Bollettino Unione Matematica Italiana

Frank Duzaar, Andreas Gastel

Minimizing p-harmonic maps at a free boundary

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **1-B** (1998), n.2, p. 391–405.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1998_8_1B_2_391_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1998.

Bollettino U. M. I.
(8) **1**-B (1998), 391-406

Minimizing p-Harmonic Maps at a Free Boundary.

FRANK DUZAAR - ANDREAS GASTEL

Sunto. – *Studiamo le proprietà di regolarità delle mappe fra varietà di Riemann che minimizzano la p-energia fra quelle che soddisfano una condizione di frontiera pazialmente libera. Proviamo che tali mappe sono Hölder continue vicino alla frontiera libera fuori di un insieme singolare, e otteniamo stime ottimali per la dimensione di Hausdorff di questo insieme singolare..*

1. – Introduction.

In this paper we investigate the regularity properties of maps $u: M \to N$ between Riemannian manifolds which minimize locally the p-energy amongst maps satisfying a partially free boundary condition $u(\Sigma) \subset \Gamma$. The *parameter domain* M for our maps is a compact connected Riemannian manifold of dimension $m \geq 2$, and the *free boundary* Σ is a non-empty, relatively open subset of ∂M. As *target manifold* N we have a compact Riemannian manifold of dimension $n \geq 1$ which we assume to be isometrically embedded in $\mathbb{R}^{n+k}$ for some $k \geq 0$. The *supporting manifold* Γ for the free boundary values is a closed submanifold of N of dimension d, $0 \leq d \leq n$. We are then interested in mappings $u: M \to N$ of Sobolev class $W^{1,p}(M, N) := \{u \in W^{1,p}(M, \mathbb{R}^{n+k}): u(x) \in N$ for almost all $x \in M\}$ which minimize locally the *p-energy*

$$E(u) = \int_M |\nabla u|^p \, dvol$$

with respect to the free boundary condition $u(\Sigma) \subset \Gamma$. Here $|\nabla u| = \left\{\sum_{i=1}^{n+k} |\nabla u^i|^2\right\}^{1/2}$. A map $u \in W^{1,p}(M, N)$ is termed to be *locally p-energy minimizing* on $M \cup \Sigma$ with respect to the free boundary condition $u(\Sigma) \subset \Gamma$ if there exists an open covering $\mathcal{X}$ of $M \cup \Sigma$ such that $E(u) \leq E(v)$ for every $v \in W^{1,p}(M, N)$ which satisfies $v(\Sigma) \subset \Gamma$ and which coincides with u outside X, for some $X \in \mathcal{X}$. A point $x \in M \cup \Sigma$ is called a *regular point* of u if u coincides with a continuous function on a neighbourhood of x in $M \cup \Sigma$. The set of regular points is denoted by $\operatorname{Reg} u$, and its complement $(M \cup \Sigma) \setminus \operatorname{Reg} u$ is termed the singular set $\operatorname{Sing} u$. By the Sobolev embedding theorem regularity in the case $p > m$ follows trivially. Therefore we restrict ourselves to the case $1 < p \leq m$. Our main result reads as follows.

THEOREM. – *If $u \in W^{1,p}(M, N)$ is locally p-energy minimizing on $M \cup \Sigma$ with respect to the free boundary condition $u(\Sigma) \subset \Gamma$, then*

$$\mathcal{H} - \dim(\Sigma \cap \operatorname{Sing} u) \leq m - [p] - 1,$$

where $[p] := \max\{l \in \mathbb{N}: l \leq p\}$. Moreover, $\Sigma \cap \operatorname{Sing} u$ is discrete in $M \cup \Sigma$ if $m - 1 \leq p < m$.

With regard to interior regularity the corresponding theorem was proved by Schoen and Uhlenbeck [9] in the quadratic case $p = 2$, and independently by Fuchs [4], Hardt and Lin [5], and Luckhaus [7] in the general case $1 < p \leq m$. Regularity for minimizing maps at a general free boundary was considered by Duzaar and Steffen [2], [3], and Hardt and Lin [6] in the case $p = 2$. Finally in [1] the first author and Grotowski obtained an optimal partial regularity result when $\partial\Gamma \neq \emptyset$ is allowed and $p = 2$ (i.e. they studied a vectorvalued thin obstacle problem).

2. – Notation and general assumptions.

First we describe our assumptions on the parameter domain $M \cup \Sigma$. We assume that $M \cup \Sigma$ is a connected Riemannian manifold with boundary $\partial M \supseteq \Sigma \neq \emptyset$ and interior M of dimension $m \geq 2$ and differentiability class 2. Introducing local coordinates around $x_0 \in \Sigma$ we specialize the parameter domain M to the unit upper half ball $B^+ := \{x \in \mathbb{R}^m: |x| < 1, x^m > 0\}$ equipped with a C^1-Riemannian metric which is close to the Euclidean metric, and Σ to its equatorial part $D = \{x \in \mathbb{R}^m: |x| < 1, x^m = 0\}$. Then, similarly to [2, section 1] and [5, section 7], we may restrict ourselves to the situation where the metric is in fact Euclidean.

Next, we specify the assumptions on the target manifold N and the supporting manifold for the free boundary values Γ. We assume that N is a compact C^2-submanifold of $\mathbb{R}^{n+k}$, that Γ is a closed submanifold of N, and that Γ as a submanifold of $\mathbb{R}^{n+k}$ is of class C^2. These assumptions imply that N admits a uniform tubular neighbourhood $\boldsymbol{U}_\sigma(N) := \{q \in \mathbb{R}^{n+k}: \operatorname{dist}(q, N) < \sigma\}$ for some $\sigma > 0$, and that the associate nearest point map $\Pi: \boldsymbol{U}_\sigma(N) \to N$ is well-defined and Lipschitz continuous with Lipschitz constants satisfying

$$\operatorname{Lip}(\Pi|_{\boldsymbol{U}_{t\varrho}(N)}) \downarrow 1 \quad \text{as } t \downarrow 0.$$

Similarly, the nearest point map onto Γ, which is Lipschitz continuous and well-defined on $\boldsymbol{U}_{\varrho}(\Gamma)$ for some $\varrho > 0$, is denoted by R and satisfies

$$\mathrm{Lip}(R|_{\boldsymbol{U}_{t\varrho}(\Gamma)}) \downarrow 1 \quad \text{as } t \downarrow 0.$$

3. – Extension and compactness.

Throughout this section we use the notation $[\![p]\!] := \min\{l \in \mathbb{N} =: p \leqslant l\}$.

LEMMA 3.1 (extension). – *For* $1 < p < \infty$, $([\![p]\!] - 1)/p < \beta < 1$, *there exist constants* $c_1(m, \beta, p)$ *and* $c_2(m, p)$ *such that whenever* $K > 0$, $\varepsilon \in]0, 1[$, $\lambda \in]0, 1]$ *and* $u, v \in W^{1,p}(S^+, \mathbb{R}^{n+k})$ *with* $u(S^+) \subset \Gamma$ *and* $v(S^+) \subset \Gamma$ *satisfy*

$$\text{(i)} \quad \int_{S^+} |\nabla u|^p + |\nabla v|^p + \frac{|u-v|^p}{\varepsilon^p} \, d\mathcal{H}^{m-1} \leqslant K^p$$

and

$$\text{(ii)} \quad d := c_1 \varepsilon^{1-\beta} \lambda^{([\![p]\!]-m)/p} K < \varrho,$$

then there exists an extension $w \in W^{1,p}([0, \lambda] \times S^+, \mathbb{R}^{n+k})$ *such that* $w(0, x) = u(x)$, $w(\lambda, x) = v(x)$ *for almost all* $x \in S^+$, $w([0, \lambda] \times \partial S^+) \subset \Gamma$, *and*

$$\int_{[0,\lambda] \times S^+} |\nabla w|^p \, d\mathcal{H}^m \leqslant c_2 (1 + \mathrm{Lip} R|_{\boldsymbol{U}_d(\Gamma)})^p \lambda \left(1 + \left(\frac{\varepsilon}{\lambda}\right)^p\right) K^p \tag{3.1}$$

and

$$\mathrm{dist}(w(t, x), \mathrm{Im} u \cup \mathrm{Im} v) \leqslant d \quad \textit{for } \mathcal{H}^m\textit{-almost-all } (t, x) \in [0, \lambda] \times S^+. \tag{3.2}$$

PROOF. – Like [DG] we assume $\lambda = 3^{-\nu}$ and decompose the unit cube $Q := [-1, 1]^{m-1}$ in $\mathbb{R}^{m-1}$ into $3^{\nu(m-1)}$ cubes of edge length 2λ. For $l = 0, \ldots, m-1$ we denote by Q^l the l-skeleton of this decomposition. Q^l is the union of the closed l-cells Q_i^l. We define

$$Z := \{x \in Q : \mathrm{dist}(x, \partial Q) \leqslant \lambda\}$$

and observe that there exists a bi-Lipschitz homeomorphism (with bi-Lipschitz constants not depending on λ) $\phi: Z \to [0, \lambda] \times S^{m-2}$ such that for $l = 1, \ldots, m-1$

$$\phi(Q^l \cap Z \backslash \partial Q) =]0, \lambda] \times \phi(Q^{l-1} \cap \partial Q). \tag{3.3}$$

The construction from [DG] yields a bi-Lipschitz homeomorphism $\psi\colon Q \to S^+$ (cf. (2.7) of [DG]) such that for $l = 0, \dots, m-1$

$$\int_{\psi(Q^l \setminus \partial Q)} |\nabla u|^p + |\nabla v|^p + \frac{|u-v|^p}{\varepsilon^p} \, d\mathcal{H}^l \leq c_3(m) \lambda^{l-m+1} K^p. \tag{3.4}$$

Interpolating linearly on $[0, \lambda] \times Y$ between u and v, i.e.

$$z(t, x) := \left(1 - \frac{t}{\lambda}\right) u(x) + \frac{t}{\lambda} v(x)$$

where $Y := \psi(Q^{[\![p]\!]-1} \setminus \partial Q)$, we obtain $z \in W^{1,p}([0, \lambda] \times \overline{Y}, \mathbb{R}^{n+k})$ satisfying

$$\int_{[0,\lambda] \times Y} |\nabla z|^p \, d\mathcal{H}^{[\![p]\!]} \leq c_4(m) \lambda^{[\![p]\!]-m+1} \left(1 + \left(\frac{\varepsilon}{\lambda}\right)^p\right) K^p \tag{3.5}$$

and

$$|z(t, x) - u(x)| \leq c_5(m, \beta, p) \varepsilon^{1-\beta} \lambda^{([\![p]\!]-m)/p} K.$$

In particular

$$\operatorname{dist}(z(t, x), \operatorname{Im} u \cup \operatorname{Im} v) \leq c_5(m, \beta, p) \varepsilon^{1-\beta} \lambda^{([\![p]\!]-m)/p} K \tag{3.6}$$

for almost every $(t, x) \in [0, \lambda] \times Y$. (3.5) follows from (3.4), and (3.6) follows from [Lu, proof of Lemma 1].

Our aim now is to deform z on a neighbourhood of $[0, \lambda] \times \partial S^+$ such that the new mapping w will obey the free boundary condition $w(t, x) \in \Gamma$ for $x \in \partial S^+$, $t \in [0, \lambda]$, in addition to (3.5) and (3.6).

Using the bi-Lipschitz homeomorphisms ϕ and ψ, we will work on $[0, \lambda] \times S^{m-2}$ instead of a neighbourhood of ∂S^+ in S^+. We define

$$\tilde{u}\colon [0, \lambda] \times S^{m-2} \to \mathbb{R}^{n+k}, \qquad \tilde{u} := u \circ \psi \circ \phi^{-1},$$

$$\tilde{v}\colon [0, \lambda] \times S^{m-2} \to \mathbb{R}^{n+k}, \qquad \tilde{v} := v \circ \psi \circ \phi^{-1}.$$

From (3.3) and the definition of Y we infer $\psi^{-1}(Y) = Q^{[\![p]\!]-1} \setminus \partial Q$, and therefore, using (3.3),

$$\varphi(Z \cap \psi^{-1}(\overline{Y})) = [0, \lambda] \times \phi(Q^{[\![p]\!]-2} \cap \partial Q) =: [0, \lambda] \times X.$$

We also define

$$\tilde{z}\colon [0, \lambda]^2 \times X \to \mathbb{R}^{n+k}, \qquad \tilde{z} := z \circ (\mathrm{id} \times (\psi \circ \varphi^{-1})).$$

Then (3.4)-(3.6) directly imply

$$\int_{[0,\lambda]\times X} |\nabla\tilde{u}|^p + |\nabla\tilde{v}|^p + \frac{|\tilde{u}-\tilde{v}|^p}{\varepsilon^p}\, d\mathcal{H}^{[\![p]\!]-1} \leqslant c_6(m)\lambda^{[\![p]\!]-m}K^p, \tag{3.7}$$

$$\int_{[0,\lambda]^2\times X} |\nabla\tilde{z}|^p\, d\mathcal{H}^{[\![p]\!]} \leqslant c_7(m)\lambda^{[\![p]\!]-m+1}\left(1+\left(\frac{\varepsilon}{\lambda}\right)^p\right)K^p, \tag{3.8}$$

and

$$\operatorname{dist}(\tilde{z}(t,r,x), \operatorname{Im}u \cup \operatorname{Im}v) \leqslant c_5(m,\beta,p)\,\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p}K \tag{3.9}$$

almost everywhere on $[0,\lambda]^2\times X$.

For $s>0$ we now define

$$\begin{aligned} a(t) &:= \frac{\lambda}{2} - \left|t-\frac{\lambda}{2}\right| \quad \text{for } 0\leqslant t\leqslant\lambda,\\ A_s &:= \{(t, sa(t)) : 0\leqslant t\leqslant\lambda\},\\ D_s &:= \{(t,r) : 0\leqslant t\leqslant\lambda,\ 0\leqslant r\leqslant sa(t)\}. \end{aligned}$$

By the coarea formula we have

$$\int_{D_\sigma\times X} |\nabla\tilde{z}|^p\, d\mathcal{H}^{[\![p]\!]} = \int_0^\sigma \int_{A_s\times X} \frac{a(t)}{\sqrt{1+s^2}}\,|\nabla\tilde{z}|^p\, d\mathcal{H}^{[\![p]\!]-1}\, ds.$$

Therefore for each $\sigma\in]0,2]$ there exists $s\in[\sigma/2,\sigma]$ such that

$$\int_{A_s\times X} \frac{a(t)}{\sqrt{1+s^2}}\,|\nabla\tilde{z}|^p\, d\mathcal{H}^{[\![p]\!]-1} \leqslant \frac{2}{\sigma}\int_{D_\sigma\times X} |\nabla\tilde{z}|^p\, d\mathcal{H}^{[\![p]\!]} \leqslant c_8(m)\sigma^{-1}\lambda^{[\![p]\!]-m+1}\left(1+\left(\frac{\varepsilon}{\lambda}\right)^p\right)K^p, \tag{3.10}$$

the last inequality following from (3.8).

Let $\mu := 1-([\![p]\!]-1)/p$, $\delta := \min\{\lambda, \varepsilon^{(1-\beta)/\mu}\}$, and $r\in]0,\delta]$. Then, for $x\in X$ the Sobolev inequality and (3.7) imply

$$|\tilde{u}(r,x)-\tilde{u}(0,x)| \leqslant c_9(m,p)r^\mu\left(\int_{[0,\lambda]\times X} |\nabla\tilde{u}|^p\, d\mathcal{H}^{[\![p]\!]-1}\right)^{1/p} \leqslant c_{10}(m,p)r^\mu\lambda^{([\![p]\!]-m)/p}K, \tag{3.11}$$

Note that $r^{\mu} \leqslant \varepsilon^{1-\beta}$ and $\delta \leqslant \lambda$. Then from (3.11) we infer

$$|\tilde{u}(r, x) - \tilde{u}(0, x)| \leqslant c_{10}(m, p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p}K. \tag{3.12}$$

Recalling (3.9) and the definition of $\tilde{u}$ we obtain for $(t, r, x) \in [0, \lambda]^2 \times X$

$$|\tilde{z}(t, r, x) - \tilde{u}(r, x)| \leqslant c_5(m, \beta, p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p}K. \tag{3.13}$$

Combining (3.12), (3.13), and assumption (ii) we infer for any $(t, r, x) \in [0, \lambda] \times [0, \sigma] \times X$

$$\operatorname{dist}(\tilde{z}(t, r, x), \Gamma) \leqslant c_1(m, \beta, p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p}K = d < \varrho, \tag{3.14}$$

which, of course, yields that $R(\tilde{z}(t, r, x))$ is well-defined for all specified arguments (t, r, x).

We now let $\sigma := 2\delta/\lambda$ (such that $(\lambda/2)\sigma = \delta$) and choose an $s \in [\sigma/2, \sigma]$ according to (3.10). In view of the inclusion $D_s \subset [0, \lambda] \times [0, s(\lambda/2)] \subset [0, \lambda] \times [0, \delta]$ we can define $\tilde{w} \in W^{1, p}([0, \lambda]^2 \times X, \mathbb{R}^{n-k})$ by

$$\tilde{w}(t, r, x) := \begin{cases} R(\tilde{z}(t, sa(t), x)) & \text{on } [0, \lambda] \times \{0\} \times X, \\ \tilde{z}(t, r, x) & \text{on } ([0, \lambda]^2 \backslash D_s \cup A_s) \times = X, \\ \dfrac{r}{sa(t)} R(\tilde{z}(t, sa(t), x)) + \left(1 - \dfrac{r}{sa(t)}\right)\tilde{z}(t, sa(t), x) & \\ & \text{on } D_s \times X. \end{cases}$$

On $D_s \times X$ we compute $\dfrac{\partial}{\partial t}\tilde{w}$, $\dfrac{\partial}{\partial r}\tilde{w}$, and $\nabla_x \tilde{w}$ and get, using $0 \leqslant r \leqslant sa(t)$ and $|a'| \equiv 1$,

$$\left|\frac{\partial}{\partial t}\tilde{w}(t, r, x)\right| \leqslant \frac{1}{a(t)}|R(\tilde{z}(t, sa(t), x)) - \tilde{z}(t, sa(t), x)| + \left|\frac{\partial}{\partial t}R(\tilde{z}(t, sa(t), x))\right| + \left|\frac{\partial}{\partial t}\tilde{z}(t, sa(t), x)\right|,$$

$$\left|\frac{\partial}{\partial r}\tilde{w}(t, r, x)\right| = \frac{1}{sa(t)}|R(\tilde{z}(t, sa(t), x)) - \tilde{z}(t, sa(t), x)|,$$

$$|\nabla_x \tilde{w}(t, r, x)| \leqslant |\nabla_x R(\tilde{z}(t, sa(t), x))| + |\nabla_x \tilde{z}(t, sa(t), x)|.$$

These inequalities together imply

$$(3.15)\qquad |\nabla \widetilde{w}(t, r, x)|^p \leqslant$$

$$c_{11}(p)\left\{\frac{1}{(sa(t))^p}\operatorname{dist}(\tilde{z}(t, sa(t), x), \Gamma)^p + (1+\operatorname{Lip}R|_{U_d(\Gamma)})^p\, |\nabla_{(t, x)}\tilde{z}(t, sa(t), x)|^p\right\}.$$

To estimate the first summand in the right hand side of (3.15) we observe that $\operatorname{dist}(\tilde{z}(t, sa(t), x), \Gamma) \leqslant |\tilde{z}(t, sa(t), x) - \tilde{u}(0, x)| \leqslant$

$$\frac{t}{\lambda}\,|\tilde{u}(sa(t), x) - \tilde{v}(sa(t), x)| + \int_0^{sa(t)} \left|\frac{\partial}{\partial r}\tilde{u}(r, x)\right| dr \leqslant$$

$$\frac{t}{\lambda}\,|\tilde{u}(sa(t), x) - \tilde{v}(sa(t), x)| = (sa(t))^{1-1/p}\left(\int_0^{sa(t)} \left|\frac{\partial}{\partial r}\tilde{u}(r, x)\right|^p dr\right)^{1/p}.$$

The same estimate with $\tilde{u}(0, x)$ replaced by $\tilde{v}(0, x)$ shows $\operatorname{dist}(\tilde{z}(t, sa(t), x), \Gamma) \leqslant$

$$\frac{\lambda - t}{\lambda}\,|\tilde{u}(sa(t), x) - \tilde{v}(sa(t), x)| + (sa(t))^{1-1/p}\left(\int_0^{sa(t)} \left|\frac{\partial}{\partial r}\tilde{v}(r, x)\right|^p dr\right)^{1/p}.$$

Both inequalities together with the definition of $a(t)$ imply for $t \in [0, \lambda]$, $x \in X$,

$$(3.16)\qquad \operatorname{dist}(\tilde{z}(t, sa(t), x), \Gamma)^p \leqslant c_{12}(p)\left\{\frac{a(t)^p}{\lambda^p}\,|\tilde{u}(sa(t), x) - \tilde{v}(sa(t), x)|^p + \right.$$

$$\left.(sa(t))^{p-1}\int_0^{sa(t)} \left|\frac{\partial}{\partial r}\tilde{u}(r, x)\right|^p + \left|\frac{\partial}{\partial r}\tilde{v}(r, x)\right|^p dr\right\}.$$

Integrating (3.16) over $D_s \times X$ we obtain, using (3.7),

$$(3.17)\qquad \int_{D_s \times X} (sa(t))^{-p}\operatorname{dist}(\tilde{z}(t, sa(t), x), \Gamma)^p\, d\mathcal{H}^{[\![p]\!]}(t, r, x) \leqslant$$

$$c_{12}(p)\left\{s^{-p}\lambda^{1-p}\int_{[0, \lambda]\times X} |\tilde{u} - \tilde{v}|^p\, d\mathcal{H}^{[\![p]\!]-1} + \lambda\int_{[0, \lambda]\times X} |\nabla\tilde{u}|^p + |\nabla\tilde{v}|^p\, d\mathcal{H}^{[\![p]\!]-1}\right\} \leqslant$$

$$c_{12}(p)c_6(m)\lambda^{[\![p]\!]-m+1}\left(1 + \frac{\varepsilon^p}{s^p\lambda^p}\right)K^p \leqslant$$

$$c_{13}(m, p)\lambda^{[\![p]\!]-m+1}\left(1 + \max\left\{\varepsilon^{p(1-(1-\beta)/\mu)}, \left(\frac{\varepsilon}{\lambda}\right)^p\right\}\right)K^p,$$

the last estimate following from $s \geqslant \sigma/2 = \delta/\lambda = \min\{1, \lambda^{-1}\varepsilon^{(1-\beta)/\mu}\}$. Since $\beta > ([\![p]\!]-1)/p$ (by assumption) we have $\varepsilon^{p(1-(1-\beta)/\mu)} < 1$, and from (3.17) we derive

$$\int_{D_s \times X} \frac{\operatorname{dist}(\tilde{z}(t, sa(t), x), \Gamma)^p}{sa(t)^p} \, d\mathcal{H}^{[\![p]\!]}(t, r, x) \leqslant \tag{3.18}$$

$$2c_{13}(m, p)\lambda^{[\![p]\!]-m+1} K^p \left(1 + \left(\frac{\varepsilon}{\lambda}\right)^p\right).$$

Now, we estimate the integral of the second summand of the right hand side of (3.15). Using (3.10) we find

$$\int_{D_s \times X} |\nabla_{(t,x)}\tilde{z}(t, sa(t), x)|^p \, d\mathcal{H}^{[\![p]\!]} \leqslant \tag{3.19}$$

$$2^{p/2} s \int_X \int_0^\lambda \frac{a(t)}{\sqrt{1+s^2}} |\nabla \tilde{z}|^p (t, sa(t), x)\sqrt{1+s^2} dt d\mathcal{H}^{[\![p]\!]-2} x =$$

$$2^{p/2} s \int_{A_s \times X} \frac{a(t)}{\sqrt{1+s^2}} |\nabla \tilde{z}|^p \, d\mathcal{H}^{[\![p]\!]-1} \leqslant c_{14}(m, p) \frac{s}{\sigma} \lambda^{[\![p]\!]-m+1}\left(1 + \left(\frac{\varepsilon}{\lambda}\right)^p\right) K^p \leqslant$$

$$c_{14}(m, p)\lambda^{[\![p]\!]-m+1}\left(1 + \left(\frac{\varepsilon}{\lambda}\right)^p\right) K^p.$$

Combining (3.15), (3.18), and (3.19) we finally arrive at

$$\int_{D_s \times X} |\nabla \widetilde{w}|^p \, d\mathcal{H}^{[\![p]\!]} \leqslant c_{15}(m, p)(1 + \operatorname{Lip} R_{|U_d(\Gamma)})^p \lambda^{[\![p]\!]-m+1}\left(1 + \left(\frac{\varepsilon}{\lambda}\right)^p\right) K^p. \tag{3.20}$$

Now, we define $w \in W^{1,p}([0, \lambda] \times \overline{Y}, \mathbb{R}^{n+k})$ compatible with the free boundary condition by

$$w := \begin{cases} \widetilde{w} \circ (id \times (\phi \circ \psi^{-1})) & \text{on } [0, \lambda] \times \overline{Y} \cap \psi(Z), \\ z & \text{on } [0, \lambda] \times Y \backslash \psi(Z). \end{cases}$$

The estimates (3.5), (3.8), and (3.20) yield

$$\int_{[0,\lambda] \times Y} |\nabla w|^p \, d\mathcal{H}^{[\![p]\!]} \leqslant c_{16}(m, p)\lambda^{[\![p]\!]-m+1}\left(1 + \left(\frac{\varepsilon}{\lambda}\right)^p\right) K^p, \tag{3.21}$$

and from (3.6), (3.9) and (3.14) we obtain

$$\operatorname{dist}(w(t, x), \operatorname{Im} u \cup \operatorname{Im} v) \leqslant c_{17}(m, \beta, p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p} K \tag{3.22}$$

for almost all $(t, x) \in [0, \lambda] \times \overline{Y}$. The inductive procedure of homogeneous extension from [Lu] together with the modifications described in [DG] can now be used to construct the map $w \in W^{1,p}([0, \lambda] \times S^+, \mathbb{R}^{n+k})$ satisfying the assertions of the lemma. ■

We apply the extension lemma to prove the following compactness theorem for p-energy minimizing maps at a free boundary.

THEOREM 3.2 (compactness). – *Suppose Γ_i, N_i, and $u_i \in W^{1,p}(B^+, N_i)$ with $u_i(D) \subset \Gamma_i$, $i \in \mathbb{N} \cup \{\infty\}$, satisfy:*

(i) *N_i admits a Lipschitz neighbourhood retraction Π_i: $\boldsymbol{U}_{\sigma_i}(N_i) \to N_i$ which satisfies $\lim_{\sigma \to 0} \mathrm{Lip}(\Pi_i \,|_{\boldsymbol{U}_\sigma(N_i)}) = 1$;*

(ii) *Γ_i admits a Lipschitz neighbourhood retraction R_i: $\boldsymbol{U}_{\varrho_i}(\Gamma_i) \to \Gamma_i$ for which*

$$\liminf_{i \to \infty} \varrho_i > 0 \quad \text{and} \quad \liminf_{i \to \infty} \mathrm{Lip}(R_i \,|_{U_t(\Gamma_i)}) < \infty \quad \text{for some } t > 0;$$

(iii) *each $v \in W^{1,p}(B^+, \mathbb{R}^{n+k})$ with $v(B^+) \subset N_\infty$ and $v(D) \subset \Gamma_\infty$ is the $W^{1,p}$-limit of maps $v_i \in W^{1,p}(B^+, \mathbb{R}^{n+k})$ with $v_i(B^+) \subset N_i$ and $v_i(D) \subset \Gamma_i$; and*

(iv) *the u_i are p-energy minimizing maps from B^+ into N_i with respect to the free boundary condition $u_i(D) \subset \Gamma_i$ for $i \in \mathbb{N}$ and converge weakly in $W^{1,p}(B^+, \mathbb{R}^{n+k})$ to u_∞.*

Then u_∞ is p-energy minimizing in $W^{1,p}(B^+, N_\infty)$ subject to the free boundary condition $u_\infty(D) \subset \Gamma_\infty$, and $u_i \to u_\infty$ strongly in $W^{1,p}(B_\sigma^+, \mathbb{R}^{n+k})$ for any $0 < \sigma < 1$.

The assertions of the compactness theorem follow from our extension lemma using a direct adaptation of the arguments from [7, p.357 ff] to the free boundary situation considered here. One merely has to replace balls B_σ and spheres S_σ by half-balls B_σ^+ and hemi-spheres S_σ^+ before applying the extension lemma.

COROLLARY 3.3. – *Suppose $(u_i)_{i \in \mathbb{N}} \subset W^{1,p}(B^+, N)$ is a sequence of p-energy minimizing maps subject to the free boundary condition $u_i(D) \subset \Gamma$, and*

$$\sup_{i \in \mathbb{N}} \int_{B^+} |\nabla u_i|^p \, dx < \infty .$$

Then there exists a subsequence (u_i) and a map $u \in W^{1,p}(B+, N)$ which *is p-energy minimizing w.r.t. the free boundary condition* $u(D) \subset \Gamma$ *such that* u_i *converges strongly in* $W^{1,p}(B_\sigma^+, \mathbb{R}^{n+k})$ *to* u *for any* $0 < \sigma < 1$.

4. – An ε-regularity theorem.

Our line of reasoning follows exactly to that of [1, Section 3] which, of course, is based on [7, Proposition 1] (see also [11, Theorem 10.3]). One merely has to replace the scaled 2-energy $\sigma^{2-m} \int_{B_\sigma^+(a)} |\nabla u|^2 dx$ by $\sigma^{p-m} \int_{B_\sigma^+(a)} |\nabla u|^p dx$ for half-balls $B_\sigma^+(a) := \{x \in \mathbb{R}^m : |x-a| < \sigma,\ x^m > 0\} \subset B^+$ centered at $a \in D$ by the scaled p-energy $\sigma^{p-m} \int_{B_\sigma^+(a)} |\nabla u|^p dx$, and, instead of [1, Lemma 3.1], to use:

LEMMA 4.1. – *Let* $\xi \in W^{1,p}(B^+, \mathbb{R}^n)$ *be p-energy minimizing with respect to the free boundary condition* $\xi(D) \subset \mathbb{R}^d \times \{0\} \subset \mathbb{R}^n$. *Then, for any* $0 < \sigma < 1$ *we have*

$$\int_{B_\sigma^+} |\nabla \xi|^p dx \leqslant c\sigma^n \int_{B^+} |\nabla \xi|^p dx,$$

where c is a constant depending on m, n and p only.

PROOF. – For $j = 1, \ldots, d$ we define $\tilde{\xi}^j$ to be the extension of ξ^j to B by even reflection (across D). Moreover, for $j = d+1, \ldots, n$ we let $\tilde{\xi}^j$ to be the extension of ξ^j to B by odd reflection. Then it is easy to check that $\tilde{\xi} := (\tilde{\xi}^1, \ldots, \tilde{\xi}^d)$ is weakly p-harmonic on B, i.e. for any $\phi \in C_0^1(B, \mathbb{R}^n)$ we have

$$\int_B |\nabla \tilde{\xi}|^{p-2} \nabla \tilde{\xi} \cdot \phi dx = 0.$$

Hence, from [12, Theorem 3.2] we infer

$$\int_{B_\sigma} |\nabla \tilde{\xi}|^p dx \leqslant c(m, n, p)\sigma^m \int_B |\nabla \tilde{\xi}|^p dx$$

which clearly yields the corresponding estimate for ξ on B_σ^+. ■

As in the case $p = 2$ [1, Theorem 3.4] (see also [2, Theorem 5.2], [6, Theorem 3.4]) we can now state an ε-regularity theorem for minimizing p-harmonic maps at a free boundary.

THEOREM 4.2. – *Given N and Γ satisfying the assumptions given in section* 2, *and* $\alpha \in]0, 1[$, *there exist constants C and ε_0 depending on α, m, n, p, N and Γ only such that every map $u \in W^{1,p}(B^+, N)$ which is p-energy minimizing w.r.t. the free boundary condition $u(D) \subset \Gamma$ and which has small scaled p-energy*

$$\varepsilon^p := \sigma^{p-m} \int_{B_\sigma^+(x_0)} |\nabla u|^p \, dx \leq \varepsilon_0^p$$

for some half-ball $B_\sigma^+(x_0) \subset B^+$, $x_0 \in D$, $|x_0| \leq 1/2$, satisfies

$$r^{p-m} \int_{B_r^+(x_1)} |\nabla u|^p \, dx \leq C\varepsilon^p \left(\frac{r}{\sigma} \right)^{p\alpha}$$

for all $x_1 \in B_{\sigma/2}^+(x_0) \cap D$, $0 < r \leq \sigma/2$. ■

Combining Theorem 4.2 with the corresponding interior result [4], [5], [7] we see that p-energy minimizing maps at a free boundary are Hölder continuous with exponent α for all $0 < \alpha < 1$ on $B_{1/4}^+$ provided $\int_{B^+} |\nabla u|^p \, dx \leq \varepsilon_1^p$ where $\varepsilon_1 > 0$ is a constant depending on α, n, m, p, N and Γ only. (Note that we also have to use the monotonicity formula Lemma 5.1.)

5. – Partial regularity.

We begin this section by deriving a *monotonicity formula* for maps $u \in W^{1,p}(B^+, N)$ which minimize the p-energy subject to the free boundary condition $u(D) \subset \Gamma$ (cf.[8], [9], [4], [5], [7] for the interior case, and [2], [6], [1] for the case $p = 2$ at the free boundary). Given $\phi \in C_0^1(B, \mathbb{R}^m)$ satisfying $\phi^m|_D = 0$ we define X_t to be the solution of $\dot{X}_t = \phi \circ X_t$ with initial condition $X_0 = \mathrm{id}$. Then $u_t := u \circ X_t$ is an admissible variation for u, and the minimizing property of u yields

$$0 = \frac{d}{dt} E(u_t) \bigg|_0 = \int_{B^+} \sum_{i,j=1}^{m} (|\nabla u|^p \delta_{ij} - p \nabla_i u \cdot \nabla_j u |\nabla u|^{p-2}) \nabla_i \varphi^j \, dx.$$

Then, exactly as in [1, Theorem 4.1] we prove

THEOREM 5.1. – *Let $a \in B^+ \cup D$ and $0 < r < \sigma < 1 - |a|$ be given. Then, any map $u \in W^{1,p}(B^+, N)$ minimizing the p-energy w.r.t. the free boundary* *condition* $u(D) \subset \Gamma$ *satisfies*:

$$(5.1) \qquad \sigma^{p-m}\left[\int_{B_\sigma^+(a)} |\nabla u|^p\,dx + \int_{B_\sigma^+(a^*)} |\nabla u|^p\,dx\right] -$$

$$- r^{p-m}\left[\int_{B_r^+(a)} |\nabla u|^p\,dx + \int_{B_r^+(a^*)} |\nabla u|^p\,dx\right] =$$

$$p\left[\int_{B_\sigma^+(a)\backslash B_r^+(a)} R^{p-m}|\nabla u|^{p-2}\left|\frac{\partial u}{\partial R}\right|^2 dx + \int_{B_\sigma^+(a^*)\backslash B_r^+(a^*)} (R^*)^{p-m}|\nabla u|^{p-2}\left|\frac{\partial u}{\partial R^*}\right|^2 dx\right],$$

where $a^* = (a^1, \ldots, a^{m-1}, -a^m)$, $R = |x-a|$, *and* $R^* = |x-a^*|$. *In particular,*

$$r^{p-m}\left[\int_{B_r^+(a)} |\nabla u|^p\,dx + \int_{B_r^+(a^*)} |\nabla u|^p\,dx\right]$$

is monotone non-decreasing on $]0, 1-|a|[$. ■

As a first consequence of the monotonicity formula we observe that the *density function*

$$\Theta_u(a) := \lim_{r\to 0} r^{p-m}\left[\int_{B_r^+(a)} |\nabla u|^p\,dx + \int_{B_r^+(a^*)} |\nabla u|^p\,dx\right]$$

is well defined for $a \in B^+ \cup D$. Note that

$$\Theta_u(a) = \begin{cases} \lim\limits_{r\to 0} r^{p-m}\displaystyle\int_{B_r(a)} |\nabla u|^p\,dx & \text{for } a \in B^+, \\ 2\lim\limits_{r\to 0} r^{p-m}\displaystyle\int_{B_r^+(a)} |\nabla u|^p\,dx & \text{for } a \in D. \end{cases}$$

THEOREM 5.2. – *Suppose that* $u_j \in W^{1,p}(B^+, N)$, $j \in \mathbb{N}$, *is p-energy minimizing subject to the free boundary condition* $u_j(D) \subset \Gamma$. *Suppose also that* $u_j \to u$ *strongly in* $L^p(B^+, \mathbb{R}^{n+k})$ *and that*

$$\sup_{j\in\mathbb{N}} \int_{B^+} |\nabla u_j|^p\,dx < \infty.$$

Then for $a \in B^+ \cup D$, $a_j \to a$ *implies*

$$\Theta_u(a) \geqslant \limsup_{j \to \infty} \Theta_{u_j}(a_j),$$

i.e. the density function is jointly upper semicontinuous.

PROOF. – In view of Corollary 3.3 we have $u_j \to u$ strongly in $W^{1,p}(B_\sigma^+, \mathbb{R}^{n+k})$ for any $0 < \sigma < 1$. For $a \in D$ consider r, $\varepsilon > 0$ such that $r + \varepsilon < 1 - |a|$. From (5.1) we infer

$$\Theta_{u_j}(a_j) \leqslant r^{p-m}\left[\int_{B_r^+(a_j)} |\nabla u_j|^p\,dx + \int_{B_r^+(a_j^*)} |\nabla u_j|^p\,dx\right] \leqslant 2r^{p-m}\int_{B_{r+\varepsilon}^+(a)} |\nabla u_j|^p\,dx$$

for any j such that $|a - a_j| < \varepsilon$. Now, the strong $W^{1,p}$-convergence of u_j on $B_{r+\varepsilon}^+(a)$ yields

$$\limsup_{j \to \infty} \Theta_{u_j}(a_j) \leqslant 2r^{p-m}\int_{B_{r+\varepsilon}^+(a)} |\nabla u|^p\,dx.$$

Letting first ε and then r tend to zero, the assertion follows. The case $a \in B^+$ follows similarly. ■

Next we discuss *tangent maps at the free boundary.* For $a \in D$, $0 < r \leqslant r_0 < 1 - |a|$ we define the rescaled map

$$u_{a,r}(x) := u(a + rx) \qquad \text{for } x \in B_{r/r_0}^+.$$

Then, the monotonicity formula and Corollary 3.3 provide as in [10, Section 3] (see also [1, Section 4]) that a sequence $(u_{a,r_i})_{i \in \mathbb{N}}$, $r_i \to 0$, converges strongly in $W^{1,p}(B_\sigma^+, \mathbb{R}^{n+k})$ for any $\sigma \in]0, \infty[$ to a map φ: $\mathbb{R}_+^m \to N$ which is p-energy minimizing w.r.t. the free boundary condition $\varphi = (\mathbb{R}^{m-1} \times \{0\}) \subset \Gamma$. Any such map is called a (free boundary) *tangent map to u at a.*

We now follow the arguments of [10, Section 3] almost verbatim. First we deduce the following important properties of tangent maps at the free boundary:

(5.2) $\qquad \Theta_\phi(0) \equiv 2r^{p-m}\int_{B_r^+} |\nabla \varphi|^p\,dx = \Theta_u(a)$ for all $r > 0$;

(5.3) $\qquad \varphi$ is homogeneous of degree zero;

(5.4) $\qquad a \in \operatorname{Reg} u \Rightarrow u$ has a constant tangent map at a; and

(5.5) $\qquad$ for $a \in B^+ \cup D$ we have: $\Theta_u(a) = 0 \Leftrightarrow a \in \operatorname{Reg} u$.

THEOREM 5.3. – $\mathcal{H}^{m-p}(\mathrm{Sing}(u)) = 0$, *in particular* $\mathrm{Sing}(u) = \emptyset$ *for* $p = m$.

Next we consider homogeneous degree zero maps $\phi\colon \mathbb{R}^m_+ \to N$ which minimize locally the p–energy subject to the free boundary condition $\phi(\mathbb{R}^{m-1} \times \{0\}) \subset \Gamma$. Then:

(5.6) $\Theta_\varphi(\cdot)$ achieves its maximum at 0;

(5.7) $\varphi \circ \tau_a = \varphi$ for any $a \in \mathcal{S}(\phi) := \{b \in \mathbb{R}^{m-1} \times \{0\} : \Theta_\phi(b) = \Theta_\phi(a)\}$;

(5.8) $\mathcal{S}(\varphi)$ is a linear subspace of $\mathbb{R}^{m-1} \times \{0\}$;

(5.9) if $\dim \mathcal{S}(\varphi) = m - [p] + 1$ then $\varphi \equiv \mathrm{const}$; and

(5.10) $\mathcal{S}(\varphi) \subset \mathrm{Sing}(\varphi)$ if φ is non-constant.

Here $\tau_a(x) := x + a$ for $x \in \mathbb{R}^m$. Finally, we return to the situation of $u \in W^{1,p}(B^+, N)$ being p–energy minimizing w.r.t. $u(D) \subset \Gamma$. For $j = 0, \ldots, m - [p]$ we define:

$$\mathcal{S}_j := \{a \in \mathrm{Sing}u \cap D \colon \dim \mathcal{S}(\phi) \leq j \text{ for every tangent map } \varphi \text{ to } u \text{ at } a\}.$$

THEOREM 5.4.

(i) $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \mathcal{S}_{m-[p]-1} \subset \mathcal{S}_{m-[p]} = \mathrm{Sing} \cap D$;

(ii) *for each* $\lambda > 0$, $\mathcal{S}_0 \cap \{b \colon \Theta_u(b) = \lambda\}$ *is discrete*; *and*

(iii) *for* $j = 0, \ldots, m - [p] - 1$ *we have* $\mathcal{H} - \dim \mathcal{S}_j \leq j$.

As an immediate consequence of Theorem 5.4, (iii) and the corresponding interior regularity result of [4], [5], [7] we obtain

THEOREM 5.5. – *Let* $u \in W^{1,p}(B_+, N)$ *be p-energy minimizing with respect to the free boundary condition* $u(D) \subset \Gamma$. *Then*

$$\mathcal{H} - \dim(\mathrm{Sing}u) \leq m - [p] - 1 \qquad \text{if } 1 < p < m - 1.$$

In the case $m - 1 \leq p < m$, $\mathrm{Sing}u \cap (D \cup B^+)$ *consists of isolated points.* ■

REFERENCES

[1] F. DUZAAR - J. F. GROTOWSKI, *Energy minimizing harmonic maps with an obstacle at the free boundary*, Manuscripta Math., **83** (1994), 291-314.

[2] F. DUZAAR - K. STEFFEN, *A partial regularity theorem for harmonic maps at a free boundary*, Asymptotic Anal., **2** (1989), 299-343.

[3] F. DUZAAR - K. STEFFEN, *An optimal estimate for the singular set of a harmonic map in the free boundary*, J. Reine Angew. Math., **401** (1989), 157-187.

[4] M. FUCHS, *p-harmonic obstacle problems*. Part I: *Partial regularity theory*, Ann. Mat. Pura Appl. (4), **156** (1990), 127-158.

[5] R. HARDT - F. H. LIN, *Maps minimizing the L^p-norm of the gradient*, Comm. Pure Appl. Math., **40** (1987), 555-588.

[6] R. HARDT - F. H. LIN, *Partially constrained boundary conditions with energy minimizing mappings*, Comm. Pure Appl. Math., **42** (1989), 309-334.

[7] S. LUCKHAUS, *Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold*, Indiana Univ. Math. J., **37** (1988), 349-367.

[8] P. PRICE, *A monotonicity formula for Yang-Mills fields*, Manuscripta Math., **43** (1983), 131-156.

[9] R. SCHOEN - K. UHLENBECK, *A regularity theorem for harmonic maps*, J. Differential Geom., **17** (1982), 307-335.

[10] L. SIMON, *Singularities of geometrical variational problems*, Regional Geometry Institute Lecture Notes, Utah (1992).

[11] K. STEFFEN, *An introduction to harmonic mappings*, Lecture Notes, SFB 256 Universität Bonn, Vorlesungsreihe, **18** (1991).

[12] K. UHLENBECK, *Regularity for a class of nonlinear elliptic systems*, Acta Mat., **138** (1977), 219-240.

F. Duzaar: Mathematisches Institut der Humboldt-Universität zu Berlin
Unter den Linden 6, D-10099 Berlin, Germany

A. Gastel: Mathematisches Institut der Heinrich-Heine-Universität Düsseldorf
Universitätsstraße 1, D-40225 Düsseldorf, Germany

*Pervenuta in Redazione
il 9 maggio 1996*