, p, \psi],\, g[\boldsymbol{v},\, \psi],\, b[\boldsymbol{v},\, \psi],\, d_1[\boldsymbol{v},\, \psi],\, d_2[\boldsymbol{v},\, \psi])$ and $(\boldsymbol{v}_0, p_0)$ is a solution to the problem (3.34). In view of Proposition 5.3 and of the continuity of the operator $\mathcal{L}$, we have

$$(5.15)\quad \begin{cases} |\mathcal{A}_1[\boldsymbol{v}, p, \psi]|_{C_s^{l+2}(\Omega_{01}, 2)} + |\mathcal{A}_2[\boldsymbol{v}, p, \psi]|_{C_{s-1}^{l+1}(\Omega_{01}, 3)} \leqslant \\ \qquad c|\psi|_{\tilde{C}_{s-1}^{l+3}(J_{d0})}(|\boldsymbol{v}|_{C_s^{l+2}(\Omega_{01}, 2)} + |p|_{C_{s-1}^{l+1}(\Omega_{01}, 3)}) + c|\boldsymbol{v}|^2_{C_s^{l+2}(\Omega_{01}, 2)}, \\ |\mathcal{A}_1[\boldsymbol{v}_1, p_1, \psi_1] - \mathcal{A}_1[\boldsymbol{v}_2, p_2, \psi_2]|_{C_s^{l+2}(\Omega_{01}, 2)} + \\ |\mathcal{A}_2[\boldsymbol{v}_1, p_1, \psi_1] - \mathcal{A}_2[\boldsymbol{v}_2, p_2, \psi_2]|_{C_{s-1}^{l+1}(\Omega_{01}, 3)} \leqslant \\ c|\boldsymbol{v}_1 - \boldsymbol{v}_2|_{C_s^{l+2}(\Omega_{01}, 2)}(|\psi_1|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)} + |\boldsymbol{v}_1|_{C_2^{l+2}(\Omega_{01}, 2)} + |\boldsymbol{v}_2|_{C_s^{l+2}(\Omega_{01}, 2)}) + \\ c|\psi_1 - \psi_2|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}(|\boldsymbol{v}_2|_{C_s^{l+2}(\Omega_{01}, 2)} + |\boldsymbol{v}_2|^2_{C_s^{l+2}(\Omega_{01}, 2)} + |p_2|_{C_{s-1}^{l+1}(\Omega_{01}, 3)}) + \\ c|p_1 - p_2|_{C_{s-1}^{l+1}(\Omega_{01}, 3)}|\psi_1|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}. \end{cases}$$

Equations (1.14) can also be written in a similar way. When we subtract from (1.14) analogous relations (1.8) for h_0, we obtain

$$(5.16)\quad \begin{cases} \dfrac{\sigma}{r}\dfrac{d}{dr}\dfrac{\psi'(r)}{(1+h_0'^2)^{3/2}} - g_0\psi(r) = f(r) + Q[\psi], \\ \psi'(0) = 0, \qquad \psi(r) \to 0, \qquad (r \to \infty), \end{cases}$$

where $f(r) = \boldsymbol{n}\cdot T(\boldsymbol{v}, p)\boldsymbol{n}|_{x_3 = h_0(r) + \psi(r)}$,

$$Q[\psi] = -\frac{3\sigma}{r}\frac{d}{dr} r\psi'^2(r)\int_0^1 (1-u)\frac{h_0'(r) + u\psi'(r)}{[1 + (h_0'(r) + u\psi'(r))^2]^{3/2}}du.$$

For arbitrary $\psi, \psi_1, \psi_2 \in \tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)$ satisfying (5.1) we have

$$|Q[\psi]|_{C_{s-1}^{l+1}(J_{d_0}, 3)} \leqslant c|\psi|^2_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)},$$

$$|Q[\psi_1] - Q[\psi_2]|_{C_{s-1}^{l+1}(J_{d_0}, 3)} \leqslant c|\psi_1 - \psi_2|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}(|\psi_1|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)} + |\psi_2|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}).$$

We denote by $\mathcal{L}_3$ a linear operator which makes correspond the solution of problem (4.1) to the function f in the right hand side of the equation and write (5.13) in the form

$$\psi = \sigma^{-1}\mathcal{L}_3(f + Q[\psi]).$$

As

$$f = (-p + \nu\boldsymbol{n}\cdot S'(\boldsymbol{v})\boldsymbol{n})|_{x_3 = h_0(r)} = \varepsilon f_0 + R_1[\boldsymbol{v}, p, \psi, \varepsilon],$$

$$f_0 = \boldsymbol{n}_0\cdot T(\boldsymbol{v}_0, p_0)\boldsymbol{n}_0|_{x_3 = h_0(r)},$$

$$R_1[\boldsymbol{v}, p, \psi, \varepsilon] = (-\mathcal{A}_2[\boldsymbol{v}, p, \psi] + \nu\boldsymbol{n}\cdot S'(\boldsymbol{v})\boldsymbol{n} - \varepsilon\nu\boldsymbol{n}_0\cdot S(\boldsymbol{v}_0)\boldsymbol{n}_0)|_{x_3 = h_0(r)} =$$

$$(-\mathcal{A}_2[\boldsymbol{v}, p, \psi] + \varepsilon\nu\boldsymbol{n}\cdot S'(\boldsymbol{v}_0)\boldsymbol{n} - \varepsilon\nu\boldsymbol{n}_0\cdot S(\boldsymbol{v}_0)\boldsymbol{n}_0) + \nu\boldsymbol{n}\cdot S'(\boldsymbol{v} - \varepsilon\boldsymbol{v}_0)\boldsymbol{n})|_{x_3 = h_0(r)},$$

we have

$$\psi = \varepsilon\psi_0 + \mathcal{A}_3[\boldsymbol{v}, p, \psi, \varepsilon]$$

where

$$\psi_0 = \mathcal{L}_3 f_0, \qquad \mathcal{A}_3[\boldsymbol{v}, p, \psi, \varepsilon] = \sigma^{-1}\mathcal{L}_3(Q[\psi] + R_1[\boldsymbol{v}, p, \psi, \varepsilon]).$$

It is easily seen that

$$|R_1[\boldsymbol{v}, p, \psi, \varepsilon]|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} \leqslant$$
$$c|\psi|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)}((|\boldsymbol{v}|_{C^{l+2}_s(\Omega_{01}, 2)} + |p|_{C^{l+1}_{s-1}(\Omega_{01}, 3)} + \varepsilon) + c|\boldsymbol{v}|^2_{C^{l+2}_s(\Omega_{01}, 2)}$$

and

$$|R_1[\boldsymbol{v}_1, p_1, \psi_1, \varepsilon] - R_1[\boldsymbol{v}_2, p_2, \psi_2, \varepsilon]|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} \leqslant$$
$$c|\psi_1 - \psi_2|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)}(|\boldsymbol{v}_2|_{C^{l+2}_s(\Omega_{01}, 2)} + |\boldsymbol{v}_2|^2_{C^{l+2}_s(\Omega_{01}, 2)} + |p_2|_{C^{l+1}_{s-1}(\Omega_{01}, 3)} + \varepsilon) +$$
$$c|\boldsymbol{v}_1 - \boldsymbol{v}_2|_{C^{l+2}_s(\Omega_{01}, 2)}(|\psi_1|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} + |\boldsymbol{v}_1|_{C^{l+2}_s(\Omega_{01}, 2)} + |\boldsymbol{v}_2|_{C^{l+2}_s(\Omega_{01}, 2)}) +$$
$$c|p_1 - p_2|_{C^{l+1}_{s-1}(\Omega_{01}, 3)}|\psi_1|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)},$$

if $\boldsymbol{v}_i, p_i, \psi_i$ satisfy the hypotheses of Proposition 5.3. Hence,

$$\text{(5.17)}\quad \begin{cases} |\mathcal{A}_3[\boldsymbol{v}, p, \psi, \varepsilon]|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} \leqslant \\ \quad c|\psi|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)}(|\boldsymbol{v}|_{C^{l+2}_s(\Omega_{01}, 2)} + |p|_{C^{l+1}_{s-1}(\Omega_{01}, 3)} + |\psi|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} + \varepsilon) + \\ \quad c|\boldsymbol{v}|^2_{C^{l+2}_s(\Omega_{01}, 2)}, \\ |\mathcal{A}_3[\boldsymbol{v}_1, p_1, \psi_1, \varepsilon] - \mathcal{A}_3[\boldsymbol{v}_2, p_2, \psi_2, \varepsilon]|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} \leqslant c|\psi_1 - \psi_2|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} \cdot \\ (|\boldsymbol{v}_2|_{C^{l+2}_s(\Omega_{01}, 2)} + |\boldsymbol{v}_2|^2_{C^{l+2}_s(\Omega_{01}, 2)} + |p_2|_{C^{l+1}_{s-1}(\Omega_{01}, 3)} + \varepsilon) + \\ c|\psi_1 - \psi_2|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)}(|\psi_1|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} + |\psi_2|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)}) + \\ c|\boldsymbol{v}_1 - \boldsymbol{v}_2|_{C^{l+2}_s(\Omega_{01}, 2)}(|\psi_1|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)} + |\boldsymbol{v}_1|_{C^{l+2}_s(\Omega_{01}, 2)} + |\boldsymbol{v}_2|_{C^{l+2}_s(\Omega_{01}, 2)}) + \\ c|p_1 - p_2|_{C^{l+1}_{s-1}(\Omega_{01}, 3)}|\psi_1|_{\tilde{C}^{l+3}_{s+1}(J_{d_0}, 3)}. \end{cases}$$

Thus, problem (1.13), (1.14) reduces to the equations

$$\boldsymbol{v} = \varepsilon\boldsymbol{v}_0 + \mathcal{A}_1[\boldsymbol{v}, p, \psi],$$

$$p = \varepsilon p_0 + \mathcal{A}_2[\boldsymbol{v}, p, \psi],$$

$$\xi = \varepsilon\xi_0 + \mathcal{A}_3[\boldsymbol{v}, p, \psi, \varepsilon],$$

or

$$V = \varepsilon V_0 + \mathcal{A}[V, \varepsilon], \tag{5.18}$$

where $V = (\boldsymbol{v}, p, \psi)$, $V_0 = (\boldsymbol{v}_0, p_0, \psi_0)$, $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$.

Equation (5.18) should be considered in the subspace $\mathcal{C}^l$ of the space

$$C_s^{l+2}(\Omega_{01}, 2) \times C_{s-1}^{l+1}(\Omega_{01}, 3) \times \tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)$$

whose elements $V = (\boldsymbol{v}, p, \psi)$ satisfy the condition $\psi(d_0) = 0$; the norm in $\mathcal{C}^l$ is defined, as usual, as the sum of norms of $\boldsymbol{v}$, p and ψ:

$$|V|_{\mathcal{C}^l} = |\boldsymbol{v}|_{C_s^{l+2}(\Omega_{01}, 2)} + |p|_{\tilde{C}_{s-1}^{l+1}(\Omega_{01}, 3)} + |\xi|_{\tilde{C}_{s+1}^{l+3}(J_{d_0}, 3)}.$$

Proposition 5.3 and inequalities (5.17) imply

$$|V_0|_{\mathcal{C}^l} \leqslant c_0 |\boldsymbol{a}|_{C^{l+2}(\Sigma_0)},$$

$$|\mathcal{A}[V]|_{\mathcal{C}^l} \leqslant c_1\big((|V|_{\mathcal{C}^l})^2 + \varepsilon |V|_{\mathcal{C}^l}\big),$$

$$|\mathcal{A}[V_1] - \mathcal{A}[V_2]|_{\mathcal{C}^l} \leqslant c_2 |V_1 - V_2|_{\mathcal{C}^l}(|V_1|_{\mathcal{C}^l} + |V_2|_{\mathcal{C}^l} + |V_1|^2_{\mathcal{C}^l} + |V_2|^2_{\mathcal{C}^l} + \varepsilon).$$

Hence, it follows from the contraction mapping principle that equation (5.18) has a unique solution satisfying the inequality

$$|V|_{\mathcal{C}^l} \leqslant \kappa(\varepsilon) = \frac{2c_0\varepsilon|\boldsymbol{a}|_{C^{l+2}(\Sigma_0)}}{1 - c_1\varepsilon + \sqrt{(1-c_1\varepsilon)^2 - 4c_0c_1\varepsilon|\boldsymbol{a}|_{C^{l+2}(\Sigma_0)}}}$$

($\kappa(\varepsilon)$ is a minimal root of the quadratic equation $c_1\kappa^2 - (1 - c_1\varepsilon)\kappa + c_0\varepsilon|\boldsymbol{a}|_{C^{l+2}(\Sigma_0)} = 0$), provided that ε is small enough:

$$4c_0c_1\varepsilon|\boldsymbol{a}|_{C^{l+2}(\Sigma_0)} < (1 - c_1\varepsilon)^2, \qquad c_2(2\kappa(\varepsilon) + 2\kappa^2(\varepsilon) + \varepsilon) < 1.$$

Theorem 1.1 is proved.

PROOF OF THEOREM 5.2. – We follow the arguments in the proof of Theorem

1.1. The mapping $x = \mathcal{Z}_2(y)$ transforms (1.15) into

$$(5.19)\quad \begin{cases} -\nu\nabla^2 \boldsymbol{v} + \nabla p = \nu(\nabla'^2 - \nabla^2)\,\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla')\,\boldsymbol{v} + (\nabla - \nabla')p \equiv \boldsymbol{f}[\boldsymbol{v}, p, \xi], \\ \nabla\cdot\boldsymbol{v} = (\nabla - \det J (J^{-1})^{\top}\nabla)\cdot\boldsymbol{v} \equiv g[\boldsymbol{v}, \xi], \\ \boldsymbol{v}|_{\Sigma_0} = 0, \quad \boldsymbol{v}\cdot n_0|_{\Gamma_0} = (I - \det J J^{-1})\,\boldsymbol{v}\cdot\boldsymbol{n}_0|_{\Gamma_0} \equiv b[\boldsymbol{v}, \xi], \\ \boldsymbol{\tau}_0^{(1)}\cdot S(\boldsymbol{v})\boldsymbol{n}_0|_{\Gamma_0} = (\boldsymbol{\tau}_0^{(1)}\cdot S(\boldsymbol{v})\boldsymbol{n}_0 - \boldsymbol{\tau}^{(1)}\cdot S'(\boldsymbol{v})\boldsymbol{n})|_{\Gamma_0} \equiv d_1[\boldsymbol{v}, \xi], \\ \int_S v_3(y)\,dy' = \varepsilon, \quad (y_3 < -1), \\ \boldsymbol{v}(y) - \boldsymbol{v}_-(y) \to 0, \quad p(y) - p_-(y) - \overline{p} \to 0, \quad (y_3 \to -\infty), \\ \boldsymbol{v}(y) \to 0, \quad p(y) \to 0, \quad (|y| \to \infty,\ y_3 > 0), \end{cases}$$

where $\xi = \alpha - \alpha_0$ (we note that the mapping $\mathcal{Z}_2$ and the vectors $\boldsymbol{n}$ and $\boldsymbol{\tau}^{(1)}$ are completely determined by ξ). It is easily seen that

$$\int_{\Omega_{02}} g[\boldsymbol{v}, \xi]\,dx = \int_{\Gamma_0} b[\boldsymbol{v}, \xi]\,dS,$$

moreover, the following proposition holds.

PROPOSITION 5.4. – 1) *For arbitrary* $\boldsymbol{v} \in C^{l+2}_{s,a}(\Omega_{02}, 2)$, $p \in C^{l+1}_{s-1,a}(\Omega_{02}, 3)$ *and arbitrary small* $\xi \in C^{l+2}_s(J_0, 1)$ (*so small that* (5.4) *is satisfied*) *there hold the estimates*

$$(5.20)\quad \begin{cases} |g[\boldsymbol{v}, \xi]|_{C^{l+1}_{s-1,a}(\Omega_{02}, 3+\beta)} + |b[\boldsymbol{v}, \xi]|_{C^{l+2}_s(\Gamma_0, 2+\beta)} + \\ \qquad |d_1[\boldsymbol{v}, \xi]|_{C^{l+1}_{s-1}(\Gamma_0, 3+\beta)} \leqslant c|\xi|_{C^{l+2}_s(J_0,1)}|\boldsymbol{v}|_{C^{l+2}_s(\Omega_{02}, 2)}, \\ |\boldsymbol{f}[\boldsymbol{v}, p, \xi]|_{C^{l}_{s-2,a}(\Omega_{02}, 4+\beta)} \leqslant \\ \qquad c|\xi|_{C^{l+2}_s(J_0,1)}\big(|\boldsymbol{v}|_{C^{l+2}_{s,a}(\Omega_{02}, 2)} + |\nabla p|_{C^{l}_{s-2,a}(\Omega_{02}, 4)}\big) + c|\boldsymbol{v}|^2_{C^{l+2}_{s,a}(\Omega_{02}, 2)}. \end{cases}$$

2) *If* $(\boldsymbol{v}_1, p_1, \xi_1)$ *and* $(\boldsymbol{v}_2, p_2, \xi_2)$ *satisfy the above hypotheses, then*

$$(5.21)\quad |\boldsymbol{f}[\boldsymbol{v}_1, p_1, \xi_1] - \boldsymbol{f}[\boldsymbol{v}_2, p_2, \xi_2]|_{C^{l}_{s-2,a}(\Omega_{02}, 4+\beta)} +$$

$$+ |g[\boldsymbol{v}_1, \xi_1] - g[\boldsymbol{v}_2, \xi_2]_{C^{l+1}_{s-1,a}(\Omega_{02}, 3+\beta)} +$$

$$|b[\boldsymbol{v}_1, \xi_1] - b[\boldsymbol{v}_2, \xi_2]|_{C^{l+2}_s(\Gamma_0, 2+\beta)} + |d_1[\boldsymbol{v}_1, \xi_1] - d_1[\boldsymbol{v}_2, \xi_2]|_{C^{l+1}_{s-1}(\Gamma_0, 3+\beta)} \leqslant$$

$$c|\xi_1 - \xi_2|_{C^{l+2}_s(J_0,1)}\big(|\boldsymbol{v}_2|_{C^{l+2}_{s,a}(\Omega_{02}, 2)} + |\boldsymbol{v}_2|^2_{C^{l+2}_{s,2}(\Omega_{02}, 2)} + |\nabla p_2|_{C^{l}_{s-2,a}(\Omega_{02}, 4)}\big) +$$

$$c|\boldsymbol{v}_1 - \boldsymbol{v}_2|_{C^{l+2}_{s,a}(\Omega_{02}, 2)}\big(|\xi_1|_{C^{l+2}_s(J_0,1)} + |\boldsymbol{v}_1|_{C^{l+2}_{s,a}(\Omega_{02}, 2)} + |\boldsymbol{v}_2|_{C^{l+2}_{s,a}(\Omega_{02}, 2)}\big) +$$

$$c|\nabla(p_1 - p_2)|_{C^{l}_{s-2,a}(\Omega_{02}, 4)}|\xi_1|_{C^{l+2}_s(J_0,1)}.$$

This proposition is proved by the same kind of calculations as Proposition 5.3 but instead of (5.14) there should be used the inequality

$$|fu|_{C^{l+i}_{s_3,a}(\Omega_{02},b_1)} \leqslant c|f|_{\overset{\circ}{C}{}^{l+2}_{s_1}(\Omega_{02},1)}|u|_{C^{l+i}_{s_3,a}(\Omega_{02},b_2)},$$

$$i\leqslant 2,\qquad b_1\leqslant b_2+1,\qquad s_3\leqslant s \text{ or } s<s_3<s+1,\qquad u|_M=0.$$

We denote by $\mathcal{L}=(\mathcal{L}_1,\mathcal{L}_2)$ a linear operator which makes correspond the solution of problem (3.30) in the domain Ω_{02} to the data $R=(\boldsymbol{f},g,b,d_1)$, according to the formulas $\boldsymbol{v}=\mathcal{L}_1R$, $p=\mathcal{L}_2R$. It is clear that (5.19) is equivalent to the equations

$$\boldsymbol{v}=\varepsilon\boldsymbol{v}_0+\mathcal{L}_1R[\boldsymbol{v},p,\xi],\qquad p=\varepsilon p_0+\mathcal{L}_2R[\boldsymbol{v},p,\xi] \tag{5.22}$$

where $R[\boldsymbol{v},p,\xi]=(\boldsymbol{f}[\boldsymbol{v},p,\xi],g[v,\xi],b[\boldsymbol{v},\xi],d_1[\boldsymbol{v},\xi])$ and $(\boldsymbol{v}_0,p_0)$ is a solution to the problem (3.35). In contrast to $\mathcal{L}_iR$, $\boldsymbol{v}_0$ and p_0 do not decay as $x_3\to-\infty$, so we introduce the functions $\boldsymbol{u}=\boldsymbol{v}-\varepsilon\boldsymbol{v}_0$, $q=p-p_0$ and write (5.22) in the form

$$\begin{cases}\boldsymbol{u}=\mathcal{L}_1R[\boldsymbol{u}+\varepsilon\boldsymbol{v}_0,q+\varepsilon p_0,\xi]\equiv\mathcal{B}_1[\boldsymbol{u},q,\xi,\varepsilon],\\ p=\mathcal{L}_2R[\boldsymbol{v}+\varepsilon\boldsymbol{v}_0,q+\varepsilon p_0,\xi]\equiv\mathcal{B}_2[\boldsymbol{u},q,\xi,\varepsilon].\end{cases} \tag{5.23}$$

Since $\boldsymbol{v}$ and p enter into $R[\boldsymbol{v},p,\xi]$ with a multiplier proportional to $J^{km}-\delta_{km}$ or to its derivatives (except for the nonlinear term $(\boldsymbol{v}\cdot\nabla')\boldsymbol{v}$), the following analogue of (5.20) holds:

$$\begin{cases}|g[\boldsymbol{u}+\varepsilon\boldsymbol{v}_0,\xi]|_{C^{l+1}_{s-1,a}/\Omega_{02},3+\beta)}+\\ \qquad |b[\boldsymbol{u}+\varepsilon\boldsymbol{v}_0,\xi]|_{C^{l+2}_{s}(\Gamma_0,2+\beta)}+|d_1[\boldsymbol{u}+\varepsilon\boldsymbol{v}_0,\xi]|_{C^{l+1}_{s-1}(\Gamma_0,3+\beta)}\leqslant\\ \qquad c|\xi|_{C^{l+2}_{s}(J_0,1)}(|\boldsymbol{u}|_{C^{l+2}_{s}(J_0,1)}+|\varepsilon|),\\ |\boldsymbol{f}[\boldsymbol{u}+\varepsilon\boldsymbol{v}_0,q+\varepsilon\boldsymbol{v}_0,\xi]|_{C^{l}_{s-2,a}(\Omega_{02},4+\beta)}\leqslant\\ \qquad c|\xi|_{C^{l+2}_{s}(J_0,1)}(|\boldsymbol{u}|_{C^{l+2}_{s,a}(\Omega_{02},2)}+|\nabla q_{C^{l}_{s-2,a}(\Omega_{02},4)}+|\varepsilon|)+\\ \qquad c(|\boldsymbol{u}|^2_{C^{l+2}_{s,a}(\Omega_{02},2)}+|\varepsilon|\,|\boldsymbol{u}|_{C^{l+2}_{s,a}(\Omega_{02},2)}+|\varepsilon|^2)\end{cases} \tag{5.24}$$

(we have taken into account that $(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0$ tends esponentially to $\varepsilon^{-2}(\boldsymbol{v}_-\cdot\nabla)\boldsymbol{v}_-=0$, as $x_3\to-\infty$, so the norm $|(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0|_{C^{l}_{s-2,a}(\Omega_{02},4)}$ is finite). Moreover, if $\boldsymbol{u}_i$, q_i, ξ_i $(i=1,2)$ satisfy the hypotheses of Proposition 5.4, then

$$|\boldsymbol{f}[\boldsymbol{u}_1+\varepsilon\boldsymbol{v}_0,q_1+\varepsilon p_0,\xi_1]-\boldsymbol{f}[\boldsymbol{u}_2+\varepsilon\boldsymbol{v}_0,q_2+\varepsilon p_2,\xi_2]|_{C^{l}_{s-2,a}(\Omega_0,4+\beta)}+ \tag{5.25}$$

$$|g[\boldsymbol{u}_1+\varepsilon\boldsymbol{v}_0,\xi_1]-g[\boldsymbol{u}_2+\varepsilon\boldsymbol{v}_0,\xi_2]_{C^{l+1}_{s-1,a}(\Omega_{02},3+\beta)}+$$

$$|b[\boldsymbol{u}_1+\varepsilon\boldsymbol{v}_0,\xi_1]-b[\boldsymbol{u}_2+\varepsilon\boldsymbol{v}_0,\xi_2]_{C^{l+2}_{s}(\Gamma_0,2+\beta)}+$$

$$|d_1[\boldsymbol{u}_1+\varepsilon\boldsymbol{v}_0,\xi_1]-d_1[\boldsymbol{u}_2+\varepsilon\boldsymbol{v}_0,\xi_2]_{C^{l+1}_{s-1}(\Gamma_0,3+\beta)}\leqslant$$

$c|\xi_1 - \xi_2|_{C_s^{l+2}(J_0,1)}(|\boldsymbol{u}_2|_{C_{s,a}^{l+2}(\Omega_{02},2)} + |\boldsymbol{u}_2|^2_{C_{s,2}^{l+2}(\Omega_{02},2)} + |\nabla q_2|_{C_{s-2,a}^{l}(\Omega_{02},4)} + |\varepsilon|) +$

$c|\boldsymbol{u}_1 - \boldsymbol{u}_2|_{C_{s,a}^{l+2}(\Omega_{02},2)}(|\xi_1|_{C_s^{l+2}(J_0,1)} + |\boldsymbol{u}_1|_{C_{s,a}^{l+2}(\Omega_{02},2)} + |\boldsymbol{u}_2|_{C_{s,a}^{l+2}(\Omega_{02},2)} + |\varepsilon|) +$

$c|\nabla(q_1 - q_2)|_{C_{s-2,a}^{l}(\Omega_{02},4)}|\xi_1|_{C_s^{l+2}(J_0,1)}\,.$

Equations (1.16) also can be written in a similar way. When we subtract (1.9) from (1.16), we obtain

$$\text{(5.26)}\qquad \begin{cases} \xi'(s) + \dfrac{\cos\alpha_0(s)}{r_0(s)}\xi(s) + \dfrac{\sin\alpha_0(s)}{r_0^2(s)}\displaystyle\int_0^s \sin\alpha_0(t)\,\xi(t)\,dt = -\sigma^{-1}f(s) + Q_1[\psi]\,, \\ \xi(0) = 0\,, \quad \xi(s)\to 0\,, \quad (s\to\infty)\,, \end{cases}$$

where

$$f(s) = \boldsymbol{n}\cdot T(\boldsymbol{v},\,p)\boldsymbol{n}|_{x_3 = x_{03}(s),\, r = r_0(s)} = \varepsilon f_0(s) + R_1[\boldsymbol{u} + \varepsilon\boldsymbol{v}_0,\, q + \varepsilon p_0,\, \xi,\, \varepsilon],$$

$$f_0(s) = \boldsymbol{n}_0\cdot T(\boldsymbol{v}_0,\, p_0)\boldsymbol{n}_0\,|_{x_3 = x_{03}(s),\, r = r_0(s)}\,,$$

$$R_1[\boldsymbol{u} + \varepsilon\boldsymbol{v}_0,\, q + \varepsilon p_0,\, \xi,\, \varepsilon] = (\mathcal{B}_2[\boldsymbol{u},\, q,\, \xi,\, \varepsilon] + \varepsilon\nu\boldsymbol{n}\cdot S'(\boldsymbol{v}_0)\,\boldsymbol{n} - \varepsilon\nu\boldsymbol{n}_0\cdot S(\boldsymbol{v}_0)\,\boldsymbol{n}_0 + \nu\boldsymbol{n}\cdot S'(\boldsymbol{u})\,\boldsymbol{n})|_{x_3 = x_{03}(s),\, r = r_0(s)}\,,$$

$$Q_1[\xi] = \frac{\xi^2(s)}{r_0(s)}\int_0^1 (1-v)\sin(\alpha_0(s) + v\xi(s))dv - \frac{\sin\alpha_0(s)}{r_0^2(s)}\int_0^s \xi^2(t)\,dt\int_0^1 (1-v)\cos(\alpha_0(t) + v\xi(t))\,dv - \frac{1}{r_0(s)}\left(\frac{\sin\alpha(s)}{r(s)} - \frac{\sin\alpha_0(s)}{r_0(s)}\right)\int_0^s \xi(t)\,dt\int_0^1 \sin(\alpha_0(t) + u\xi(t))\,du\,.$$

Hence, (5.26) is equivalent to

$$\xi = \varepsilon\xi_0 + \mathcal{B}_3[\boldsymbol{u},\, q,\, \xi,\, \varepsilon]$$

where $\xi_0 = \sigma^{-1}\mathcal{L}_3 f_0$, $\mathcal{B}_3[\boldsymbol{u},\, q,\, \xi,\, \varepsilon] = \sigma^{-1}\mathcal{L}_3(Q_1 + R_1[\boldsymbol{u},\, q,\, \xi,\, \varepsilon])$, and $\mathcal{L}_3$ is a linear operator which assigns the solution of (4.2) to $g(s)$.

Thus, we have reduced Problem 2 to the equation

$$\text{(5.27)}\qquad V \equiv (\boldsymbol{u},\, q,\, \xi) = \varepsilon V_0 + \mathcal{B}[V,\, \varepsilon]$$

with $V_0 = (0,\, 0,\, \xi)$, $\mathcal{B} = (\mathcal{B}_1,\, \mathcal{B}_2,\, \mathcal{B}_3)$ in the space

$$\mathcal{C}_a^l = C_{s,a}^{l+2}(\Omega_{02},\, 2) \times \widehat{C}_{s-1,a}^{l+1}(\Omega_{02},\, 3) \times \widehat{C}_s^{l+2}(J_0,\, 1)$$

where $\widehat{C}_s^{l+2}(J_0,\, 1)$ is the subspace of $C_s^{l+2}(J_0,\, 1)$ whose elements satisfy the condition $\xi(0)=0$ and $\widehat{C}^{l+1}_{s-1,a}(\Omega_{02}, 3)$ is the space of functions such that

$$|\nabla q|_{C^l_{s-2,a}(\Omega_{02},4)} < \infty\,, \qquad q(x) \to 0 \ (|x| \to \infty\,, x_3 > 0)$$

(but $q(x)$ may tend to a constant, as $x_3 \to -\infty$). Just as in the preceding theorem, it is easy to verify that $\mathcal{B}$ is a contraction operator in $\mathcal{C}^l_a$, and equation (5.27) has a unique solution satisfying the estimate

$$|\boldsymbol{u}|_{C^{l+2}_{s,a}(\Omega_{02},2)} + |\nabla q|_{C^l_{s-2,a}(\Omega_{02},4)} + |\xi|_{C^{l+2}_s(J_0,1)} \leq c|\varepsilon|\,.$$

Hence, Problem 2 also has a unique small solution. The theorem is proved.

Appendix: proof of inequality (2.7).

Let $A_r = \{x \in R^3_+ : r < 1 + |x| \leq 2r\}$, $B_{r,\lambda} = \mp r \in R^3_+ : r/2 + \lambda < 1 + |x| \leq 4(r-\lambda)\}$, $B'_{r,\lambda} = B_{r,\lambda} \cap \partial R^3_+$, $\lambda \in (0, r/2)$, and let $\zeta(x,\lambda)$ be a smooth cut-off function equal to one for $x \in B_{r,\lambda}$, to zero for $x \in R^3_+ \backslash B_{r,\lambda/2}$, and satisfying the inequality

$$|D^j \zeta(x,\lambda)| \leq c(j)\,\lambda^{-|j|}\,.$$

If $(\boldsymbol{v}, p)$ is a solution of (2.1), then $\boldsymbol{u} = \boldsymbol{v}\zeta(x,\lambda)$, $q = (p-\overline{p})\zeta(x,\lambda)$, $\overline{p} = \mathrm{Const}$, satisfy the relations

$$-\nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{f}\zeta - 2\nu\nabla\boldsymbol{v}\nabla\zeta - \nu\boldsymbol{v}\nabla^2\zeta + (p-\overline{p})\nabla\zeta\,,$$

$$\nabla\cdot\boldsymbol{u} = g\zeta + \nabla\zeta\cdot\boldsymbol{v}\,,$$

$$u_3|_{x_3=0} = b(x')\,\zeta\,, \qquad S_{j3}(\boldsymbol{u})|_{x_3=0} = d_j\zeta + v_j\frac{\partial\zeta}{\partial x_3} + v_3\frac{\partial\zeta}{\partial x_j}\bigg|_{x_3=0}\,, \qquad j=1,2\,.$$

Classical Schauder estimate for this problem gives

$$[\boldsymbol{v}]^{(l+2)}_{B_{r,\lambda}} + [\nabla p]^{(l)}_{B_{r,\lambda}} \leq [\boldsymbol{u}]^{(l+2)}_{R^3_+} + [\nabla q]^{(l)}_{R^3_+} \leq$$

$$c\Bigg([\boldsymbol{f}\zeta]^{(l)}_{R^3_+} + [g\zeta]^{(l+1)}_{R^3_+} + [b\zeta]^{(l+2)}_{R^2} + [\boldsymbol{d}'\zeta]^{(l+1)}_{R^2} +$$

$$[2\nu\nabla\boldsymbol{v}\nabla\zeta + \nu\boldsymbol{v}\nabla^2\zeta - (p-\overline{p})\nabla\zeta]^{(l)}_{R^3_+} + [\boldsymbol{v}\cdot\nabla\zeta]^{l+1}_{R^3_+} + \sum_{j=1}^{2}\left[v_j\frac{\partial\zeta}{\partial x_3} + v_3\frac{\partial\zeta}{\partial x_j}\right]^{(l+1)}_{R^2}\Bigg).$$

To estimate the norms in the right-hand side, we use well known interpolation inequalities. For instance, we have

$$[\boldsymbol{f}\zeta]^{(l)}_{R^3_+} \leqslant [\boldsymbol{f}]^{(l)}_{B_{r,\lambda/2}} + c\sum_{k=0}^{[l]}\Big(\lambda^{-k}[\boldsymbol{f}]^{(l-k)}_{B_{r,\lambda/2}} + \lambda^{l-k}\max_{B_{r,\lambda/2}}|D^j\boldsymbol{f}(x)|\Big) \leqslant$$

$$c\Big([\boldsymbol{f}]^{(l)}_{B_{r,\lambda/2}} + \lambda^{-l}\max_{B_{r,\lambda/2}}|\boldsymbol{f}(x)|\Big).$$

The norms of $g\zeta$, $b\zeta$, $\boldsymbol{d}'\zeta$ are estimated in a similar way. The norms of expressions containing $\boldsymbol{v}$ and $p-\overline{p}$ are evaluated with the help of interpolation inequalities with a small parameter ε:

$$[\nabla\boldsymbol{v}\nabla\zeta]^{(l)}_{B_{r,\lambda/2}} \leqslant \varepsilon[\boldsymbol{v}]^{(l+2)}_{B_{r,\lambda/2}} + c(\varepsilon)\,\lambda^{-l-2}\max_{B_{r,\lambda/2}}|\boldsymbol{v}(x)|\,,$$

$$[(p-\overline{p})\nabla\zeta]^{(l)}_{B_{r,\lambda/2}} \leqslant \varepsilon[\nabla p]^{(l)}_{B_{r,\lambda/2}} + c(\varepsilon)\,\lambda^{-l-1}\max_{B_{r,\lambda/2}}|p(x)-\overline{p}|\,.$$

If we choose the constant $\overline{p}$ in such a way that $\int_{B_{r,\lambda/2}}(p(x)-\overline{p})\,dx=0$, then

$$\max_{B_{r,\lambda/2}}|p(x)-\overline{p}| \leqslant cr\max_{B_{r,\lambda/2}}|\nabla p(x)| \leqslant cr(\nu\max_{B_{r,\lambda/2}}|\nabla^2\boldsymbol{v}(x)| + \max_{B_{r,\lambda/2}}|\boldsymbol{f}(x)|)\,.$$

Now, we evaluate $|\nabla^2\boldsymbol{v}(x)|$ with the help of the same kind of interpolation inequality, i.e.,

$$\max_{B_{r,\lambda/2}}|\nabla^2\boldsymbol{v}(x)| \leqslant \varepsilon_1\lambda^l\frac{\lambda}{r}[\boldsymbol{v}]^{(l+2)}_{B_{r,\lambda/2}} + c(\varepsilon_1)\lambda^{-2-2/l}r^{2/l}\max_{B_{r,\lambda/2}}|\boldsymbol{v}(x)|\,,$$

choose ε_1 in an appropriate way and collect all the terms. This gives

$$[\boldsymbol{v}]^{(l+2)}_{B_{r,\lambda}} + [\nabla p]^{(l)}_{B_{r,\lambda}} \leqslant c\big([\boldsymbol{f}]^{(l)}_{B_{r,\lambda/2}} + \lambda^{-l-1}r\max_{B_{r,\lambda/2}}|\boldsymbol{f}(x)| + [g]^{(l+1)}_{B_{r,\lambda/2}} +$$

$$\lambda^{-l-1}\max_{B_{r,\lambda/2}}|g(x)| + [b]^{(l+2)}_{B'_{r,\lambda/2}} + \lambda^{-l-2}\max_{B'_{r,\lambda/2}}|b(x')| + [\boldsymbol{d}']^{(l+1)}_{B'_{r,\lambda/2}} + \lambda^{-l-1}\max_{B'_{r,\lambda/2}}|\boldsymbol{d}'(x')|\big) +$$

$$\varepsilon[\boldsymbol{v}]^{l+2}_{B_{r,\lambda/2}} + \varepsilon[\nabla p]^{(l)}_{B_{r,\lambda/2}} + c'(\varepsilon)\lambda^{-(2+l)(1+1/l)}r^{1+2/l}\max_{B_{r,\lambda/2}}|v(x)|\,.$$

Multiplying this inequality by $\lambda^{(2+l)(1+1/l)}$, we obtain

$$F(\lambda) \leqslant c\varepsilon 2^{(2+l)(1+1/l)}F(\lambda/2) + K(\lambda)$$

with

$$F(\lambda) = \lambda^{(2+l)(1+1/l)}\big([\boldsymbol{v}]^{(l+2)}_{B_{r,\lambda}} + [\nabla p]^{(l)}_{B_{r,\lambda}}\big),$$

$$K(\lambda) = c\lambda^{(2+l)(1+1/l)}\big([\boldsymbol{f}]^{(l)}_{B_{r,\lambda/2}} + \lambda^{-l-1}r\max_{B_{r,\lambda/2}}|\boldsymbol{f}(x)| +$$

$$[g]^{(l+1)}_{B_{r,\lambda/2}} + \lambda^{-l-1}\max_{B_{r,\lambda/2}}|g(x)| + [b]^{(l+2)}_{B'_{r,\lambda/2}} + \lambda^{-l-2}\max_{B'_{r,\lambda/2}}|b(x')| +$$

$$[d']^{(l+1)}_{B'_{r,\lambda/2}} + \lambda^{-l-1}\max_{B'_{r,\lambda/2}}|\boldsymbol{d}'(x')|\big) + c'(\varepsilon)\,\lambda^{-(2+l)(1+1/l)}r^{1+2/l}\max_{B_{r,\lambda/2}}|\boldsymbol{v}(x)| \leqslant$$

$$c(r/2)^{(2+l)(1+1/l)}\Big([\boldsymbol{f}]^{(l)}_{B_r}+r^{(-l)}\max_{B_r}|\boldsymbol{f}(x)|+$$

$$[g]^{(l+1)}_{B_r}+r^{-l-1}\max_{B_r}|g(x)|+[b]^{(l+2)}_{B'_r}+r^{-l-2}\max_{B'_r}|b(x')|+$$

$$[\boldsymbol{d}']^{(l+1)}_{B'_r}+r^{-l-1}\max_{B'_r}|\boldsymbol{d}'(x')|\Big)+c'(\varepsilon)r^{1+2/l}\max_{B_r}|v(x)|\equiv K_0\,.$$

Hence, taking ε sufficiently small we arrive at

$$F(\lambda)\leqslant\frac{1}{2}F\left(\frac{\lambda}{2}\right)+K_0\,,$$

which implies $F(\lambda)\leqslant 2K_0$, i.e., (2.7).

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