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Projective Normality of Abelian Varieties with a Line Bundle of Type $(2, \ldots)$.

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Sunto. – *Sia X una varietà abeliana e L un fibrato in rette ampio di tipo $(2, 2d_2, \ldots, 2d_g)$ su X; sia φ_L l'applicazione associata a L. In questo lavoro si dimostra il seguente fatto*: *se $d_i \leq 2$ per qualsiasi i, L non è mai normalmente generato* (*quindi, se φ_L è un embedding, $\varphi_L(X)$ non è proiettivamente normale*); *negli altri casi invece L è normalmente generato per $(X, c_1(L))$ generico nello spazio dei moduli delle varietà abeliane polarizzate di tipo $(2, 2d_2, \ldots, 2d_g)$.*

1. – Introduction.

Let X be an abelian variety with an ample line bundle L of type $(\delta_1, \ldots, \delta_g)$, with $\delta_i \mid \delta_{i+1}$, and let φ_L be the associated rational map. In this paper we examine the problem whether $\varphi_L(X)$ is projectively normal in the case where $\delta_1 = 2$.

It is well known that if $\delta_1 \geq 3$, φ_L is an embedding and $\varphi_L(X)$ is projectively normal (see Theorem 7.3.1 in [L-B]).

Besides, in [Laz], Lazarsfeld proved that, if X is an abelian surface, L is of type $(1, d)$, $|L|$ has not fixed components and φ_L is birational onto its image, then $\varphi_L(X)$ is projectively normal for d odd ≥ 7 and d even ≥ 14.

Here we examine the case of an abelian variety with an ample line bundle L of type $(2, 2d_2, \ldots, 2d_g)$; we know that in this case there exists an ample line bundle M, of type $(1, d_2, \ldots, d_g)$, s.t. $L = M^2$ (see for instance [L-B] Lemma 2.5.6). We prove the following fact: if $d_i \leq 2$ for every i, then L is never normally generated (thus, if φ_L is an embedding[1]), $\varphi_L(X)$ is not projectively normal); otherwise (that is $\exists i$ s.t. $d_i > 2$) L is normally generated for generic $(X, c_1(L))$ in the moduli space of polarized abelian varieties of type $(2, 2d_2, \ldots, 2d_g)$.

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(1) We recall the following Ohbuchi's theorem (see [Oh1]): let M be an ample line bundle on an abelian variety X; M^2 is very ample iff (X, M) is not isomorphic to the product of two abelian varieties with line bundles (X_i, M_i), $i = 1, 2$, with $\dim X_1 > 0$ and $h^0(X_1, M_1) = 1$.

NOTATIONS AND DEFINITIONS. – We collect here some notations and standard definitions we will use in all the paper.

- $\boldsymbol{X}$, $\boldsymbol{V}$, $\boldsymbol{\Lambda}$; X is a complex torus equal to V/Λ where V is a complex vector space and Λ a lattice in V.
- $\boldsymbol{X_n}$ is the set of n-torsion points of X.
- $\boldsymbol{\varphi_L}$ is the rational map associated to a line bundle L on X.
- $\boldsymbol{t_x}$ is the translation on X by the point x.
- $\widehat{\boldsymbol{X}}$ is the dual complex torus of X.
- ϕ_L is the homomorphism $X \to \widehat{X}$, $x \mapsto t_x^* L \otimes L^{-1}$, where L is a line bundle on X.
- $\boldsymbol{K(L)}$ is the kernel of ϕ_L; it does depend only on H, the first Chern class of L, thus we denote $K(L)$ also as $K(H)$; if L is nondegenerate then $K(L)$ is a finite group isomorphic to $(\boldsymbol{Z}/d_1 \oplus \ldots \oplus \boldsymbol{Z}/d_g)^2$ with $d_i \mid d_{i+1}$; we say that L is of *type* $(d_1, \ldots, d_g)$.
- $\boldsymbol{\Lambda(L)}$ or $\boldsymbol{\Lambda(H)} = \{v \in V \mid \mathrm{Im}H(v, \Lambda) \subset \boldsymbol{Z}\}$ where L is a line bundle and H its first Chern class; (we recall that $K(H) = \Lambda(H)/\Lambda$).
- $[\boldsymbol{y}]$ means the class in X of a point $y \in V$.
- Suppose H is a non degenerate hermitian form on V, $E = \mathrm{Im}H$ and $E(\Lambda, \Lambda) \subset \boldsymbol{Z}$.

A direct sum decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ is called a decomposition for H (or for E) if Λ_1 and Λ_2 are isotropic with respect to E; a real vector space decomposition $V = V_1 \oplus V_2$, with V_1 and V_2 real vector subspaces of V, is called a decomposition for H (or for E) if $(V_1 \cap \Lambda) \oplus (V_2 \cap \Lambda)$ is a decomposition of Λ for H.

Choose a decomposition of V for H: $V = V_1 \oplus V_2$. Let L_0 be the unique line bundle with Chern class H and semicharacter $\chi_0 \colon V \to \boldsymbol{C}_1$, $\chi_0(v) = e^{\pi i E(v_1, v_2)}$, where $v = v_1 + v_2$ and $v_i \in V_i$. For every L with Chern class H there is a point $c \in V$, uniquely determined up to translation by elements of $\Lambda(H)$, s.t. $L = t_{[c]}^* L_0$ (see [L-B]) Lemma 3.1.2); c is called the *characteristic* of L with respect to the chosen decomposition.

Besides we denote $\boldsymbol{\Lambda(L)_i} = \Lambda(L) \cap V_i$ and $\boldsymbol{K(L)_i} = \Lambda(L)_i/\Lambda$.

- A line bundle L on X is called *symmetric* if $(-1)_X^*(L) \simeq L$, where $(-1)_X$ is the multiplication by -1 on X. A line bundle L with $c_1(L) = H$ is symmetric if and only if the characteristic of L with respect to some decomposition of V for H is in $(1/2)\Lambda(H)$ (see [L-B], Chapter 4, § 6 and § 7, for a reference on symmetric bundles).

Let $\pi \colon L \to X$ be a symmetric line bundle on X. A biholomorphic map $f \colon L \to L$ is called isomorphism of L over $(-1)_X$ if $\pi \circ f = (-1)_X \circ \pi$ and the induced map from the fibre of L over x to the fibre over $-x$ is $\boldsymbol{C}$-linear $\forall x \in X$.

The isomorphism f is called normalized if the induced map on the fibre of L over 0 is the identity. For any symmetric line bundle there is a unique normalized isomorphism $(-1)_L\colon L \to L$ over $(-1)_X$ (see [L-B] Lemma 4.6.3); it induces an involution on $H^0(L)$.

In [N-R] Nagaraj and Ramanan gave the following definition: an ample symmetric line bundle L of type $(1, \dots, 1, 2, \dots, 2)$ on an abelian variety is said *strongly symmetric* if $(-1)_L$ acts on $H^0(L)$ as Identity or as -Identity.

- A line bundle L on X is called *normally generated* if it is very ample and $\varphi_L(X)$ is projectively normal. We have that L is normally generated iff it is ample and the natural maps $S^n H^0(X, L) \to H^0(X, L^n)$ are surjective for all $n \geq 2$ (see [L-B], Chapter 7, § 3 and [M], p. 38).

2. – The main result.

Before to state the theorem we quote some propositions of [B-L-R], which will be useful to prove the theorem, and we make some remarks.

We quote the following facts and lemmas from [B-L-R].

A polarized abelian variety (X, M) of type $(d_1, \dots, d_g)$ admits an isogeny onto a principally polarized abelian variety $\pi\colon (X, M) \to (Y, P)$ s.t. $\pi^* P = M$ and, let $\hat{\pi}\colon \hat{Y} \to \hat{X}$ be the dual isogeny, $\ker \pi$ and $\ker \hat{\pi}$ are isomorphic to $\bigoplus_{i=1}^{g} \boldsymbol{Z}/d_i$. The isogeny π determines the subgroup $Z := \phi_P^{-1}(\ker \hat{\pi}) \simeq \bigoplus_{i=1}^{g} \boldsymbol{Z}/d_i$ in Y. Conversely any subgroup Z of a principally polarized abelian variety (Y, P) determines an isogeny $\pi\colon X \to Y$: the dual of the isogeny $Y \simeq \hat{Y} \to \hat{X} := Y/Z$.

LEMMA 1 (Lemma 1.1 in [B-L-R]). – *Let Z be a cyclic subgroup of order d of a principally polarized abelian variety (Y, P) and $\pi\colon X \to Y$ the associated isogeny. Then $M = \pi^*(P)$ is of type $(1, \dots, 1, d)$.*

LEMMA 2 (part a) of Lemma (1.2) in [B-L-R]). – *Let $\pi\colon (X, M) \to (Y, P)$ be an isogeny onto a principally polarized abelian variety (Y, P) associated to a finite subgroup $Z \subset Y$. There is a canonical decomposition*

$$H^0(M) \simeq \bigoplus_{z \in Z} H^0(t_z^* P)$$

induced by the embeddings $\pi^*\colon H^0(t_z^* P) \to H^0(M)$.

We recall also that if M is an ample line bundle then M^2 is normally generated if and only if the map $H^0(M^2) \otimes H^0(M^2) \to H^0(M^4)$ is surjective (see [Ko] or [L-B], Chapter 7, § 3).

We finish these preliminaries stating the following two remarks:

REMARK 1. – *Let X be an abelian variety of dimension g with an ample line bundle L of type $(d_1, \ldots, d_g)$; if $d_1 \cdot \ldots \cdot d_g < 2^{g+1} - 1$, then L is not normally generated.*

In fact, if we call $d = d_1 \cdot \ldots \cdot d_g$, we have $\dim S^2 H^0(L) = (d(d+1))/2$ and $\dim H^0(L^2) = 2^g d$, thus $\dim S^2 H^0(L) < \dim H^0(L^2)$ if $d < 2^{g+1} - 1$.

REMARK 2. – *If a polarized abelian variety (X, M) is a product of two polarized abelian varieties (X_1, M_1) and (X_2, M_2), then M^2 is normally generated if and only if M_i^2 is normally generated for $i = 1, 2$.*

In fact, if (X, M) is isomorphic to $(X_1 \times X_2, p_1^* M_1 \otimes p_2^* M_2)$, where $p_i \colon X_1 \times X_2 \to X_i$, $i = 1, 2$, are the obvious projections, we have that $H^0(X_1 \otimes X_2, p_1^* E_1 \otimes p_2^* E_2) \simeq H^0(X_1, E_1) \otimes H^0(X_2, E_2)$ for any line bundle E_i on X_i.

Thus we have the following commutative diagram:

$$\begin{array}{ccccc} H^0(p_1^* M_1^2 \otimes p_2^* M_2^2) & \otimes & H^0(p_1^* M_1^2 \otimes p_2^* M_2^2) & \to & H^0(p_1^* M_1^4 \otimes p_2^* M_2^4) \\ \wr\| & & \wr\| & & \wr\| \\ (H^0(M_1^2) \otimes H^0(M_2^2)) & \otimes & (H^0(M_1^2) \otimes H^0(M_2^2)) & \to & (H^0(M_1^4) \otimes H^0(M_2^4)) \end{array}$$

The map of the first row is surjective if and only if the maps $H^0(M_i^2) \otimes H^0(M_i^2) \to H^0(M_i^4)$ for $i = 1, 2$ are surjective.

THEOREM. – *Fix $d_2, \ldots, d_g \in \boldsymbol{N}$ with $1 \leq d_2 \leq \ldots \leq d_g$.*

Let X be an abelian variety of dimension g and M an ample line bundle on X of type $(1, d_2, \ldots, d_g)$; set $L = M^2$

If $d_i \leq 2$ for every i, then L is never normally generated (thus, if φ_L is an embedding, $\varphi_L(X)$ is not projectively normal).

Otherwise (that is $\exists i$ s.t. $d_i > 2$) L is normally generated for generic $(X, c_1(L))$ in the moduli space of polarized abelian varieties of type $(2, 2d_2, \ldots, 2d_g)$.

PROOF. – Observe that L is normally generated if and only if L' is normally generated where L' is a line bundle with the same Chern class of L, that is it is obtained from L by a translation.

As we already recalled, L is normally generated if and only if the multiplication map $H^0(M^2) \otimes H^0(M^2) \to H^0(M^4)$ is surjective ([Ko] or Chapter 7, §3 in [L-B]).

By one of Ohbuchi's theorems ([Oh2] or Chapter 7, §2 in [L-B]) this is equivalent, once a decomposition of V for $c_1(M)$ is fixed, to see that $|M|$ has no base point in $t_{[c]}^* K(M^2)$ where $c \in V$ is the characteristic of M.

• *Case M is of type* $(1, \dots, 1)$: $\varphi_L(X)$ is not an embedding (see, for instance, [L-B], Chapter 4, §8).

• *Case M is of type* $(1, \dots, 1, 2, \dots, 2)$, (more precisely M is of type $(d_1, \dots, d_g)$ with $d_i = 1$ for $i = 1, \dots, s$, $d_g = 2$ for $i = s+1, \dots, g$, $1 \leq s < g$): L is never normally generated.

We can suppose the characteristic of M is zero with respect to some decomposition of V for $c_1(M)$.

We state that there is a base point of $|M|$ belonging to X_2. Obviously it suffices to consider the case of type $(1, 2, \dots, 2)$, i.e. $s = 1$ (in fact in general we can find an isogeny $\pi: (X', M') \to (X, M)$ with (X', M') of type $(1, 2, \dots, 2)$ and $M' = \pi^* M$ and if the statement is true for (X', M') then it is true for (X, M)). One easily sees (for example using the inverse formula [L-B] 4.6.4) that the line bundle M is strongly symmetric since its characteristic is zero. By [N-R] Proposition 2.7, the base locus of an indecomposable strongly symmetric line bundle of type $(1, 2, \dots, 2)$ is not empty and is contained in the set of 2-torsion points. Thus there is a base point of $|M|$ in X_2.

Since $X_2 \subset K(M^2)$, Ohbuchi's theorem ([Oh2] or 7.3.1 [L-B]) yields the result.

• *Case M is of type* $(1, \dots, 1, d)$, $d \geq 3$: L is normally generated for generic $(X, c_1(L))$ in the moduli space of polarized abelian varieties of this type.

We have only to exhibit an example of abelian variety (X, L) of this type s.t. L is normally generated. In fact: consider the moduli space of polarized abelian varieties $(X, c_1(L))$ of fixed type $(2, 2d_2, \dots, 2d_g)$; the subset of the ones s.t. L is not normally generated is a closed subset, because L is not normally generated if and only if $|M|$ has base point in $t_{[c]}^* K(M^2)$, where $c \in V$ is the characteristic of M.

We apply the quoted lemmas of [B-L-R]. The example we exhibit is the same of [B-L-R], Theorem 1c). Let us call (Y, P) a product of g principally polarized elliptic curves of characteristic zero (fixed a decomposition of the lattices) $(E_1, P_1) \times \dots \times (E_g, P_g)$ with $P_i = (Q_i)$; let $\underline{Q} = (Q_1, \dots, Q_g)$; observe that $\underline{Q} \in Y_2$. Let us consider in Y an element $(z_1, \dots, z_g)$ with $z_i \neq 0$ of order d for every i. Let Z be the subgroup of Y generated by $(z_1, \dots, z_g)$; let (X, M) and $\pi: X \to Y$ be respectively the polarized abelian variety of type $(1, , \dots, 1, d)$ and the isogeny determined by Z. Observe that if we take a decomposition of the lattice of X compatible with the chosen decomposition of the lattice of Y (see [L-B] p. 160), 0 is a characteristic of M.

The base locus of $|M|$ is the inverse image by π of the base locus of $\bigoplus_{z \in Z} H^0(t_z^* P)$.

The base locus of $\oplus_{z \in Z} H^0(t_z^* P)$ is the empty set if $d > g$; while, if $g = d$, it is the following set:

$$\{\underline{Q} + (\sigma(1) z_1, \sigma(2) z_2, \dots, \sigma(d) z_d) \mid \sigma \in \mathcal{S}_d\};$$

and, more generally, if $g \geq d$, it is:

$$\{\underline{Q}+(x_1, \ldots, x_{i_1-1}, \sigma(1)z_{i_1}, x_{i_1+1}, \ldots, x_{i_2-1}, \sigma(2)z_{i_2}, x_{i_2+1}, \ldots, x_{i_d-1},$$
$$\sigma(d)z_{i_d}, x_{i_d+1}, \ldots, x_g)\,|\, i_1, \ldots, i_d \in \{1, \ldots, g\},\ i_1 < i_2 < \ldots < i_d,\ \sigma \in \mathcal{S}_d,\ x_k \in E_k\}.$$

Observe that $K(M) = \pi^{-1}(Z)$.

Since

$$2(x_1, \ldots, x_{i_1-1}, \sigma(1)z_{i_1}, x_{i_1+1}, \ldots, x_{i_2-1}, \sigma(2)z_{i_2}, x_{i_2+1}, \ldots, x_{i_d-1},$$
$$\sigma(d)z_{i_d}, x_{i_d+1}, \ldots, x_g) \notin Z$$

for all $i_1, \ldots, i_d \in \{1, \ldots, g\}$, with $i_1 < i_2 < \ldots < i_d$, $\sigma \in \mathcal{S}_d$, $x_k \in E_k$, we conclude that there is no base point of $|M|$ in $K(M^2)$.

- *Case M is of type* $(1, \ldots, 1, 2, \ldots, 2, d_{k+1}, \ldots d_g)$, $d_i \geq 3$, *for* $i > k$, $k < g$, (more precisely, let the type of M be $(d_1, \ldots, d_g)$ with $d_i = 1$ for $i = 1, \ldots, s$, $d_i = 2$ for $i = s+1, \ldots, k$, $d_i \geq 3$, for $i = k+1, \ldots, g$, $k < g$, $s \geq 1$): L is normally generated for generic $(X, c_1(L))$ in the moduli space of polarized abelian varieties of this type.

We have only to exhibit an example of abelian variety (X, L) of this type s.t. L is normally generated:

consider (X, M) equal to the product of some polarized abelian varieties (X_j, M_j):

(X_1, M_1) of dimension $s+1$ and of type $(1, \ldots, 1, d_{k+1})$ s.t. M_1^2 is normally generated,

(X_j, M_j) elliptic curves of type (2) for $j = 2, \ldots, k-s+1$,

(X_j, M_j) elliptic curves of type (d_{j+s}) for $j = k-s+2, \ldots, g-s$;

by Remark 2, our example is s.t. L is normally generated, thus we conclude. Q.E.D.

REMARK 3. – Observe that if X is an abelian surface the proof of the result is more simple; in fact, if M is of type $(1, d)$ and $|M|$ has not fixed components, we have:

if $d \geq 3$, $|M|$ has no base point, by Lemma 1.2 in Chapter 10 in [L-B]; thus $L = M^2$ is normally generated; suppose $d = 2$; we can suppose the characteristic of M equal to 0; by Lemma 1.2 in Chapter 10 in [L-B] $|M|$ has exactly four base points; in [L-B], Chapter 10, Example 1.4, Lange and Birkenhake remarked that these points belong to $K(M^2)$: in fact let b be a base point of $|M|$; also $-b$ is a base point of $|M|$ (since M is symmetric since the characteristic of M is 0); but $K(M)$ has the same cardinality of the set of base points and acts on the set of base points by translations, thus it acts transitively on the set of base points; thus $2b \in K(M)$; thus $b \in K(M^2)$, thus L is not normally generated.

We can apply the same argument whenever the dimension of the base locus of $|M|$ is zero and the cardinality of the base locus of $|M|$ is equal to the cardinality of $K(M)$. For instance suppose X is a threefold and M is of type $(1, 1, 3)$ and the dimension of the base locus of $|M|$ is zero. $|M|$ has at most $M^3 = 18$ base points (exactly 18 if counted with multiplicity). Since $K(M)$ acts on the set of base points by translations, $\#K(M) = 9$ must divide the cardinality of the set of base points. Then the cardinality of the set of base points of $|M|$ must be either 9 or 18. In the first case there is only one orbit of $K(M)$ and we have at once that L is not normally generated; while in the second case there are two orbits of $K(M)$: only if each orbit is symmetric we can conclude at once that L is not normally generated.

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