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Contractions of Smooth Varieties II: Computations and Applications.

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Sunto. – *Una contrazione su una varietà proiettiva liscia X è data da una mappa $\varphi: X \to Z$ propria, suriettiva e a fibre connesse in una varietà irriducibile normale Z. La contrazione si dice di Fano-Mori se inoltre $-K_X$ è φ-ampio. Nel lavoro, naturale seguito e completamento delle ricerche introdotte in* [A-W3], *si studiano diversi aspetti delle contrazioni di Fano-Mori attraverso esempi* (*capitolo 1*) *e teoremi di struttura* (*capitoli 3 e 4*). *Si discutono anche alcune applicazioni allo studio di morfismi birazionali propri tra varietà complesse non singolari* (*capitolo 2*).

Introduction.

Let X be a smooth complex variety. A *contraction* of X is a proper map $\varphi: X \to Z$ onto a normal irreducible variety Z with connected fibers. The contraction φ is called *a Fano-Mori contraction* (or also *extremal contraction*) if the anti-canonical divisor $-K_X$ is φ-ample. If $K_X = \varphi^* K_Z$ then φ will be called a *crepant contraction.*

The present paper is a sequel, a sort of appendix, to [A-W3] where we give a general layout for the study of Fano-Mori contractions of smooth varieties with small fibers. In [A-W3] however we did not tackle a few of the associated problems to which we return in the present paper.

The paper is divided into four sections. In the first we describe some examples of Fano Mori contractions. The existence of the contractions which appear in our classification theorem was first proved in Section 3 of [A-W3]. Here we present them in an alternative way with the use of local coordinates and we end up with equations giving them explicitly. This explicit description allows to verify the geometric picture which we presented in Section 6 of [ibid]. We find it very instructive.

The second section deals with some general applications to the study of birational map between smooth complex varieties. In particular we characterize birational morphisms whose inverse have an exceptional center of dimension less or equal than two (Proposition 2.2) which improves our earlier result from [A-W1].

The classification of elementary Fano-Mori contractions of a smooth 3-fold is one of the famous results of S. Mori ([Mo]). In the third section we slightly generalize this result to the non elementary case. Although this seems to be known to the experts, we present it here because on one hand we do not know a precise reference for it and on the other hand it is a good test for our general approach which we introduced in the section 4 of [A-W3].

Moreover Proposition 3.1 is the starting point for getting a complete classification of Fano-Mori contractions of smooth 4-fold which contract a divisor to a curve which is done in the fourth section. This classification was first obtained by H. Takagi in [Ta]; his arguments are related to local computations after noticing that the contraction is locally given by blowing-up a smooth curve in a hypersurface of $\boldsymbol{C}^5$ (see Lemma 4.4). Our approach is different and it is in the spirit of the paper [A-W3]. Namely, after describing the reduced structure of each fiber, we want to describe its normal bundle and then obtain the classification (for instance the fact that the fiber structures are reduced follows from the spannedness of the normal bundle).

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1. – Fano-Mori contractions via local coordinates.

In [A-W3] we show several examples of Fano-Mori contractions. If the fiber F is smooth a simple way to construct a contraction having it as a fiber is by taking the total space of a semiample vector bundle on F; this will be the conormal bundle. More precisely let $\mathcal{E}$ be a vector bundle over a smooth variety F and let $\boldsymbol{V}(\mathcal{E}) := \mathrm{Spec}\,(\mathrm{Sym}\,(\mathcal{E}))$ be the *total space of the dual* $\mathcal{E}^*$, where $\mathrm{Sym}\,(\mathcal{E}) = \bigoplus_{m \geqslant 0} S^m \mathcal{E}$ is the symmetric algebra of sections of $\mathcal{E}$. Note that $\boldsymbol{P}(\mathcal{O} \oplus \mathcal{E}) := \mathrm{Proj}\,(S(\mathcal{E} \oplus \mathcal{O}_F))$ is the projective closure of $\boldsymbol{V}(\mathcal{E})$.

Assume that $\mathcal{E}$ is semiample, i.e. $S^k(\mathcal{E})$ is generated by global sections for some $k > 0$. Then we let

$$\varphi : \boldsymbol{V}(\mathcal{E}) \to Z = \mathrm{Spec}\,(\bigoplus_{k \geqslant 0} H^0(F, S^k(\mathcal{E}))\,,$$

be the map associated to the evaluation of sections of $S^k(\mathcal{E})$. It is a contraction and it gives the collapsing of the zero section, call it F_0, of the total space $\boldsymbol{V}(\mathcal{E})$ to the vertex z of a cone Z. Alternatively, the map φ is the restriction of the map given by the tautological bundle ξ on $\mathrm{Proj}\,(S(\mathcal{E} \oplus \mathcal{O}_F))$; the normal bundle of F_0 into $\boldsymbol{V}(\mathcal{E})$ is $\mathcal{E}^*$.

If $-K_F - \det \mathcal{E}$ is ample then the contraction φ is a Fano-Mori contraction. Note also that the map φ is birational if and only if ξ is big, equivalently if the top Segre class of $\mathcal{E}$ is positive (if rank $\mathcal{E} = 2$ then $c_1^2 - c_2 > 0$). ξ, equivalently $\mathcal{E}$, is ample if and only if the map φ is an isomorphism outside F_0 (Grauert criterium).

In Section 3 of [A-W3] we gave several examples based on this construction and on the list of Fano bundles studied in [S-W]. We also considered complete intersections in projective bundles. Below we give an explicit computation of these examples in terms of local coordinates.

First let us recall that the vector bundle $\mathcal{S}$ is defined in the following way: Let $\boldsymbol{Q}_4 \simeq Gr(1, 3)$ be the smooth 4-dimensional quadric, identified with the Grassmaniann of lines in $\boldsymbol{P}^3$ or of linear planes in the linear space $W \simeq \boldsymbol{C}^4$. Let $\mathcal{S}$ be the universal sub-bundle which we call *the spinor bundle* on $\boldsymbol{Q}_4$. Let Y be a (codimension 1 or 2) linear section $i: Y \hookrightarrow \boldsymbol{Q}_4$. We will call again, by abuse of notation, $\mathcal{S} = i^*(\mathcal{S})$ *the spinor bundle* (see [A-W3]).

We choose a basis (w_0, w_1, w_2, w_3) of W and the coordinates z_i so that any $w \in W$ is written as $w = \sum z_i w_i$. Let us consider the Plücker embedding $\boldsymbol{Q}^4 \subset \boldsymbol{P}(\wedge^2 W)$ in which $\boldsymbol{Q}^4$ is the set of primitive 3-forms $\boldsymbol{Q}^4 = \{\omega \in \wedge^2 W : \omega \wedge \omega = 0\}$. We use Plücker coordinates $[t_{01}, t_{02}, t_{03}, t_{12}, t_{13}, t_{23}]$ associated to the above chosen coordinates, so that $\omega = \sum t_{ij} \cdot w_i \wedge w_j$. In these coordinates the quadric is given by the equation

$$q(t_{ij}) := t_{01} t_{23} - t_{02} t_{13} + t_{12} t_{03} = 0 \,. \tag{1}$$

The total space $\boldsymbol{V}(\mathcal{S}^*)$ of the bundle $\mathcal{S}$ is given tautologically in $\boldsymbol{Q}^4 \times W$ as the set $\{(\omega, w) \colon \omega \wedge w = 0\}$. For the above choice of coordinates this is written as follows

$$\begin{cases} w_3^* = t_{01} z_2 - t_{02} z_1 + t_{12} z_0 = 0 \,, & w_2^* = -t_{01} z_3 + t_{03} z_1 - t_{13} z_0 = 0 \,, \\ w_1^* = t_{02} z_3 - t_{03} z_2 + t_{23} z_0 = 0 \,, & w_0^* = -t_{12} z_3 + t_{13} z_2 - t_{23} z_1 = 0 \,. \end{cases} \tag{2}$$

Now we will construct a 3-dimensional family of codimension 2 smooth intersections and hence of 4-dimensional contractions. We introduce 3 new coordinates: p, r and s and two polynomials which are linear in t and give the following equations:

$$\begin{cases} f_1 = t_{01} - p t_{23} + z_1 t_{12} + z_2 t_{23} + r z_3 t_{13} = 0 \,, \\ f_2 = t_{02} - s t_{13} + z_0 t_{03} + z_3 t_{13} = 0 \,. \end{cases} \tag{3}$$

On $\boldsymbol{Q}_0$ these equation cut out a net of quadrics: for $p = s = 0$ the intersection is a reducible quadric, for $ps = 0$ this is a quadric cone, while for $ps \neq 0$ this is a smooth quadric.

Let $\mathcal{X}$ be a variety defined in $\boldsymbol{P}(\wedge^2 W) \times W \times \boldsymbol{C}^3$ by equations (1), (2) and (3). We have a projection on to the last three coordinates $\pi: \mathcal{X} \to \boldsymbol{C}^3$ the fibers of which we denote by X_{prs}. The projection onto W, $\phi: \mathcal{X} \to W$ can be viewed as a 3-dimensional family of birational contractions $\varphi_{prs}: X_{prs} \to W$. Computing the ideal of 5×5 minors of the Jacobi matrix of the above polynomials one can verify that $\mathcal{X}$ and X_{000} are smooth. We did it with the help of the program *Computations in Commutative Algebra* [CoCoa]. Thus, for p, r and s close to 0 the variety X_{prs} is smooth.

This example can be described from the point of view of the target. That is, we consider the birational map $W \times \boldsymbol{C}^3 - \to \mathcal{X}$ and describe its exceptional locus. If we fix z_i and (p, r, s) generally then the system of linear equations in t_j consisting of (2) and (3) is of rank 5 and thus it will have a unique line of solutions (which also satisfy (1)). Thus, if $(z_i) \neq 0$ then the exceptional locus of $W \times \boldsymbol{C}^3 - \to \mathcal{X}$ is by given the ideal of 5×5 minors of the following matrix

$$(4) \qquad \begin{pmatrix} z_2 & -z_1 & 0 & z_0 & 0 & 0 \\ -z_3 & 0 & z_1 & 0 & -z_0 & 0 \\ 0 & z_3 & -z_2 & 0 & 0 & z_0 \\ 0 & 0 & 0 & -z_3 & z_2 & -z_1 \\ 1 & 0 & 0 & z_1 & r \cdot z_3 & z_2 - p \\ 0 & 1 & z_0 & 0 & z_3 - s & 0 \end{pmatrix}.$$

Over $W \times \boldsymbol{C}^3$, however, the scheme described by this ideal is non-reduced as it has imbedded component at $(z_i) = 0$. The dominant component is described by the following functions which we obtained using [CoCoa]:

$$g_1 := z_0 z_3^3 r + z_0 z_1 z_2 z_3 + z_2 z_3^2 r + z_1 z_2^2 - z_0^2 z_3 - z_1 z_3^2 + z_1 z_3 s - z_0 z_2,$$
$$g_2 := -z_0 z_1^2 z_3 + z_0 z_2 z_3^2 - z_0 z_3^2 p - z_1^2 z_2 + z_2^2 z_3 - z_2 z_3 p + z_0 z_1,$$
$$g_3 := -z_0 z_1 z_3^2 r - z_0 z_2^2 z_3 - z_1 z_2 z_3 r + z_0 z_2 z_3 p + z_0^2 z_1 - z_2^3 + z_1^2 z_3 - z_1^2 s + z_2^2 p,$$
$$g_4 := z_1^2 z_3^2 - z_2 z_3^3 - z_0 z_3^2 r - z_1^2 z_3 s + z_2 z_3^2 s + z_3^3 p - z_3^2 sp - z_0 z_1 z_2 + z_0^2,$$
$$g_5 := -z_0^2 z_3^2 r - z_0^2 z_1 z_2 z - z_1^2 z_2 z_3 + z_2^2 z_3^2 + z_1^2 z_2 s - z_2^2 z_3 s - z_2 z_3^2 p + z_2 z_3 sp + z_0^3 + z_0 z_1 z_3 - z_0 z_1 s,$$
$$g_6 := z_0^2 z_1^2 + z_1^3 z_3 - z_0^2 z_2 z_3 - z_1 z_2 z_3^2 - z_0 z_1 z_3 r - z_1^3 s + z_1 z_2 z_3 s + + z_0^2 z_3 p + z_1 z_3^2 p - z_1 z_3 sp - z_0 z_2^2 + z_0 z_2 p\ .$$

We have thus proved the following result (see also (3.2.2) in [A-W3])

PROPOSITION (1.1). – *Let $(S_{prs}, 0)$ be an analytic germ at the origin of the surface in the germ of $W = (\boldsymbol{C}^4, 0)$ defined by the ideal $I_{prs} = (g_1, \ldots, g_6)$. If* *we blow-up* $(\boldsymbol{C}^4, 0)$ *along* $(S_{prs}, 0)$, $\varphi_{prs}: X_{prs} \to W$, *we obtain a (local) Fano-Mori contraction of a smooth 4-fold* X_{prs} *which is a smooth blow-up outside the origin and with a special fiber* $\varphi_{prs}^{-1}(0)$ *which is reduced and isomorphic to a two dimensional quadric (for* $p = s = 0$ *it is a reducible quadric, for* $ps = 0$ *it is a quadric cone, while for* $ps \neq 0$ *it is a smooth quadric). Moreover the family* $\pi : \mathcal{X} = \bigcup_{prs} X_{prs} \to \boldsymbol{C}^3$ *is a smooth, 3-dimensional family of manifolds admitting a birational contraction* $\phi: \mathcal{X} \to W \times \boldsymbol{C}^3$ *whose restriction over* $(p, r, s) \in \boldsymbol{C}^3$ *is the contraction* φ_{prs}.

Proceeding further with the analysis of this example we recover the description obtained in the section 6 [A-W3]; this time however this is given in explicit coordinates. So we compute the tangent cone of the exceptional set along $(z_i) = 0$ by taking the lowest degree terms in each of the above polynomials (we treat variables p, s and r as coefficiants, i.e. they have degree 0). If either p or s is not zero then the result is the following ideal

$$(sz_1 z_3 - z_0 z_2,\ -pz_2 z_3 + z_0 z_1,\ -sz_1^2 + pz_2^2,\ -spz_3^2 + z_0^2).$$

If both p and s are non-zero then it is the ideal of two lines:

$$\sqrt{s}\cdot z_1 - \sqrt{p}\cdot z_2 = \sqrt{p}\cdot\sqrt{s}\cdot z_3 + z_0 = 0 \quad \text{and} \quad \sqrt{s}\cdot z_1 + \sqrt{p}\cdot z_2 = \sqrt{p}\cdot\sqrt{s}\cdot z_3 - z_0 = 0$$

where $\sqrt{s}$ denotes one of the roots of the non-zero number. Over the set $\{p\cdot s = 0\} \setminus \{p = s = 0\}$ the two lines collapse to one line on which we have the double structure; we recall that over this set the special fiber of the contraction φ_{prs} is equal to the quadric cone.

This fits to the description of the contraction φ which is provided in Section 6 of [A-W3]. Namely, let us use the notation from [loc. cit.] and consider the resolution of φ which we denote by $\widehat{\varphi}$ and which is defined on the blow-up of X along the special fiber F. The map $\widehat{\varphi}$ over the exceptional divisor of the blow-up F is equal to the evaluation map of the relative sheaf $\mathcal{O}(1)$ related to the conormal of F, c.f. Proposition 5.4(d), [A-W3]. On the other hand it is known that the evaluation map on the projectivisation of the spinor bundle over a 2-dimensional (possibly singular) irreducible quadric is an inversion of a blow-up of two lines in $\boldsymbol{P}^3$ which coincide if the quadric is singular.

For $s = p = 0$ the special fiber of φ is a reducible quadric and accordingly the tangent cone is different and it is generated by the following functions

$$(z_0^2,\ z_0 z_1,\ z_0 z_2,\ -z_1 z_2 z_3 r - z_2^3 + z_1^2 z_3)$$

which defines a scheme structure over a plane rational cubic

$$z_0 = z_1 z_3(z_1 - rz_2) - z_2^3 = 0$$

with an embedded point at $z_0 = z_1 = z_2 = 0$ which is also the singularity of the cubic. For $r \neq 0$ the singularity is a node while for $r = 0$ it degenerates to a cusp.

Again, this can be explained very well on the level of the description of φ which is provided in [A-W3, (6.9)] — let us recall that we use the notation consistent with this reference. The reducible quadric $F = \{t_{01} = t_{02} = t_{03} t_{12} = 0\}$ has two components: $F_1 = \{t_{01} = t_{02} = t_{12} = 0\}$ and $F_2 = \{t_{01} = t_{02} = t_{03} = 0\}$ such that $\mathcal{S}_{|F_1} = \mathcal{O} \oplus \mathcal{O}(-1)$ and $\mathcal{S}_{|F_2} = \Omega \boldsymbol{P}^2(1)$. This is because over F_1 equations (2) give

$$t_{03} z_1 - t_{13} z_0 = -t_{03} z_2 + t_{23} z_0 = t_{13} z_2 - t_{23} z_1 = 0$$

and that is the blow-up of W along the line $z_0 = z_1 = z_2 = 0$, while over F_2 they reduce to a single equation $-t_{12} z_3 + t_{13} z_2 - t_{23} z_1 = 0$.

In [loc. cit.] it is noted that the geometry of the tangent cone is related to the scheme of jumping lines of the normal bundle of the components F_1 and F_2. We do not repeat all the argument. We merely recall that $N_{F_2/X}$ and $N_{F_1/X}$ can be recovered from $N_{F_1 \cup F_2/X} \simeq \mathcal{S}$ via a «vector bundle surgery» which was described in Section 2 of [A-W3] and which depends on the locus of singular points C_s in the blow-up $\widehat{X}$ of X along F. The curve C_s is the resolution of the singularity of the tangent cone of S. In particular, the type of the singularity of the tangent cone depends on the intersection of C_s with a fiber of $\widehat{\varphi}$ and it can be related to the jumping lines of the bundle $N_{F_1/X}$.

The bundle $N_{F_1/X}$ is not stable but it is semistable and it has the following presentation on the plane F_1:

$$0 \to \mathcal{O} \to N_{F_1/X}(1) \to \mathcal{J}_Q \to 0$$

where Q is a 0-dimensional scheme of length 2 supported on the line $F_1 \cap F_2$ which in our coordinates is $t_{01} = t_{02} = t_{03} = t_{12} = 0$. Thus the scheme of jumping lines of $N_{F_1/X}$ depends on the position of Q and as the result we have

CLAIM (1.1.1). – *The singularity of the* (*reduced*) *tangent cone of S is either a cusp or a node depending on whether Q is supported at one or, respectively, two points.*

Now we let [CoCoa] verify the above claim by computing the bundle N_{F_1}. We consider the exact sequence of bundles over F_1:

$$0 \to N_{F_1/X} \to N_{F_1/\boldsymbol{P}^5 \times W} \simeq \mathcal{O}(1)^3 \oplus \mathcal{O}^4 \to (N_{X/\boldsymbol{P}^5 \times W})_{|F_1} \to 0$$

where the map $\mathcal{O}(1)^3 \oplus \mathcal{O}^4 \to (N_{X/\boldsymbol{P}^4 \times W})_{|F_1}$ is described as follows

$$\frac{\partial}{\partial t_{ij}} \to \frac{\partial q}{\partial t_{ij}} \frac{\partial}{\partial q} + \sum_m \frac{\partial w_m^*}{\partial t_{ij}} \frac{\partial}{\partial w_m^*} + \sum_n \frac{\partial f_n}{\partial t_{ij}} \frac{\partial}{\partial f_n} \quad \text{for } (i, j) = (0, 1), (0, 2), (1, 2),$$

$$\frac{\partial}{\partial z_k} \to \frac{\partial q}{\partial z_k} \frac{\partial}{\partial q} + \sum_m \frac{\partial w_m^*}{\partial z_k} \frac{\partial}{\partial w_m^*} + \sum_n \frac{\partial f_n}{\partial z_k} \frac{\partial}{\partial f_n} \quad \text{for } k = 0, \ldots, 3 .$$

Therefore, passing to graded modules over graded rings we will consider the graded module M over the graded ring $R = \boldsymbol{C}[t_{03}, t_{13}, t_{23}, h]$ generated by the rows of the following Jacobi matrix

$$\begin{bmatrix} t_{23} & 0 & 0 & 0 & 1 & 0 \\ -t_{13} & 0 & 0 & 0 & 0 & 1 \\ t_{03} & 0 & 0 & 0 & 0 & 0 \\ 0 & ht_{13} & ht_{23} & 0 & 0 & ht_{03} \\ 0 & -ht_{03} & 0 & ht_{23} & 0 & 0 \\ 0 & 0 & -ht_{03} & -ht_{13} & ht_{23} & 0 \\ 0 & 0 & 0 & 0 & rht_{13} & ht_{13} \end{bmatrix}$$

where variable h has been added to shift the gradation of the last four rows with respect to the first three rows — which is related to the fact that the first three rows are the image of $\mathcal{O}(1)$. The variable r is in these computations treated as a constant. This way we let [CoCoa] compute the resolution of M:

$$0 \to R(-7) \to R(-4) \oplus R(-5) \oplus R(-6) \to R^3(-1) \oplus R^4(-2) \to M \to 0$$

in which the last nontrivial map comes from $\mathcal{O}(1)^3 \oplus \mathcal{O}^4 \to (N_{X/\boldsymbol{P}^5 \times W})_{|F_1}$. Thus the first syzygy of this resolution is the bundle $N_{F_1/X}$ and therefore we can compute its associated module explicitely. As the result we find out that the unique section of $N_{F_1/X}(1)$ vanishes at $t_{03} = t_{13}(t_{13} - rt_{23}) = 0$.

REMARK (1.1.2). – We would like to note the following interesting phenomenon which is apparent from our computations. Namely, in the case of the reducible quadric, although the normal bundle of the total fiber F remains the same and it is all the time equal to $\mathcal{S}_{|F}$, the normal of each of these components varies. In particular for different values of r the neighbourhood (analytic or formal) of the special fiber in X_r varies.

With this we conclude the discussion on complete intersection in vector bundles. The next series of examples will concern double covering technique. In [A-W3] this is covered in Section (3.5). We begin by recalling some generalities.

Let $\psi : Y \to Z'$ be a Fano-Mori contraction of a smooth variety Y. Assume that L is a ψ-ample line bundle and that $-K_Y - 2L = \psi^*(L')$ for some line bundle L' over Z'. If there is a smooth divisor $B \in |2L|$ then we can construct a double covering $\pi : X \to Y$ which is branched along B. The variety X is smooth and $-K_X = \pi^*(L + \psi^*(L'))$, therefore $-K_X$ is $\psi \circ \pi$-ample.

If ψ is of fiber type then fibers of $\varphi := \psi \circ \pi$ are connected and thus $\varphi : X \to Z'$ is a good contraction. If ψ is birational then we take the connected part of the Stein factorisation of $\psi \circ \pi$ (the finite part of the Stein factorisation $\pi' : Z \to Z'$ is a double covering branched along $\psi(B)$).

Note that a similar construction with L such that $K_X + L = \psi^*(L')$ leads from a good birational contraction ψ to a crepant contraction $\varphi : X \to Z$.

Passing to the concrete calculations we begin with conic fibrations of a 4-fold with an isolated two dimensional fiber which is a quadric. As it follows from [A-W3] the conormal bundle of the quadric is the pull back of $\boldsymbol{TP}^2(-1)$ via some double covering of $\boldsymbol{P}^2$. Consequently, the base of our double covering will be the total space of the bundle $\boldsymbol{TP}^2(-1)$, which constitutes the (only) example of a smooth non-equidimensional 4-dimensional scroll with an isolated 2-dimensional fiber over the origin.

Let $[t_0, t_1, t_2]$ be homogeneus coordinates on $\boldsymbol{P}^2$ and (z_0, z_1, z_2) be the affine coordinates on $\boldsymbol{C}^3$, then we set

$$Y := \{f := t_0 z_0 + t_1 z_1 + t_2 z_2 = 0\} \subset \boldsymbol{C}^3 \times \boldsymbol{P}^2$$

and $\psi : Y \to \boldsymbol{C}^3$ is the projection. If $p : Y \to \boldsymbol{P}^2$ is the other projection then by adjunction we have $-K_Y = p^* \mathcal{O}(2)$ and thus we will follow the above construction scheme for $L = p^* \mathcal{O}(1)$.

PROPOSITION (1.2). – *Let Y and L be as above and take B a smooth divisor in $|2L|$; let also X be, as explained above, a double covering of Y branched along B. We have a conic fibration, $\varphi : X \to \boldsymbol{C}^3$ with a special fiber over the origin isomorphic to a reduced quadric. The special fiber will depend on the choice of B and how it intersects with the special fiber of ψ. The following table illustrates the possible choices of B. In the first column we indicate the intersection of B with the fiber of ψ over the origin, in the second the special fiber of φ and in the last the corresponding polynomial g which defines B.*

Number	$B \cap \psi^{-1}(0)$	$\varphi^{-1}(0)$	g
(1)	*smooth conic*	*smooth quadric*	$(t_0^2 + t_1 t_2)$
(2)	*reducible conic*	*singular quadric*	$(t_0 t_1) + z_1 t_2^2$
(3)	*double line*	*reducible quadric*	$(t_0^2) + z_0 t_1^2 + z_1 t_2^2$

PROOF. – The only thing to prove is the smoothness of B in all the cases of the table. It follows from the computation of the Jacobi matrix of f and g: for instance in the last case, it is

$$\begin{pmatrix} t_0 & t_1 & t_2 & z_0 & z_1 & z_2 \\ t_1^2 & t_2^2 & 0 & 2t_0 & 2t_1 z_0 & 2t_2 z_1 \end{pmatrix}$$

and has rank two on any point of $B \cap \psi^{-1}(0)$. One can also check that there exists no smooth branch divisor which contains the 2-dimensional fiber (so that the case of a «double $\boldsymbol{P}^2$» can not be produced this way.

Similarly we produce examples of divisorial elementary contraction of a 4-fold with quadric fibers. Let $\psi \colon Y \to \boldsymbol{C}^4$ be the blow up of $\boldsymbol{C}^4$ with coordinates $(z_0, \dots, z_3)$ along the line $w_0 = w_1 = w_2 = 0$. In local coordinates Y is the submanifold of $\boldsymbol{C}^4 \times \boldsymbol{P}^2$ defined as

$$Y = \{((w_0, w_1, w_2, w_3), [t_0, t_1, t_2]) \in \boldsymbol{C}^4 \times \boldsymbol{P}^2 \colon w_0 t_1 - w_1 t_0 = w_1 t_2 - w_2 t_1 = w_0 t_2 - w_2 t_0 = 0\}$$

Let $p : Y \to \boldsymbol{P}^2$ be the projection. Then, by adjunction, $-K_Y = p^* \mathcal{O}(2)$ and thus again we take $L = p^* \mathcal{O}(1)$. If we take a smooth divisors B in $2L$ and we perform the above general construction we will obtain a divisorial contraction with exceptional divisor contracted to the line and with general fiber a quadric (smooth or singular). Both general and special fiber will depend on the choice of B. The subsequent table illustrates the possible choices of B. In the first column of the table we indicate the intersection $B \cap \psi^{-1}(0)$, in the second column we show the intersection with all other fibers and in the last column the polynomial g defining B.

No.	$B \cap \psi^{-1}(0)$	$B \cap$ *general fiber*	*Analytic equation of B*
(1)	*smooth conic*	*smooth conic*	$t_0^2 + t_1^2 + t_2^2$
(2)	*reducible conic*	*smooth conic*	$(t_0^2 + t_1 t_2 + w_3^m (t_2^2)$ *with* $m \geq 1$
(3)	*reducible conic*	*reducible conic*	$t_0^2 + t_1^2 + w_2 (t_2^2)$
(4)	*double line*	*reducible conic*	$t_0^2 + w_3^m (t_1^2) + w_2 t_2^2$
(5)	*double line*	*smooth conic*	$t_0^2 + w_1 (t_1^2 + t_2^2) + w_2 (t_1^2 + t_2^2) + w_3^m t_1^2 + w_3 t_2^2$

The image Z of the extremal contraction $\varphi : X \to Z$ that we construct is a double cover of $\boldsymbol{C}^4$ ramified along the set $\psi(B) \subset \boldsymbol{C}^4$ which is locally given by a function f. Thus Z is given in $\boldsymbol{C}^5$ by the equations $w_4^2 = f$. For instance in the first case we have that Z is given in $\boldsymbol{C}^5$ by the equation $w_4^2 = f = w_0^2 + w_1^2 + w_2^2$. Note also that the special fiber of φ is a smooth quadric, respectively a singular irreducible quadric or a reducible quadric, if $B \cap \psi^{-1}(0)$ is a smooth conic, resp. a reducible conic or a double line.

It is straightforward at this point to prove the following

PROPOSITION (1.3). – *Let Y and B as above. The divisorial contractions $\varphi\colon X \to Z$ constructed with X as a double cover of Y branched along B give, as B varies as in the above table, all the possibilities listed in the next theorem* (4.1).

2. – Characterisations of blow-ups.

In this section we give two characterizations of blow-down morphisms. Before stating the results we recall that a contraction $\varphi\colon X \to Z$ is divisorial if its exceptional set $E(\varphi) = \{x \in X : \dim \varphi^{-1}(\varphi(x)) > 0\}$ is a divisor and φ is elementary if $\operatorname{Pic} X/\varphi^*(\operatorname{Pic} Z) \simeq \boldsymbol{Z}$. The definition of log terminal singularities can be found in [K-M-M]. The first result is known in small dimensions, see e.g. [Wi, proof of 2.2, pp. 568-569].

Using relative base-point-freeness, namely the Theorem 5.1 of [A-W1], one proves the following general characterisation of blow-ups:

PROPOSITION (2.1). – *Let $\varphi\colon X \to Z$ be a contraction of a variety X with at most log terminal singularities. Assume that φ is elementary and of divisorial type. Assume moreover that the exceptional divisor, call it E, is a Cartier divisor. Let $L = \mathcal{O}(-E)$ be the line bundle associated to $-E$ and let $r \geq 0$ be a rational number such that $K_X + rL$ is numerically trivial along fibers of φ: that is $(K_X + rL)_{|F} \equiv 0$ for any fiber F of φ. If for any fiber F of the contraction φ we have $\dim F \leq (r+1)$ then φ is the blow-up of Z along $\varphi(E) \subset Z$ which is a reduced subscheme defined by the push-forward of the ideal of E, that is $\varphi_* \mathcal{O}(-E) \hookrightarrow \varphi_* \mathcal{O}_X = \mathcal{O}_Z$.*

PROOF. – Let $\widetilde{X}$ be the blow-up of Z along $\varphi(E)$. By a version of base-point-free theorem, (1.3.4) in [A-W3], L is φ-very ample. That is, possibly tensoring with the pull back of a very ample line bundle on Z, we can assume that L is very ample. On the other hand the surjectivity of the map $\varphi^* \varphi_*(L) \to L$ [A-W1, 5.1] implies that the sheaf $\varphi^{-1}(\varphi_*(\mathcal{O}(-E))) \cdot \mathcal{O}_X$ is invertible and therefore, by the universal property of the blow-up, there exists a morphism (over Z) from X into $\widetilde{X}$. This map is actually given in a neighbourhood of E by the evaluation of sections of $L = \mathcal{O}(-E)$. Thus, since L is φ-very ample, this map is an isomorphism.

The following characterisation of blow-ups is an improvement of Corollary 4.11 in [A-W1].

PROPOSITION (2.2). – *Let $\varphi : X \to Z$ be a proper birational morphism between smooth varieties. Put $S := \{z \in Z \mid \dim \varphi^{-1}(z) \geq 1\}$ and $E = \varphi^{-1}(S)$. Assume that $(E)_{\mathrm{red}}$ is an irreducible divisor and that $\dim S \leq 2$. Then φ is the blow up of Z at S and either S is smooth or $\dim X = 4$ and S is a non normal surface with isolated singularities. In the last case the fibers over the singular points have dimension two and together with their normal bundle they were classified in the main theorem of* [A-W3] (*in particular the example in the above proposition* (1.1) *gives such a map*).

PROOF. – Let $\varphi^* : \mathrm{Pic}(Z) \to \mathrm{Pic}(X)$ be the pull-back map. By assumption on the irreducibility of $(E)_{\mathrm{red}}$ and the smoothness of Z (actually it is enough to assume that Z is factorial) the cokernel of this map is generated by E. Let then $L := -E$; L is φ-very ample and φ is a birational Fano Mori contraction. If $\varphi_{|E}$ is equidimensional the proposition follows immediately from the Theorem (4.1) in [A-W1] (see the Corollary (4.11) in [ibidem] which is more general then the actual proposition in this case). If it is not equidimensional then we have some isolated fibers of dimension $(n-2)$, then we apply the corollary (5.8.1) in [A-W3] to reduce to the case $n = 4$ which is exhaustively described in [A-W3].

3. – Fano-Mori contractions of a smooth 3-fold.

Before stating the main result of this section let us recall the notation. Firstly, an r-th Hirzebruch surface $\boldsymbol{F}_r$ is a rational minimal ruled surface $\boldsymbol{F}_r \to \boldsymbol{P}^1$ with a fiber of the ruling denoted by f and the unique section C_0 which satisfies $C_0^2 = -r$. Moreover, any Cartier divisor (equivalently, a line bundle) on $\boldsymbol{F}_r$ is rationally equivalent to $aC_0 + bf$ for some $a, b \in \boldsymbol{Z}$. Secondly, a rational cone $\boldsymbol{S}_r \subset \boldsymbol{P}^r$ is a projective cone over an r-th Veronese emmbedding $\boldsymbol{P}^1 \hookrightarrow \boldsymbol{P}^{r-1}$ with the induced line bundle $\mathcal{O}_{\boldsymbol{S}_r}(1)$ which generates $\mathrm{Pic}(\boldsymbol{S}_r)$. Now we can state the following:

PROPOSITION (3.1). – *Let φ be a Fano-Mori contraction of a smooth 3-fold with an isolated two dimensional fiber F. Let $L = -K_X$.*

i) *If φ is birational then (F, L_F) is one of the following pairs*

$$(\boldsymbol{P}^2, \mathcal{O}(1)), \quad (\boldsymbol{P}^2, \mathcal{O}(2)), \quad (\boldsymbol{S}_2, \mathcal{O}_{\boldsymbol{S}_2}(1)), \quad (\boldsymbol{F}_0, C_0 + f), \quad (\boldsymbol{F}_1, C_0 + 2f),$$

or $\boldsymbol{P}^2 \bigcup_{C_0} \boldsymbol{F}_2$, that is a union of $\boldsymbol{P}^2$ and $\boldsymbol{F}_2$ which meet transversally along a rational curve which is a line in $\boldsymbol{P}^2$ and C_0 in $\boldsymbol{F}_2$, and then $L_{|\boldsymbol{P}^2} = \mathcal{O}(1)$, $L_{|\boldsymbol{F}_2} = C_0 + 3f$. In the case when $F = \boldsymbol{F}_1$ the contraction φ contracts also a smooth divisor $E \subset X$ to a smooth curve and $E \cap \boldsymbol{F}_1 = C_0 \subset \boldsymbol{F}_1$. In the other cases the fiber F is an isolated positive dimensional fiber of φ.

ii) *If φ is of fiber type with generic fiber of dimension* 1 (*a conic fibration*) *then the pair* (F, L_F) *is either* $(\boldsymbol{F}_0, C_0 + 2f)$ *or* $F = \boldsymbol{F}_0 \bigcup_{C_0} \boldsymbol{F}_1$, *that is a union of* $\boldsymbol{F}_0$ *and* $\boldsymbol{F}_2$ *which meet transversally along* C_0, *and then* $L_{|\boldsymbol{F}_0} = C_0 + f$, $L_{|\boldsymbol{F}_1} = C_0 + 2f$.

REMARK. – Let us note that all the above cases exist. Indeed, the construction of elementary contractions with appropriate fibers is done by Mori in [Mo]. All cases which are not in the Mori's list are obtained from Mori's contractions. Namely, the cases $(\boldsymbol{F}_1, C_0 + 2f)$ and $\boldsymbol{P}^2 \cup \boldsymbol{F}_2$ are obtained from $(\boldsymbol{P}^2, \mathcal{O}(1))$ by a blow-up of a curve. The case $(\boldsymbol{F}_0, C_0 + 2f)$ is just a product of $\boldsymbol{P}^1$ and a simple blow-up of a surface while the case of $\boldsymbol{F}_0 \cup \boldsymbol{F}_1$ is obtained by blowing a line C_0 in the previous one. Alternatively, the last case is the blow-up of the quadric cone singularity of the scroll

$$\boldsymbol{P}^2 \times \boldsymbol{C}^2 \supset \{[t_0, t_1, t_2], (z_1, z_2) : t_1 z_1 + t_2 z_2 = 0\} \to \boldsymbol{C}^2$$

c.f. Example (3.4.3) and (3.2) in [B-W].

PROOF. – The list of possible components of a fiber is set up in Table III of [A-W3]. Also, because of (4.7.1) and (4.10.1) in [ibidem] neither $(\boldsymbol{P}^2, \mathcal{O}(2))$ nor $\boldsymbol{S}_2$ is a component of a reducible fiber and — the fact which will be used constantly in our arguments — no three components meet at one point.

Let us discuss first the question of reducible fibers: assume therefore that F contains at least two intersecting components, F_1 and F_2. Then $F_1 \cap F_2 := R$ is a line relative to L and the two components intersect in R tranversally (see Lemma (4.7.1) in [ibidem]).

We use now the intersection theory of divisors to prove a useful formula. Let $N_{R/X}$ denotes the normal bundle of R in X; we have $N_{R/X} = N_1 \oplus N_2$, where N_i denotes the normal bundle of R in F_i, and $\deg(\det(N_{R/X})) = -1$, by adjunction formula. It follows that

$$-1 = \deg(\det(N_{R/X})) = \deg N_1 + \deg N_2 = F_1 \cdot R + F_2 \cdot R\ .$$

Since $F_i \cdot R = (K_{F_i} - K_X) \cdot R = K_{F_i} \cdot R + 1$, we obtain $K_{F_1} \cdot R + K_{F_2} \cdot R = -3$.

The curve R is a line relatively to L, therefore it is just a line in $\boldsymbol{P}^2$ while it can be linearly equivalent to either f or C_0 in $\boldsymbol{F}_r$, the last case only if $L_{\boldsymbol{F}_r} = C_0 + (r+1)f$. This implies that $K_{F_i} \cdot R = -3$, if $F_i \simeq \boldsymbol{P}^2$, and $K_{F_i} \cdot R = (r-2)$ or -2, if $F_i \simeq \boldsymbol{F}_r$. This observation added to the above formula gives only two possibilities (up to possible renumeration of components), namely the following:

(i) $F_1 = \boldsymbol{P}^2$ and $F_2 = \boldsymbol{F}_2$, R is a line in $\boldsymbol{P}^2$ and the section C_0 in $\boldsymbol{F}_2$;

(ii) $F_1 = \boldsymbol{F}_1$ and $F_2 = \boldsymbol{F}_r$, R is the section C_0 in $\boldsymbol{F}_1$ and a fiber f in $\boldsymbol{F}_r$.

If we are in the case (ii) we can apply an argument of deforming a rational curve in F as developed in the section 4 of [ibidem]. In particular, we apply the inequality (4.5.1) of [ibidem], namely

$$\dim_{[C]} \mathrm{Hilb}(F) = \dim_{[C]} \mathrm{Hilb}(X) \geq L \cdot C ,$$

where C is a rational smoothable curve of degree ≥ 2 and $\dim_{[C]} \mathrm{Hilb}(F)$, respectively $\dim_{[C]} \mathrm{Hilb}(X)$, is the dimension of a component of the Hilbert scheme containing C in F, resp. in X. Take thus a rational curve of degree 3 which is the union of $f \subset \boldsymbol{F}_1$ and of $C_0 + f \subset \boldsymbol{F}_r$ (none of these curves is contained in another component because our «no three meet» rule). Since $\dim_{[C]} \mathrm{Hilb}(F) \leq 2$ for $r > 0$, we obtain a contradiction unless $r = 0$. Then $F = \boldsymbol{F}_0 \cup \boldsymbol{F}_1$. If the fiber F has at least three components, by the above, we must have that two of them are $\boldsymbol{F}_1$ and one is $\boldsymbol{F}_0$, the $\boldsymbol{F}_1$ are disjoint and they intersect with the $\boldsymbol{F}_0$ along two disjoint fibers of its ruling. Deforming a degree 3 rational curve in F which is the union of the section C_0 in $\boldsymbol{F}_0$ and of two fibers, one in each $\boldsymbol{F}_1$, we obtain a contradiction, as above.

Now we know all possible fibers and it remains to distinguish the fibers of birational contractions from these of fiber type contractions. To this end we note that if the fiber in question has ample conormal bundle (i.e. $\mathcal{O}_F(-F)$ is ample) then it is an isolated positive dimensional fiber of the contraction (see also (6.1) in [ibidem]). Indeed, the limit of 1-dimensional fibers approaching the fiber F would produce a curve in F whose intersection with F would be zero. Since $\mathcal{O}_F(-F)$ can be computed easily by adjunction, we find out all the pairs (F, L_F) which may be non-isolated positive dimensional fibers; these are the following:

$$(\boldsymbol{F}_0, C_0 + 2f), \quad (\boldsymbol{F}_1, C_0 + 2f), \quad (\boldsymbol{F}_0 \cup_{C_0} \boldsymbol{F}_1, L_{\boldsymbol{F}_0} = C_0 + f,\ L_{\boldsymbol{F}_1} = C_0 + 2f).$$

Moreover, only in the case $(\boldsymbol{F}_1, C_0, +2f)$ the normal $F_F = C_0 + f$ has zero intersection with the unique curve C_0 while in the remaining two cases the locus of curves which have intersection 0 with F coincides with F. This distinguishes the birational and fiber case.

4. – Fano-Mori divisorial contractions of a smooth 4-fold.

In the present section we prove the following

THEOREM (4.1). – *Let $\varphi: X \to Z$ be an elementary Fano-Mori contraction of a smooth projective variety X of dimension* 4; *assume that φ is birational and that it contracts a divisor E to a curve C. Then*

(*a*) *The curve C is smooth and $\varphi: X \to Z$ is the blow-up of Z along C.*

(*b*) *$g := \varphi_{|E}: E \to C$ is either a $\boldsymbol{P}^2$-bundle or a quadric bundle.*

(*c*1) *If E is a $\boldsymbol{P}^2$-bundle then the normal bundle of each fiber in X is* *either* $\mathcal{O}(-1)\oplus\mathcal{O}$ *or* $\mathcal{O}(-2)\oplus\mathcal{O}$; *in particular all fibers of* φ *are reduced and with no embedded components. In the first case* Z *is smooth and* φ *is the smooth blow-up; in the second* $C=\operatorname{Sing} Z$ *and* Z *is locally isomorphic to* $S_2\times \boldsymbol{C}$ *where* S_2 *is the germ of singularity obtained by contracting the zero section in the total space of the bundle* $\mathcal{O}(2)$ *over* $\boldsymbol{P}^2$.

($c2$) *If* E *is a quadric bundle then the general fiber is irreducible and isomorphic to a two dimensional, possibly singular, quadric. Isolated special fibers can occur and they are isomorphic either to a singular quadric or to a reduced but reducible quadric* (*i.e. union of two* $\boldsymbol{P}^2$ *intersecting along a line*); *in particular there are no special fibers which are isomorphic to a double plane. The normal bundle of each fiber is* $\mathcal{O}(-1)\oplus\mathcal{O}$. *Locally* Z *can be described as a hypersurface of* $\boldsymbol{C}^5$; *in the following table we give a list of possibilities for* $Z=V(g)\subset\boldsymbol{C}^5$ *according to the described combinations of general and special fibers. We choose coordinates* $(z_1, z_2, z_3, z_4, z_5)$ *such that* $C=\{z_1=z_2=z_3=z_4=0\}\subset\boldsymbol{C}^5$.

Number	*Special fiber*	*General fiber*	$g=$ *analytic equation of* Z	
(1)	$\boldsymbol{F}_0$	$\boldsymbol{F}_0$	$z_1^2+z_2^2+z_3^2+z_4^2$	
(2)	$\boldsymbol{S}_2$	$\boldsymbol{F}_0$	$z_1^2+z_2^2+z_3^2+z_5^m z_4^2$	$m\geqslant 1$
(3)	$\boldsymbol{S}_2$	$\boldsymbol{S}_2$	$z_1^2+z_2^2+z_3^2+z_4^3$	
(4)	$\boldsymbol{P}^2\cup\boldsymbol{P}^2$	$\boldsymbol{S}_2$	$z_1^2+z_2^2+z_3^3+z_4^3+z_3^2 z_5^m$	$m\geqslant 1$
(5)	$\boldsymbol{P}^2\cup\boldsymbol{P}^2$	$\boldsymbol{F}_0$	$z_1^2+z_2^2+z_3^3+z_4^3+z_3^2 z_5^m+z_3 z_4 f(z_5)+z_4^2 g(z_5)$ with $z^m g(z)\neq f(z)^2/4$	

PROOF. – Note that the assumption $\operatorname{Pic}(X)/\varphi^*(\operatorname{Pic}(Z))\simeq\boldsymbol{Z}$ implies that all non trivial fibers are of dimension two and that E is a prime divisor.

Let H be the pull back of a very ample divisor on Z and let X' be a general element of the linear system $|H|$. Then X' is a smooth 3-fold and $\varphi' := \varphi_{|X'}\colon X'\to Z'$ is a birational extremal contractions which contracts a general fiber F' to a point and this is, at least locally, the only non trivial fiber of φ' (see [A-W3],(1.3.1)). Now we use the proposition of the previous section which gives 6 possibilities for the pair $(F', -K_{X'|F'})$.

The pair $(\boldsymbol{F}_1, C_0+2f)$ is ruled out since in this case F' is not the only non trivial fiber. The case of the reducible fiber $\boldsymbol{P}^2\cup\boldsymbol{F}_2$ cannot occur either. In fact in this case $-K_{F'|\boldsymbol{F}_2}$ is equal to $-C_0-4f$ and this is impossible because, by adjunction, $-K_{F'}=(-K_X-E)_{F'}$ and thus $-K_{F'}$ must be the sum of two ample divisors. Thus $(F', -K_{X'|F'})$ is one of the following pairs

$$(\boldsymbol{P}^2, \mathcal{O}(1)),\quad (\boldsymbol{P}^2, \mathcal{O}(2)),\quad (\boldsymbol{F}_0, \mathcal{O}(1)),\quad (\boldsymbol{S}_2, \mathcal{O}(1)).$$

If $(F', -K_{X'|F'})=(\boldsymbol{P}^2, \mathcal{O}(2))$ then, relatively with the respect to φ it holds

$2E = K_X$. Thus we can apply Theorem (4.1.iii) in [A-W1] and conclude that $C \subset Z$ are both smooth and that φ is the smooth blow-up.

From now on we consider the other cases and we let $-K_X = -E := L$. Proposition (2.1) implies that in this case φ is the blow-up of Z along C. Moreover, by the proposition (1.3.4) in [A-W3], L is φ very ample. This, together with the equidimensionality of the fibers of $E \to C$, implies that E is a $\boldsymbol{P}^2$-bundle or a quadric bundle over C.

LEMMA (4.2). – *The normal bundle of each fiber is $\mathcal{O}(-2) \oplus \mathcal{O}$ in the $\boldsymbol{P}^2$-bundle case and $\mathcal{O}(-1) \oplus \mathcal{O}$ in the quadric bundle case. This in particular implies that there are no fibers with the non reduced structure.*

PROOF. – In order to do this we apply a type of argument developed in [A-W3] — all quotations in this proof refer to this paper. Let us fix a fiber F: first we want to prove the spannedness of the conormal bundle of F using Lemma (5.7.4). If F is $\boldsymbol{P}^2$ or an irreducible quadric we combine (5.7.4) with Lemma (2.7), resp. (2.11) (which we can apply by (1.2.2)) and we get our conclusion with the following exceptions: $F = \boldsymbol{P}^2$ and $N_{F/X} = \mathcal{O}(-3) \oplus \mathcal{O}(1)$, $F = \boldsymbol{F}_0$ and $N_{F/X} = \mathcal{O}(-2, -1) \oplus \mathcal{O}(1, 0)$ or $F = \boldsymbol{S}_2$ and $N_{F/X} = \mathcal{E}$ where $\mathcal{E}$ is as in the second sequence in (2.11.b.i′). But these cannot occur since, by the general theory of deformations (see [Ko] (I.2.15)) this will implies that F deforms in at least a two dimensional family in X (in fact $h^1(N_{F/X}) = 0$ and $h^0(N_{F/X}) \geq 2$).

If F is the reducible quadric and the conormal fails to be spanned then the restriction of $N_{F/X}$ to each line of a component $F_1 \simeq \boldsymbol{P}^2$ must be $\mathcal{O}(-2) \oplus \mathcal{O}(1)$. We can apply the theorem of Van de Ven to claim that $(N_{F/X})_{|F_1} = \mathcal{O}(-2) \oplus \mathcal{O}(1)$. Thus, in view of (2.3) if we blow-up the other component and consider the strict transform of F_1 then its normal bundle would be $\mathcal{O}(-3) \oplus \mathcal{O}$. Now we see that the strict transform would move which is clearly impossible since the general fibers are quadrics.

We conclude then applying the Lemmas (2.6) and (2.8) (note, for instance in the quadric bundle case, that the other two possibilities in (2.8) occur respectively in the fiber type case (see (5.9.6)) and in the birational case with isolated two dimensional fiber (see (5.7.6)).

LEMMA (4.3). – *The curve C is smooth.*

PROOF. – We give two different proofs of this lemma.

1-st proof. Let $L := -K_X + mH$, where H is the pull-back of a very ample line bundle on Z and $m >> 0$; by the proposition (1.3.4) in [A-W3] (see also [A-W1], theorem (5.1)) the line bundle L is very ample. Let X' be a general section of $|L|$: X' is a smooth 3-fold and $\varphi' := \varphi_{|X'}: X' \to Z'$ is a crepant birational contraction which contracts the irreducible divisor $E' := E \cap X$ to the curve $C \subset Z' := \varphi'(X') \subset Z$ (see (1.3.2) or (4.4) in [A-W3]). These maps were studied by P. H. Wilson in [Wi]; in particular from his results it follows that C is a smooth curve.

2-nd proof. Take any fiber F of φ and let $\mathcal{H}$ be a component of the Hilbert scheme of X parametrizing the deformations of F in X. Let $\mathcal{F} \subset \mathcal{H} \times X$ be the incidence variety. Since $H^1(F, N_{F/X}) = 0$ it follows that $\mathcal{H}$ is smooth at F whereas from the evaluation map $H^0(F, N_{F/X}) \otimes \mathcal{O}_F \to N_{F/X}$ we infer that the map $\mathcal{F} \to X$ is an immersion onto $E \subset X$. Thus $\varphi_E \colon E \to C$ can be identified with $\mathcal{F} \to \mathcal{H}$ so that (in a neighbourhood of F) $C \simeq \mathcal{H}$ is smooth.

LEMMA (4.4) (see [Ta]). – *Locally Z is a hypersurface in $\boldsymbol{C}^5$.*

PROOF. – Since φ is the blown-up of Z along C we have that

$$\bigoplus_{n \geqslant 0} (I_C^n / I_C^{n+1}) = \bigoplus_{n \geqslant 0} \varphi_* \mathcal{O}_E(-nE)$$

as $\mathcal{O}_C$ algebras. In particular this implies that (I_C / I_C^2) is locally free of rank $\leqslant 4$.

Finally, it is easy and straightforward to check that blowing up the hypersufaces Z as in the table of the proposition along the curve C one obtains the prescribed Fano-Mori divisorial contractions.

Note that the table can also be obtained as in the second part of Section 1 (Proposition (1.2)), following the construction of double coverings in [A-W3] (3.5.4).

REMARK. – In this paper we did not consider the problem of the analytic unicity of the contractions, i.e. the unicity modulo analytic changes of coordinates in an analytic neighborhood of a fixed fiber.

For instance, refering to the first part of section 1, one can ask the following. Let $\varphi \colon X \to Z$ be a Fano-Mori divisorial contractions of a 4-fold X which is a blow-up of a surface with an isolated non normal singularity; if the fiber over the singularity is a quadric, is the germ of the surface around the singular point analytic equivalent to the one described in the proposition (1.1)?

Similarly one can ask if, modulo analytic changes of coordinates, the hypersurfaces $Z = V(g) \subset \boldsymbol{C}^5$ in the table of proposition (4.1) are all the possible ones.

This second question seems to be easier than the first one. In particular, in the notation of the proposition (4.1), consider the map $Z = V(g) \to C$. Since C is a smooth curve this map is flat and we can think at it as an infinitesimal deformation of the fiber over the origin which is a threefold with an isolated hypersurface singularity at the origin. Since in this case the miniversal deformation is well known one should be able to describe all possible one dimensional deformations (which remain singular), i.e. all $Z = V(g)$.

This problem will be dealt with, in a broader context, in the forthcoming thesis of G. Occhetta

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