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Halphen Gaps and Good Space Curves.

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Sunto. – *In questo articolo dimostriamo l'esistenza di curve "buone e generali" di grado d e genere g che giacciono su di una superfice liscia di grado s, per ogni $s \geqslant 4$, $d \geqslant \binom{s}{2}$, e g in un certo intervallo vicino al genere massimo..*

Introduction.

In recent years there has been a great interest in space curves and its classification. Inspired by Halphen [16], Gruson and Peskine managed to determine the possible degree and genus of a smooth connected curve in $\boldsymbol{P}^3$, cf. [15]. Earlier they found a least upper bound $G(d, s)$ for the genus of a smooth connected curve, not contained in a surface of degree less than s (in range C, see (1.4)). The interesting existing problem of Halphen of finding all (d, g, s), $0 \leqslant g \leqslant G(d, s)$, such that there exists a smooth connected space curve of degree d and genus g, not lying on a surface of degree less than s, is however not solved, although Walter and Ballet have given substantial contributions in their papers [30] and [1]. A triple (d, g, s) for which there does not exist a curve with invariants d, g and s as in Halphen's problem, is called an Halphen gap, and it is conjectured [14] that there are no Halphen gaps for $g \leqslant G(d, s) - (s-1)(s-2)/2$, $s \geqslant 3$. Walter [30] proposes the slightly less optimistic bound $G_0(d, s)$ in his:

CONJECTURE 0.1. – *Let $s \geqslant 4$, $d \geqslant \binom{s-1}{2}$, and g be integers such that*

$$(s-4)d + 1 - \binom{s-1}{3} \leqslant g \leqslant G_0(d, s) :=$$

$$:= \frac{d^2}{2s} + \frac{(s-4)d}{2} + 1 - \frac{1}{4}\binom{s+1}{3} + \frac{\mu(s-\mu)}{2s}$$

where μ is the unique integer such that $0 \leqslant \mu < s$ and $\mu \equiv d - s(s-3)/2$ (mod s). Then there exists a smooth connected curve of degree d and genus g on a smooth surface S of degree s in $\boldsymbol{P}^3$.

Along with this search for characterizing triples (d, g, s) for which there exist curves, there has been a desire, for each (d, g) where the Hilbert scheme $H(d, g)_S$ of smooth connected curves of degree d and genus g is non-empty, to find good general curves with predictable properties. When the genus g is small compared to the degree d, Ballico and Ellia succeeded to prove the existence of a non-special, smooth connected curve of maximal rank (1.1), with the expected number of moduli (2.12), whose corresponding irreducible component of the Hilbert scheme $H(d, g)_S$ is generically smooth of dimension $4d$. More recently Walter and Fløystad essentially extended this result to the A' or FW-range (i.e. a range asymptotically given by $g^2 \leqslant 8d^3/9$, see (1.9) and (1.10)). Even though there is still a part of this range where the proof has not yet appeared in the literature, the existing problem of good curves in the FW-range are now quite well understood, cf. [3], [4], [28], [26], [11], [12], [31] and [32].

In the *compliment of the FW-range* (FW^C-range) there is quite a lot of components [9], and one might still hope that there is at least one which behaves well and where the general curve has nice properties. Hartshorne [17] proposes to consider superficially general curves, i.e. curves lying on a surface of the largest possible degree $s(d, g)$ allowed (1.1), as the natural candidates of the good general curves. For the components of such curves in $H(d, g)_S$, Hartshorne asks for the dimension and wonders if there always exist a generically smooth component among them. Ballico and Ellia [5] asks if or when such curves sit on a *smooth* surface of degree $s(d, g)$, in the same paper where they investigates for which d, g (e.g. in the FW^C-range) it is natural to hope of existence of curves of maximal rank. In [23] we add to this discussion of existence two remarks which seem relevant for the FW^C-range. Firstly, instead of maximal rank, we propose to look for the existence of curves C of maximal rank in some surface S of minimal degree containing C (2.11iv). Secondly we consider an infinitesimal variant for a curve on a smooth surface to correspond to a general component in the Noether-Lefschetz locus (see (2.2)). It turns out that such an "infinitesimally general curve in the NL-locus" is almost always unobstructed (2.7), and the dimension of the corresponding component of $H(d, g)_S$ is found. Inspired by Walter's conjecture above and some further remarks in his paper [30] (see (1.6)) where he indicates that the curves of (0.1) should correspond to general components in the NL-locus, we propose, believing that the curves of (0.1) should be infinitesimally general in the NL-locus as well, the following

CONJECTURE 0.2. – *Let* $s \geqslant 4$, $d \geqslant \binom{s+2}{2}$ *and* g *be integers such that*

$$g \geqslant sd - \binom{s+3}{3} + 2.$$

If $g \leqslant G_0(d, s)$ *then there exists a generically smooth component* V *in the Hilbert scheme* $H(d, g)_S$ *of smooth connected space curves of dimension* $\dim V = (4-s)d + \binom{s+3}{3} + g - 2$ *whose generic curve is smooth of maximal corank and sits on a smooth surface of degree* s *in* $\boldsymbol{P}^3$.

Note that the lower bound of the genus in (0.2) corresponds to a part of the upper bound of the FW-range (see (1.9)), i.e. the union of natural parts of the ranges of (0.2) is the FW^C-range. Indeed one may easily prove that the union of the ranges: $G_0(d, s+1) < g \leqslant G_0(d, s)$, $d \geqslant \binom{s+2}{2}$, for various $s \geqslant 1$ is precisely the FW^C-range, making of course these ranges the most interesting subranges of the conjecture 0.2. In [23] we conjectured (0.2) under the hypothesis: $d > s^2$ and $g \geqslant sd - \binom{s+3}{3} + 3$, which now seems too restrictive because we can show, under weak conditions, that a counterexample to (0.2) in the range $\binom{s+2}{2} < d \leqslant s^2$ leads to a counterexample in the range $d > s^2$. For the reliability of the conjecture we remark that it is consistent with the general principles of Hartshorne in [17] and it is related to his Conjecture 4.4 on the normal bundle (see (2.6)), but it avoids the counterexamples of [33]. Finally note that the Conjectures 0.1 and 0.2 above are true for $s = 4$ by [30] and [23], Prop. 2.1.

In this paper we prove a part of the conjectures above for $s \geqslant 5$, i.e. we prove the existence of generic curves of maximal corank as in (0.1) and (0.2) for every (d, g, s) satisfying $\max\left\{G_1(d, s), (s-3)d + 1 - \binom{s}{3}\right\} < g \leqslant G_0(d, s)$. For a definition of $G_1(d, s)$ we refer to Section 3 (see (3.2)), and to Theorem 3.3 and Remark 3.5 for precise statements. Since $G_0(d, s) - G_1(d, s) \sim s^3/18$ (asymptotically), there is still a large range left where the conjectures are open. Even though the range of existence is somewhat restricted, we think it is an interesting contribution to both conjectures. Indeed (0.1) is related to Halphen gaps, and such gaps are supposed to lie near the maximum genus $G(d, s)$. Since our existence result holds for any g in a range near this maximum, there are no gaps so close to the maximum. The ranges of Walter and Ballet in their papers [30] and [1] are not so close to $G(d, s)$.

It is, however, the questions concerning “good” curves and the second conjecture, which turned out to be the most difficult to handle. In the first two sections we discuss thoroughly relevant concepts and results. Since we show the existence of curves by a linkage argument, linking to any curve in the FW-range, we had to develope a criterion (2.8) which give conditions on a curve and complete intersection containing it, such that the linked curve is infinitesimally general in the NL-locus. This led to Theorem 2.9 which tells essentially that "good" maximal rank curves in the FW-range link to "good" generic, unobstructed curves of maximal corank in the remaining range. Note that the curves we construct are obviously superficially general if d is large enough, and they give the good answers to Hartshorne's and Ballico's, Ellia's questions for superficially general curves in this case. The curves turn also out to be general in the sense that they have a surjective Brill-Noether map (of the hyperplane section divisor, see (2.13)). Moreover since we also in this paper prove (0.1) and "(0.2) except for the maximal corank property" for many other values of (d, g), mainly satisfying $3d^2/8(s+1)+(s-4)d/2+1<g\leqslant G_0(d, s)$, cf. (3.8), we think we have given strong evidence to both conjectures in its most interesting range: $G_0(d, s+1)<g\leqslant G_0(d, s)$, $d\geqslant\binom{s+2}{2}$.

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1. – Preliminaries.

(1.1) In this paper we work over an algebraically closed field of *characteristic zero*. By a space curve C we mean an equidimensional, locally Cohen Macaulay, generically a complete intersection, closed subscheme in $\boldsymbol{P}^3$ of dimension 1, with sheaf ideal $\mathcal{J}_C$ and normal sheaf $\mathcal{N}_C$. Let $H(d, g)$ be the Hilbert scheme of curves C in $\boldsymbol{P}^3$ with Hilbert polynomial $\chi(\mathcal{O}_C(v))=dv+1-g$, and *let* $H(d, g)_S$ the open subscheme of smooth connected curves. A curve C is said to be of *maximal corank* [25] (resp. *maximal rank*), if it has the following property;

$$\textit{either } H^1(\boldsymbol{P}^3, \mathcal{J}_C(v))=0 \textit{ or } H^1(C, \mathcal{O}_C(v)) \ (\textit{resp. } H^0(\boldsymbol{P}^3, \mathcal{J}_C(v)))=0, \quad \textit{for every integer } v$$

Let $e(C)=\max\{k\,|\,H^1(\mathcal{O}_C(k))\neq 0\}$ be the *index of speciality* of the curve C, and let $s(C)=\min\{k\,|\,H^0(\mathcal{J}_C(k))\neq 0\}$. If $d\geqslant 3$, $g\geqslant 0$ are given integers, we define the "*postulation index*" $s(d, g)$ to be the smallest integer k such that *every* smooth connected curve is contained in a surface of degree k, i.e. $s(d, g)=\max\{s(C)\,|\,C\in H(d, g)_S\}$.

DEFINITION 1.2. – *A smooth connected curve C is said to be superficially general if $s(C)=s(d, g)$.*

LEMMA 1.3. – *Let C' be a curve (effective divisor) of degree d' and (arith-*

metic) genus g', on a smooth surface S of degree s in $\boldsymbol{P}^3$, let H be a hyperplane section and let C be a member of $|C' + nH|$, $n \geqslant 0$, of degree d and genus g. Then

i) $$d = d' + ns, \qquad g = g' + nd' + \frac{1}{2} sn(s-4+n);$$

ii) $H^1(\mathcal{J}_{C'}(v-n)) \cong H^1(\mathcal{J}_C(v))$ *for any integer* v,

$H^1(\mathcal{O}_{C'}(v-n)) \cong H^1(\mathcal{O}_C(v))$ *for* $v > s-4+n$;

iii) $e(C) = e(C') + n$ *provided* $e(C') \geqslant s-4$. *In this case*

C has maximal corank if and only if C' has maximal corank.

Indeed if we use the adjunction formula $2g - 2 = C.(C+K)$, the fact that the sheaf ideal $\mathcal{J}_{C/S} = O_S(-C)$ satisfies $\mathcal{J}_{C/S} = \mathcal{J}_{C'/S}(-n)$ by assumption, the exact sequence $0 \to \mathcal{J}_{C/S} \to \mathcal{O}_S \to \mathcal{O}_C \to 0$ and the corresponding one for $\mathcal{J}_{C'/S}$, one may easily prove (1.3), cf. [23], Lemma 2.2, for details.

(1.4) Now let $d \geqslant 3$, $s \geqslant 2$ be given integers. We define the maximum genus $G(d, s)$ to be the largest genus of a smooth connected curve in $\boldsymbol{P}^3$ of degree d, not contained in a surface of degree $< s$. Recall that the value of $G(d, s)$ is less or equal to $d(s-1) - \binom{s+2}{3} + 1$ in range $A: (s^2+4s+6)/6 \leqslant d < (s^2+4s+6)/3$, and equality (which gives us the conjectured value of $G(d, s)$) holds at least in the "upper half part" of this range (cf. [11] and [2]). In range $B: (s^2+4s+6)/3 \leqslant d \leqslant s(s-1)$ and $s \geqslant 5$, the conjectured value (cf. [19]) of $G(d, s)$ is known to be true in the case $d \geqslant s^2 - 6s + 14$, cf. [7] and [29]. Finally in range C: $s(s-1) < d$, we have by [13] a complete description:

$$G(d, s) = \frac{d^2}{2s} + \frac{(s-4)d}{2} + 1 - h$$

where $h = (1/2)f(s - f - 1 + f/s)$ *and* $f \equiv d(\operatorname{mod} s)$, $0 \leqslant f < s$. Compare $G(d, s)$ with $G_0(d, s)$. In this range the smooth connected curves C with genus $g = G(d, s)$ satisfy $s(C) = s$, i.e. $s(d, g) = s$.

DEFINITION 1.5. – (d, g, s) *is said to be an Halphen gap provided* $0 \leqslant g \leqslant G(d, s)$ and $s(d, g) < s$.

Halphen gaps are known to exist for every $s \geqslant 4$, and the known gaps are quite close to $G(d, s)$, cf. [8]. As pointed out in the introduction, we will in this paper show the absence of Halphen gaps in a certain range by proving a part of the Conjecture 0.1 of [30], which in particular conjectures no Halphen gaps

(d, g, s) provided $G(d, s+1) < g \leqslant G_0(d, s)$. In [30] Walter asserts more

CONJECTURE 1.6 (Walter). – *Let $s \geqslant 4$, d and g be integers such that*

$$(s-4)d + 1 - \binom{s-1}{3} \leqslant g \leqslant (s-3)d + 1 - \binom{s}{3}$$

Then there exists a smooth connected curve C of degree d and genus g on a smooth surface S of degree s in $\boldsymbol{P}^3$, satisfying $H^1(\mathfrak{I}_C(s-4)) = 0$ and $H^1(\mathcal{O}_C(s-3)) = 0$.

Due to the vanishing of $H^1(\mathfrak{I}_C(s-4))$, Walter indicates that the conjectured curves should give general components of the NL-locus, a remark which inspired us to look into the corresponding infinitesimal variant which we present in the next section.

For later use, we observe that if the Conjecture 1.6 is true in such a way that the existing curves are of maximal corank, then the following twisted variant of (1.6) holds:

(1.6V) *Let $s \geqslant 4$, $v \geqslant 0$, d and g be integers such that*

$$(s-4+v)d+1-\binom{s+v-1}{3}+\binom{v-1}{3} < g \leqslant (s-3+v)d+1-\binom{s+v}{3}+\binom{v}{3}.$$

Then there exists a smooth connected curve C of degree d and genus g on a smooth surface S of degree s in $\boldsymbol{P}^3$, satisfying $H^1(\mathfrak{I}_C(t)) = 0$ for $t \leqslant s-4+v$ and $H^1(\mathcal{O}_C(s-3+v)) = 0$.

(1.7) To prove this, let $C' \in H(d', g')_S$ be a curve as in (1.6) of maximal corank and take a smooth connected member C of the linear system $|C' + vH|$ (cf. [30], Lemma 3.4, to see the smoothness of C). Using (1.3ii) and (1.3iii) we see that C satisfies the cohomological conditions of (1.6V). Moreover to show that the genus g of C belongs to the range given by (1.6V), we first remark:

(*) Let $g_v(d, s) = (s-3+v)d+1-\binom{s+v}{3}+\binom{v}{3}$. Then, by Riemann-Roch's theorem, the equation $g = g_v(d, s)$ is equivalent to $\chi(\mathfrak{I}_{C/S}(s-3+v)) = 0$.

So, since the genus g' of C' satisfies $g_{-1}(d', s) < g' \leqslant g_0(d', s)$ by (1.6) and $\mathfrak{I}_{C'/S} = \mathfrak{I}_{C/S}(v)$ by assumption, we get $g_{v-1}(d, s) < g \leqslant g_v(d, s)$, i.e. the range of (1.6V), as required.

Now we claim that (1.6V) *implies* (0.1), cf. [30]. Indeed the claim follows at once from

LEMMA 1.8. – *The graph of $g = G_0(d, s)$ is just the largest piecewise linear curve defined by the lines $g = g_v(d, s)$, $v \geqslant -1$, of* (1.6V) *in the d, g-plane, i.e. the piecewise linear curve drawn between the points R_v, $v \geqslant 0$, where R_v is the point of intersection of the lines $g = g_{v-1}(d, s)$ and $g = g_v(d, s)$.*

PROOF (sketch). – If $R_0 = (d_0, g_0)$ and $R_v = (d, g)$, we have by (1.7*) and (1.3i): $d = d_0 + vs$, $g = g_0 + vd_0 + sv(s - 4 + v)/2$. Eliminating v, we get the following curve through the points $R_v = (d, g)$ in the d, g-plane;

$$g = \frac{d^2}{2s} + \frac{(s-4)d}{2} - \left(\frac{d_0^2}{2s} + \frac{(s-4)d_0}{2} - g_0\right).$$

Inserting for (d_0, g_0) in terms of s, we see that $-(d_0^2/2s + (s-4)d_0/2 - g_0)$ is equal to $1 - \binom{s+1}{3}/4 + (s-1)/2s$, i.e. g is exactly equal to the expression of $G_0(d, s)$ of (0.1) with $\mu = 1$. Finally if we compute and add to g the correction term ε given by $g + \varepsilon = g_v(d, s)$ between R_v and R_{v+1}, we get exactly $g + \varepsilon = G_0(d, s)$ for $1 \leqslant \mu \leqslant s$, and we are done.

(1.9) Consider the line l_v $(v \geqslant 0)$ with equation $g = vd + 1 - \binom{v+3}{3}$ in the d, g-plane, and let P_v be the point of intersection between the lines l_{v-1} and l_v. Let $C_{\mathrm{FW}}(d)$ be the piecewise linear curve between the points $P_v(v > 0)$, and $g = G_{\mathrm{FW}}(d)$ the equation of $C_{\mathrm{FW}}(d)$. The *FW-range is defined to be*

$$\{(d, g) \mid d \geqslant 1,\ 0 \leqslant g \leqslant G_{\mathrm{FW}}(d)\}.$$

Note that a smooth curve C of degree d and genus g satisfies $g = vd + 1 - \binom{v+3}{3}$ if and only if the Euler characteristic $\chi(\mathfrak{I}_C(v)) = 0$. We therefore denote the equation of the line l_v by $\chi(\mathfrak{I}(v)) = 0$.

For the FW-range we have the following result, stated here weaker than announced [32], but in a form we need. It is partially proved in [3], [4], [28], [26], [11] and [31]. Using for instance Fløystad's work [11], we get the result below in the range

$$nd + 1 - \binom{n+3}{3} < g \leqslant (n-1)d + 1 - \binom{n+2}{3}, \qquad d \geqslant (n^2 + 4n + 3)/4$$

for any integer $n \geqslant 2$. Since a published version of [32] has not yet appeared, we state it as

THEOREM/CONJECTURE 1.10. – *For any (d, g) in the FW-range, there exists a smooth maximal rank curve C of degree d and genus g such that $e(C) \leqslant s(C) - 2$ and $H^1(\mathfrak{N}_C) = 0$.*

2. – Good general space curves.

Superficially general curves (1.2) should correspond to "good, general" curves of the Hilbert scheme $H(d, g)_S$ of smooth connected space curves. Another "general curve philosophy" can be obtained from the concept of a general component in the Noether-Lefschetz locus, i.e. the locus of smooth surfaces of Picard group larger than $\mathbb{Z}$. In this section we consider such general curves (called infinitesimally general in the NL-locus), we discuss thoroughly relevant concepts and results and we prove that curves obtained by linkage from a nice curve of maximal rank are general in this sense (Theorem 2.9). Moreover we show that curves of the Conjectures 0.1 and 1.6 which are infinitesimally general in the NL-locus, almost always correspond to components of $H(d, g)_S$, which satisfy the claims of the Conjecture 0.2 (see (2.10)), making the concept of such a general curve an important tool to prove the main existing results of this paper. Finally we include a simple criterion for a curve to have a surjective Brill-Noether map (2.13).

(2.1) Let C be a curve (i.e. effective divisor) sitting on a smooth surface S of degree s in $\boldsymbol{P}^3$, and let $\mathcal{N}_S$ (resp. $\mathcal{N}_{C/S}$) be the normal sheaf of $S \subseteq \boldsymbol{P}^3$ (resp. $C \subseteq S$). Consider the map α_C: $H^0(\mathcal{N}_S) \to H^0(\mathcal{O}_S(s-4))^\vee$ defined as the composition of maps in the diagram

$$\begin{tikzcd}
H^0(\mathcal{N}_S) \cong H^0(\mathcal{O}_S(s)) \arrow[d, "\gamma_2"] & & H^0(\mathcal{O}_S(s-4))^\vee \\
H^0(\mathcal{N}_C) \arrow[r, "\gamma_1"] & H^0(\mathcal{O}_C(s)) \arrow[r] & H^1(\mathcal{N}_{C/S}) \cong H^0(\mathcal{O}_C(s-4))^\vee \arrow[u]
\end{tikzcd}$$

where the horizontal maps are induced by the exact sequence $0 \to \mathcal{N}_{C/S} \to \mathcal{N}_C \to \mathcal{O}_C(s) \to 0$ and $\mathcal{N}_{C/S} \cong \omega_C(4-s)$, ω_C the dualizing sheaf. We denote by α'_C: $H^0(\mathcal{N}_S) \to H^0(\mathcal{O}_C(s-4))^\vee$ the corresponding composition of maps into $H^0(\mathcal{O}_C(s-4))^\vee$.

DEFINITION 2.2. – *Let C be a curve sitting on a smooth surface S of degree s in $\boldsymbol{P}^3$. $C \subseteq S$ is said to be infinitesimally general in the Noether-Lefschetz locus* (*inf. general in the NL-locus*) *provided the map α_C is surjective and $H^1(\mathcal{I}_C(s-4)) = 0$.*

It is easy to see that if $C \subseteq S$ is inf. general in the NL-locus, then S belongs to a so-called general component of the NL-locus, i.e. it has the largest possible codimension $\binom{s-1}{3}$ in the Hilbert scheme Hilb(s) of smooth surfaces of degree s in $\boldsymbol{P}^3$, cf. [10] and [6] and see for instance [23], p. 324, for an interpretation of $H^1(\mathcal{I}_C(s-4)) = 0$.

REMARK 2.3. – The different constructions of α_C of [10] imply that α_D and $n\alpha_C$ coincide provided $D \in |nC + mH|$, n, m are integers. So, if $n = 1$ and if we suppose $H^1(\mathcal{I}_C(s-4)) = H^1(\mathcal{I}_C(s-4-m)) = 0$, we can by (1.3) say that the property of being inf. general in the NL-locus is invariant under "adding m hyperplane sections".

(2.4) Let $D(d, g; s)$ denote the Hilbert flag scheme (i.e. the incidence correspondence given a scheme structure inherited from deformation theory) parametrizing objects $(C \subseteq S)$ where C is a curve of degree d and arithmetic genus g and S is a *smooth* surface of degree s in $\boldsymbol{P}^3$. It is proved in [20], (Theorem 1.3.2), that *coker* α'_C (cf. (2.1)) *contains the obstructions of deforming the object* $(C \subseteq S)$ *in* $\boldsymbol{P}^3$. We denote this obstruction group by $A^2(C \subseteq S)$. If $A^1(C \subseteq S)$ denotes the tangent space of $D(d, g; s)$ at $(C \subseteq S)$, one gets easily from $A^1(C \subseteq S) = \ker(\gamma_1, \gamma_2)$ and $A^2(C \subseteq S) = \operatorname{coker} \alpha'_C$ an exact sequence

$$(*) \quad A^1(C \subseteq S) \xrightarrow{T} H^0(\mathcal{N}_C) \to H^1(\mathcal{I}_C(s)) \to A^2(C \subseteq S) \to H^1(\mathcal{N}_C) \xrightarrow{l} H^1(\mathcal{O}_C(s))$$

where $l = l_{C/S}$ is surjective. $T = T_{\mathrm{pr}_1}$, the tangent map of the natural projection $\mathrm{pr}_1 \colon D(d, g; s) \to H(d, g)$, is injective in case C sits on only one surface of degree s. For a later observation we remark that if S is not smooth, there still exist tangent and obstruction groups $A^i(C \subseteq S)$, $i = 1, 2$, for deforming an object $(C \subseteq S)$ (the definition of $A^2(C \subseteq S)$ is slightly changed, cf. [22], page 132) such that $(*)$ is exact.

REMARK 2.5. – i) From the description of $A^2(C \subseteq S)$ in (2.4) and Definition 2.2 we see that *the groups* $A^2(C \subseteq S)$ *and* $H^i(\mathcal{I}_C(s-4))$ *for* $i = 0$ *and* 1 *vanish if and only if* $C \subseteq S$ *is inf. general in the* NL-*locus.*

ii) Using for instance the exact sequence (2.4*), we get: *Any maximal rank curve C which sits on a smooth surface S of degree s in* $\boldsymbol{P}^3$ *and satisfies* $H^1(\mathcal{N}_C) = 0$, *must also satisfy* $A^2(C \subseteq S) = 0$.

(2.6) Now if $H^i(\mathcal{I}_C(s-4)) = 0$ for $i = 0, 1$ and $H^1(\mathcal{I}_C(s)) = 0$, we see by (2.5i) and (2.4*) that a curve C, lying on a smooth surface of degree s, is *inf. general in the NL-locus if and only if the map* $l \colon H^1(\mathcal{N}_C) \xrightarrow{l} H^1(\mathcal{O}_C(s))$ *of* (2.4) *is an isomorphism.* Note that the map l was considered by Hartshorne in his paper [17] where he looked for good properties of a "general" curve, and his Conjecture 4.4 on the range of the vanishing of $H^1(\mathcal{N}_C)$ was based on the principle that l should be an isomorphism for some curve C with $e < s$. Even though there are a few counterexamples to his conjecture [33], one might, also for these values of (d, g), hope that there exists curves with $e \geqslant s$ for which el is an isomorphism. We have checked (using Rem. 3.5) that this is true for the counterexample of [33] which is within the range of the Conjecture 0.2. Therefore, in view of Hartshorne's principle above, it is reasonable to believe that there should exist inf. general curves in the NL-locus for every (d, g) in the compliment of the FW-range, giving further evidence to the Conjecture 0.2.

To see the importance of this "inf. general" concept for the Conjecture 0.2 and for Theorem 2.9, we recall the following result (which one may deduce from the fact that $H^1(\mathcal{J}_C(s)) = 0$ implies the smoothness of the morphism pr_1 above, cf. [21], Lemma A10, 7 and 8 for details);

PROPOSITION 2.7. – *Let C be a curve on a smooth surface $S \subseteq \boldsymbol{P}^3$, $s = \deg S$, and suppose $A^2(C \subseteq S) = 0$ and $H^1(\mathcal{J}_C(s)) = 0$. Then C belongs to a generically smooth component V of the Hilbert scheme $H(d, g)$ of dimension*

$$\dim V = (4-s)d + \binom{s+3}{3} + g - 1 - h^0(\mathcal{J}_C(s)).$$

Moreover the generic curve C_1 of V sits on some smooth surface S_1 of degree s and satisfies $A^2(C_1 \subseteq S_1) = 0$ as well.

We also need the following lemma to prove the main Theorem 2.9 of this section.

LEMMA 2.8. – *Let C be a curve (an effective divisor) on a smooth surface $S \subseteq \boldsymbol{P}^3$ of degree $s \leqslant s(C) + 3$, and suppose $A^2(C \subseteq S) = 0$. Let T be a surface of degree t containing C, properly intersecting S, and let C' be the linked curve. If $H^1(\mathcal{J}_C(t)) = 0$, then $C' \subseteq S$ is infinitesimally general in the NL-locus.*

To prove the lemma, one may use the isomorphism $A^2(C \subseteq S \cap T) \cong A^2(C' \subseteq S \cap T)$ of [22], Cor. 2.14 ($A^2(C \subseteq S \cap T)$ is the obstruction group of deforming $C \subseteq S \cap T$), and some exact sequence ([22], 1.11) similar to (2.4*), involving $A^2(C \subseteq S \cap T)$ instead of $A^2(C \subseteq S)$. This was in fact our original proof of Lemma 2.8 and hence of Theorem 2.9, and it had the advantage of applying to any surface S provided the linkage was geometric (in the conclusion we had to replace the inf. general property by the vanishing of the three groups of (2.5i)). However, in case S is smooth, we have discovered the following simple proof which just uses (2.3):

PROOF. – Since $H^0(\mathcal{J}_C(s-4)) = 0$ by assumption, we get $\operatorname{coker} \alpha_C = 0$ because $\operatorname{coker} \alpha'_C = A^2(C \subseteq S) = 0$, cf. (2.4) and (2.1). Hence $\operatorname{coker} \alpha_{C'} = 0$ by (2.3) because $C' \equiv tH - C$, cf. [18], § 4. We conclude by (2.2) because $h^1(\mathcal{J}_{C'}(s-4)) = h^1(\mathcal{J}_C(t)) = 0$.

THEOREM 2.9. – *Let C be a curve of maximal rank sitting on a smooth surface $S \subseteq \boldsymbol{P}^3$ of degree $s \leqslant s(C) + 3$, satisfying $H^1(\mathcal{N}_C) = 0$ and $e(C) \leqslant s(C) - 2$.*

Let T be a surface of degree t which contains C and intersects S properly, and let C' be the linked curve. Then

i) *$C' \subseteq S$ is infinitesimally general in the Noether-Lefschetz locus,*

ii) *C' is of maximal corank,*

iii) *$H^0(\mathfrak{I}_{C'/S\cap T}(v+1)) = 0$ and $H^1(\mathfrak{I}_{C'}(v)) = 0$ for $v \leqslant s+t-4-s(C)$.*

In particular if $t \geqslant s(C)+4$, then $C' \subseteq \boldsymbol{P}^3$ belongs to a generically smooth component V of the Hilbert scheme $H(d', g')$, of dimension $\dim V = (4-s)d' + \binom{s+3}{3} + g' - 2$, *and the generic curve of V sits on some smooth surface of degree s and satisfies* i), ii) *and* iii) *as well.*

PROOF. – To see the important i) we just combine (2.5ii) and (2.8). Next since we have $\mathfrak{I}_{C'/S\cap T} \cong \omega_C(4-s-t)$ almost by the definition of C', we get $h^0(\mathfrak{I}_{C'/S\cap T}(v+1)) = h^1(\mathcal{O}_C(s+t-5-v)) = 0$ for $v \leqslant s+t-4-s(C)$, and the rest of ii) and iii) follows by similar linkage arguments because C has maximal rank. Finally we conclude by (2.7), observing that $A^2(C' \subseteq S) = 0$ by (2.5i).

We have now the concepts and basic results to treat the Conjecture 0.2 successfully. Indeed, putting (1.6V), (1.8) and (2.7) together, we get the following result ([23], Rem. 4.4), needed in the next section to prove (0.2) partially;

PROPOSITION 2.10. – *If* (1.6V) *is true for $v \geqslant 4$ and the special case: $v = 3$ and $g = g_3(d, s)$, in such a way that the existing curve and surface, $C \subseteq S$, is infinitesimally general in the* NL-*locus and satisfies $H^0(\mathfrak{I}_{C/S}(s)) = 0$, then the Conjecture* 0.2 *holds.*

PROOF. – The case $v \geqslant 4$ is clear from (2.5i) and (2.7). In the special case, we have $H^0(\mathfrak{I}_{C/S}(s)) = 0$ by assumption and $H^1(\mathcal{O}_C(s)) = 0$ by (1.6V). But now $g = g_3(d, s)$ leads to $\chi(\mathfrak{I}_{C/S}(s)) = 0$ by (1.7*), hence to $H^1(\mathfrak{I}_C(s)) = 0$. So (2.7) applies also in the special case.

REMARK 2.11. – The most interesting range of the Conjectures 0.1 and 0.2 seems to be

$$(*) \qquad G_0(d, s+1) < g \leqslant G_0(d, s), \qquad d \geqslant \binom{s+2}{2}$$

since one proves easily that the union of the ranges given by $(*)$ for $s \geqslant 1$ is precisely the compliment of the FW-range. For any (d, g) in this compliment FW^C, there is a unique number s so that $(*)$ holds. Now we think a good generic curve, hence a nice component V of $H(d, g)_S$ in the FW^C-range, should have the following properties:

i) *the generic curve* C *lies on a smooth surface* S *of the degree* s, *given by* $(*)$, *and satisfies* $H^0(\mathfrak{I}_{C/S}(s)) = 0$;

ii) $C \subseteq S$ *is infinitesimally general in the Noether-Lefschetz locus*;

iii) C *has maximal corank*;

iv) C *has maximal rank in* S, *i.e. either* $H^0(\mathfrak{I}_{C/S}(v)) = 0$ *or* $H^1(\mathfrak{I}_C(v)) = 0$ *for any integer* v.

Especially when such generic curves C are superficially general (e.g. $g > G(d, s+1)$), they seem particularly interesting. To indicate why, observe that i) and iii) imply $H^1(\mathfrak{I}_C(v)) = 0$ for $v \leqslant s$, because $(*)$ leads to $\chi(\mathfrak{I}_{C/S}(s)) \geqslant 0$. Hence (2.7) applies to nice components V as above, i.e. V is generically smooth and dim V is known. By the discussion of the introduction of [23], one may prove that *the minimum value of the dimension of the components of* $H(d, g)_S$ *is equal to this number* $\dim V$ *provided* $d > s^2$ *and* $g > G(d, s+1)$. Now if the Conjecture 0.2 is true in the range $(*)$, then we have a good candidate of such a nice generic curve. Finally observe that the condition $H^0(\mathfrak{I}_{C/S}(s)) = 0$ of i), which is not automatically satisfied in the case $d \leqslant s^2$, is quite natural under the assumption ii). Indeed using the isomorphism $l_{C/S}$ of (2.4) and for instance an easy linkage argument, one may prove that $H^0(\mathfrak{I}_{C/S}(s)) \neq 0$ implies $H^1(\mathcal{O}_C(s)) = 0$, a vanishing which is somewhat unexpected in the FW^C-range, cf. [33].

(2.12) Finally we recall another "general concept" of a space curve. Indeed one knows that the injectivity of the Brill-Noether map

$$\mu_0 : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C)$$

gives general moduli of a component V of $H(d, g)$, i.e. a generically smooth natural map $\mu|_V : V \hookrightarrow H(d, g) \to M_g$, into the moduli space of smooth curves of genus g. As explained by Sernesi, the surjectivity of μ_0, together with the vanishing of $H^1(\mathfrak{I}_C(1))$, is also of some importance because it leads to "right" fiberdimension of $\mu|_V$, hence to the expected dimension of the image $\mu(V)$ which is $4d - 15 + \varepsilon$ in case $\dim V = 4d + \varepsilon$, cf. [28] or [26] which treats the case $\varepsilon = 0$ (the expected number of moduli). The components we find in the next section have almost always $\varepsilon > 0$, but still "right" fiberdimension, due to

PROPOSITION 2.13. – *Let* $C \subseteq \boldsymbol{P}^3$ *be a curve satisfying* $H^0(\mathfrak{I}_C(3)) = 0$ *and* $H^1(\mathfrak{I}_C(v)) = 0$ *for* $v \leqslant 2$. *Then the Brill-Noether map* μ_0 *above is surjective.*

PROOF. – Let $M = \oplus_v H^1(\mathfrak{I}_C(v))$ and $A = \oplus_v H^0(\mathcal{O}_C(v))$ and let $L_0 \to M$ be a minimal presentation of graded $R = k[X_0, \ldots, X_4]$-modules. Since we have an exact sequence $R \to A \to M \to 0$, we can find a minimal presentation of the form $L_0 \oplus R \to A$, giving rise to a minimal graded resolution

$$(*) \qquad 0 \to L_2 \to L_1 \to L_0 \oplus R \to A \to 0$$

cf. [25], pp. 46-47. Note that $L_0 \oplus R \cong \oplus R(-e_i) \oplus R$ where $e_i > 2$ by assumption. If a relation involves only the minimal generator of R, its degree must be $\geqslant s(C) > 3$. Hence, since the resolution is minimal, $L_2 \cong \oplus R(-n_i)$ with $n_i > 4$. Now sheafifying $(*)$ and splitting into two short exact sequences, we get, by applying $\mathrm{Hom}_{O_P}(-, \omega_P)$, an exact sequence

$$0 \to \widetilde{L}_0^{\vee}(-4) \oplus \mathcal{O}_P(-4) \to \widetilde{L}_1^{\vee}(-4) \to \widetilde{L}_2^{\vee}(-4) \to \omega_C \to 0$$

cf. [11], Cor. 5.8. Taking cohomology, we see it suffices to prove the surjectivity of

$$H^0(\mathcal{O}_P(1)) \otimes H^0(\widetilde{L}_2^{\vee}(-5)) \to H^0(\widetilde{L}_2^{\vee}(-4))$$

which is trivially true because $n_i > 4$.

3. – Existence of good curves near the maximum genus.

Thanks to Theorem 2.9 and 1.10 we are now able to prove the Conjecture 1.6 in such a way that the existing curves and surfaces satisfy the assumptions of (2.10), in a certain range near the maximum genus. Hence we have a proof (cf. Rem. 3.5) for the conjectures of the introduction in this range. Note that the following proposition is true for $s = 4$ without the claim $g \geqslant (2s+v-5)d - s(s+v-1)(2s+v-5)/2$ (and with smooth curves also for $v = 0$) by [23].

PROPOSITION 3.1. – *Let $s \geqslant 5$, $v \geqslant 1$, (resp. $v = 0$), d and g be integers such that*

$$(s-4+v)d+1-\binom{s+v-1}{3}+\binom{v-1}{3} < g \leqslant (s-3+v)d+1-\binom{s+v}{3}+\binom{v}{3}.$$

If furthermore $g \geqslant (2s+v-5)d - s(s+v-1)(2s+v-5)/2$ and (1.10) holds, then there exists a smooth connected curve C, (resp. a reduced curve), of degree d and genus g on a smooth surface S of degree s in $\boldsymbol{P}^3$. Moreover $C \subseteq S$ is infinitesimally general in the NL-*locus and satisfies $H^1(\mathfrak{I}_C(t)) = 0$ for $t \leqslant s-4+v$, $H^1(\mathcal{O}_C(s-3+v)) = 0$ and $H^0(\mathfrak{I}_{C/S}(s-3+v)) = 0$.*

PROOF OF (3.1). – Let A_n be the subrange of the FW-range given by

$$nd+1-\binom{n+3}{3}<g\leqslant(n-1)d+1-\binom{n+2}{3}$$

i.e. by $\chi(\mathfrak{I}(n-1))\leqslant 0$ and $\chi(\mathfrak{I}(n))>0$, cf. (1.9). Then for any $(d,g)\in A_n$, there exists by (1.10) a smooth maximal rank curve C such that $e(C)\leqslant s(C)-2$. It follows that $\chi(\mathfrak{I}_C(s(C)))>0$ and $\chi(\mathfrak{I}_C(s(C)-1))\leqslant 0$, and we get $s(C)=n$. Since by Castelnuovo-Mumford, $\mathfrak{I}_C(n+1)$ is globally generated, there exists a smooth surface S of degree $s=n+1$ containing C. If T is another surface of degree $n+v$ (v as in Prop. 3.1) containing C and intersecting S properly, then the linked curve C' is smooth provided $v>0$, and reduced otherwise [27]. Moreover we *claim* that C' belongs to the range of Proposition 3.1. Indeed since $\mathfrak{I}_{C/S\cap T}\cong\omega_{C'}(3-v-2n)$ and $n=s-1$, we get (for $v>0$);

$$\chi(\mathfrak{I}_C(n))=h^0(\mathfrak{I}_{C/S\cap T}(n))-h^1(\mathfrak{I}_C(n))+h^1(\mathcal{O}_C(n))=$$

$$=h^1(\mathcal{O}_{C'}(n+v-3))-h^1(\mathfrak{I}_{C'}(n+v-3))+h^0(\mathfrak{I}_{C'/S\cap T}(n+v-3))=$$

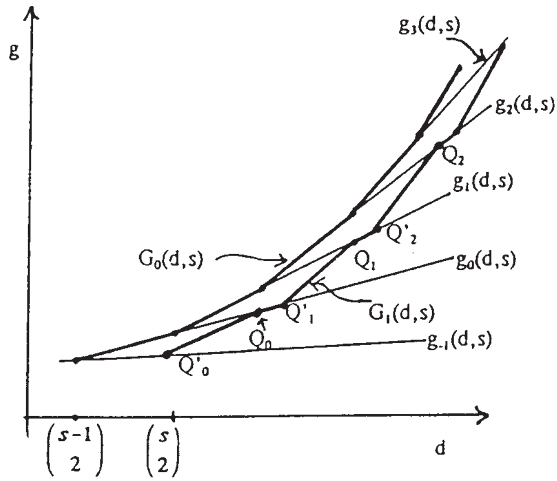
$$=\chi(\mathfrak{I}_{C'/S}(s+v-4))$$

$\big(\chi(\mathfrak{I}_C(n))=\chi(\mathfrak{I}_{C'/S}(s-4))$ for $v=0$ as well$\big)$ and similarly; $\chi(\mathfrak{I}_C(n-1))=\chi(\mathfrak{I}_{C'/S}(s+v-3))$. The genus g' of the linked curve is given by

$$g'=g+\frac{(d'-d)(n+1+n+v-4)}{2}=g+\frac{(2d'-s(s+v-1))(2s+v-5)}{2},$$

i.e. $g'\geqslant(2s+v-5)d'-s(s+v-1)(2s+v-5)/2$ because $g\geqslant 0$. Consulting (1.7*), the claim follows easily, and we conclude by Theorem 2.9.

(3.2) Let $Q_v=(d_v,g_v)$, resp. $Q'_v=(d'_v,g'_v)$, be the point where the line $g=(2s+v-5)d-s(s+v-1)(2s+v-5)/2$ intersect the line $g=g_v(d,s):=(s-3+v)d+1-\binom{s+v}{3}+\binom{v}{3}$, resp. the line $g=g_{v-1}(d,s)$, and let $g=G_1(d,s)$ be the equation of the piecewise linear curve between the points Q'_v, Q_v, Q'_{v+1}, $v\geqslant 0$. Recall that $G_0(d,s)$ is given by (0.1) or (1.8) and we get precisely the piecewise linear curves drawn in the diagram.



Now (3.1), (the case $s=4$ is taken care of in [23]), (3.2), (2.10) and its proof, and (2.13) give immediately the main theorem of this paper;

THEOREM 3.3. – *Let $s \geqslant 4$, $d \geqslant \binom{s}{2}$, and $g \geqslant (s-3)d+1-\binom{s}{3}$ be integers such that*

$$G_1(d,\,s) < g \leqslant G_0(d,\,s)$$

and suppose that (1.10) *holds, i.e. for any (d',g') in the FW-range, we suppose there exists a smooth maximal rank curve D of degree d' and genus g' such that $e(D) \leqslant s(D)-2$ and $H^1(\mathcal{N}_D)=0$. Then there exists a smooth connected curve C of degree d and genus g on a smooth surface S of degree s in $\boldsymbol{P}^3$. The curve C has maximal corank and $C \subseteq S$ is infinitesimally general in the Noether-Lefschetz locus. If furthermore*

$$g \geqslant sd - \binom{s+3}{3} + 2$$

then we can suppose C is the generic curve of a generically smooth component V in the Hilbert scheme $H(d,\,g)_S$ of smooth connected space curves, of dimension

$$\dim V = (4-s)d + \binom{s+3}{3} + g - 2$$

whose image in the moduli scheme M_g of smooth curves has dimension $\dim V - 15$.

COROLLARY 3.4. – *Let $s \geqslant 4$, $d \geqslant \binom{s}{2}$, and $g \geqslant (s-3)d+1-\binom{s}{3}$ be inte-*

gers, and suppose (1.10) *holds. Then the Conjectures* 0.1 *and* 0.2 *are true in the range* $G_1(d,s) < g \leqslant G_0(d,s)$. *In particular if*

$$G_2(d,s) = \frac{d^2}{2s} + \frac{(s-4)d}{2} - \frac{7s^3 - 8s^2 - 20s - 60}{72}$$

then both conjectures hold in the range

$$G_2(d,s) \leqslant g \leqslant G_0(d,s).$$

PROOF OF (3.4). – By (3.3) it suffices to prove $G_1(d,s) < G_2(d,s)$. Let $Q_v = (d_v, g_v)$ be the point of intersection between the lines $g = (s-3+v)d + 1 - \binom{s+v}{3} + \binom{v}{3}$ and $g = (2s+v-5)d - s(s+v-1)(2s+v-5)/2$. One checks that $d_v = sv + (5s^2 - 8s - 3)/6$. This gives d_v, hence also g_v, in terms of s and v only. Eliminating v, we get the following connection between d_v and g_v;

$$g_v = G_2(d_v, s) - \frac{1}{8s}$$

with $G_2(d,s)$ as in (3.4). Since the derivative of $G_2(d,s)$ with respect to d is $\leqslant 2s+v-5$ for $d \leqslant d_v$, we get $G_1(d,s) < G_2(d,s)$ and we are done.

REMARK 3.5. – Theorem 1.10 is not yet published, and we have to shrink the ranges of (3.1), (3.3) and (3.4) if we should use published results [3], [4], [28], [26], [11] and [31] only. For instance recall that Fløystad [11] proves the existence of smooth curves of maximal rank with $H^1(\mathcal{I}_C(-1)) = 0$ in A_n (cf. the proof of (3.1)) provided $d \geqslant (s^2 + 2s)/4$. It follows that if we in Proposition 3.1 replace $g \geqslant (2s+v-5)d - s(s+v-1)(2s+v-5)/2$ by

$$d \leqslant sv + \frac{3s^2 - 6s}{4}$$

any conclusion of (3.1) holds by [11] and the proof of (3.1) in this restricted range. Similarly if we replace $G_1(d,s)$ in (3.3) and (3.4) by $G_1'(d,s)$, defined by replacing $g = (2s+v-5)d - s(s+v-1)(2s+v-5)/2$ by $d = sv + (3s^2 - 6s)/4$ in (3.2), then (3.3), resp. (3.4), holds without assuming (1.10) in the range $G_1'(d,s) < g \leqslant G_0(d,s)$, resp. $G_2'(d,s) < g \leqslant G_0(d,s)$ where

$$G_2'(d,s) = \frac{d^2}{2s} + \frac{(s-4)d}{2} + 1 - \frac{7s^3 - 12s^2 - 4s}{96}.$$

(3.6) Observe that we have proved Prop. 3.1 by taking suitable curves C from the FW-range, contained in some complete intersection of bidegree $(s(C)+1, s(C)+v)$, and then performed the linkage to a curve C_v'. We could, however, prove the result by the linkage above with $v=0$, and after that "added v hyperplane sections", i.e. taken D from $|C'+vH|$, $v>0$, where $C'=C_0'$. In particular we can suppose the system $|C'+H|$ contains a smooth connected member. Following this idea further, it is natural to ask if members D of the linear system

$$|\alpha C'+\beta H|, \qquad \textit{for } \alpha \geqslant 2 \textit{ and } \beta \geqslant 0$$

give new curves proving both conjectures of the introduction more completely. We shall see that this is almost the case, at least if β is large enough. Since it is usually easy to see when such systems contain a smooth member (they do for instance in the case $\alpha \leqslant \beta$), we concentrate on proving that the curves have the other "good" properties of Section 2, i.e. we concentrate on verifying Conjecture 0.2. We succeed only partially since we have not been able to prove that the generic curves are of maximal corank. And the values of $(d, g) = (d(D), g(D))$ we get by varying α and β are still far from covering the conjectured range. Loosely speaking, we get a netting (grid) in the d, g-plane, mainly in the range $3d^2/8(s+1)+(s-4)d/2+1<g\leqslant G_1(d, s)$, whose intersections correspond to the values of (d, g) where there exist curves for which both conjectures are true (except for maximal corank). More precisely if $D\in |\alpha C'+\beta H|$, we get $d=\alpha d'+\beta s$. Therefore, using the adjunction formula, $2g-2=(\alpha C'+\beta H).(\alpha C'+\beta H+K)$ where $K=(s-4)H$, we deduce

$$(*)\quad \begin{cases} d=\alpha d'+\beta s \quad \textit{and} \\ g=1+\alpha^2(g'-1)+\alpha d'\left(\beta-\dfrac{(\alpha-1)(s-4)}{2}\right)+\dfrac{\beta s(\beta+s-4)}{2}. \end{cases}$$

Our main contribution is (3.8) below which is a consequence of

PROPOSITION 3.7. – *Let C be a curve of degree d and genus g, contained in a smooth surface $S\subseteq \boldsymbol{P}^3$ of degree s, and suppose $g>(s-4)d-\binom{s-1}{3}$ and $C\not\equiv tH$ (linear equivalence), where H a hyperplane section. Moreover suppose $C\subseteq S$ is infinitesimally general in the NL-locus and satisfies*

$$H^1(\mathfrak{J}_C(t))=0 \qquad \textit{for } t\leqslant s-4.$$

Let v_0 be an integer $\geqslant 2\binom{s-1}{3}/d$, let $\alpha\geqslant 1$, $\beta\geqslant v_0$ be integers and let D be a member of $|\alpha C+\beta H|$. Then $D\subseteq S$ is infinitesimally general in the NL-locus and satisfies

$$H^0(\mathfrak{J}_{D/S}(t))=0 \quad \textit{for } t\leqslant s-4+\beta \quad \textit{and} \quad H^1(\mathfrak{J}_D(t))=0 \ \textit{for } t\leqslant s-4+\beta-v_0.$$

COROLLARY 3.8. – *Let $s \geqslant 4$ be an integer and let (d, g) be given by* (3.6*) *for some $\alpha \geqslant 2$ and $\beta \geqslant 2(s+3)/3$, $\alpha \leqslant \beta$, and some d' and g' in the range*

$$(s-4)d' + 1 - \binom{s-1}{3} \leqslant g' \leqslant (s-3)d' + 1 - \binom{s}{3},$$

$$g' \geqslant (2s-5)d' - \frac{s(2s-5)(s-1)}{2}.$$

If (1.10) *holds, then the Hilbert scheme $H(d, g)_S$ contains a non-empty generically smooth component of dimension $(4-s)d + g + \binom{s+3}{3} - 2$ whose generic curve D is infinitesimally general in the NL-locus and satisfies $H^0(\mathfrak{I}_{D/S}(s)) = 0$ and $H^1(\mathfrak{I}_D(t)) = 0$ for $t \leqslant s$.*

LEMMA 3.9. – *Let C, v_0 and α be as in Prop.* 3.7. *Then*

$$H^0(\mathfrak{I}_{\alpha C/S}(s-4)) = 0 \quad \text{and} \quad H^1(\mathfrak{I}_{\alpha C/S}(t)) = 0 \quad \text{for } t \leqslant s-4-v_0.$$

PROOF OF (3.9). – Consider the exact sequence

$$0 \to \mathcal{O}_S(-\alpha C + tH) \to \mathcal{O}_S(-C + tH) \to \mathcal{O}_{(\alpha-1)C}(-C + tH) \to 0.$$

Since $O_S(-\alpha C) = \mathfrak{I}_{\alpha C/S}$, we get the vanishing of $H^0(\mathfrak{I}_{\alpha C/S}(s-4))$ from $H^0(\mathfrak{I}_{C/S}(s-4)) = 0$, cf. (2.5i). Next we *claim* that $H^0(\mathcal{O}_{(\alpha-1)C}(-C+tH)) = 0$ for $t \leqslant s-4-v_0$. To see this it suffices to prove

$$(-C+tH).\, C \leqslant 0, \quad \textit{i.e.} \quad td \leqslant C^2$$

because the case $td = C^2$ and $H^0(\mathcal{O}_{(\alpha-1)C}(-C+tH)) \neq 0$ leads to $C \equiv tH$. Since C^2 is equal to $C^2 + C.\,K - (s-4)C.\,H = 2g - 2 - (s-4)d$, we must prove $(s-4-v_0)d \leqslant 2g - 2 - (s-4)d$. By assumption $g - 1 \geqslant (s-4)d - \binom{s-1}{3}$, and it suffices to prove

$$(s-4-v_0)d \leqslant 2(s-4)d - 2\binom{s-1}{3} - (s-4)d$$

which is equivalent to the assumption $v_0 \geqslant 2\binom{s-1}{3}/d$, and the claim follows. Now using the claim, we get for any $t \leqslant s-4-v_0$ an exact sequence

$$0 \to H^1(\mathcal{O}_S(-\alpha C + tH)) \to H^1(\mathcal{O}_S(-C+tH)) \to H^1(\mathcal{O}_{(\alpha-1)C}(-C+tH)).$$

Since $H^1(\mathfrak{I}_{C/S}(t)) \cong H^1(\mathfrak{I}_C(t)) = 0$ for $t \leqslant s-4-v_0$ by assumption, we conclude easily.

PROOF OF (3.7). – First note that the vanishing of $H^i(\mathfrak{I}_{D/S}(t))$ for $i = 0, 1$ follows easily from the lemma because $\mathfrak{I}_{D/S} = \mathcal{O}_S(-\alpha C - \beta H) = \mathfrak{I}_{\alpha C/S}(-\beta)$, hence

$$H^1(\mathfrak{I}_D(t)) \cong H^1(\mathfrak{I}_{D/S}(t)) \cong H^1(\mathfrak{I}_{\alpha C/S}(t-\beta)) = 0 \qquad \text{for } t - \beta \leqslant s - 4 - v_0,$$

and correspondingly for $H^0(\mathfrak{I}_{D/S}(t))$. By (2.3) we have coker $\alpha_C =$ coker $\alpha_D = 0$, i.e. D is inf. general in the NL-locus because $H^1(\mathfrak{I}_D(s-4)) = 0$ by the assumption $\beta \geqslant v_0$. This completes the proof.

PROOF OF (3.8). – Observe that for any (d', g') in the range of the corollary there exists a curve C', inf. general in the NL-locus for some smooth surface S of degree s, satisfying

$$d' \geqslant \binom{s-1}{2}, \qquad H^1(\mathfrak{I}_{C'}(t)) = 0 \qquad \text{for } t \leqslant s-4, \quad H^0(\mathfrak{I}_{C'/S}(s-4)) = 0$$

by (3.1). Now let v_0 be the smallest integer such that $v_0 \geqslant 2(s-3)/3$. We will apply Prop. 3.7 to this curve C' with v_0 as above and $\beta \geqslant 4 + v_0$. Since C' is not a multiple of a hyperplane section, it suffices to check $v_0 d' \geqslant 2\binom{s-1}{3}$. This follows easily from $d' \geqslant \binom{s-1}{2}$. Then, by Proposition 3.7, a general member D of $|\alpha C' + \beta H|$ is inf. general in the NL-locus and satisfies

$$H^0(\mathfrak{I}_{D/S}(s)) = 0 \qquad \text{and} \qquad H^1(\mathfrak{I}_D(t)) = 0 \qquad \text{for } t \leqslant s.$$

Note that D is smooth because $\alpha \leqslant \beta$. The conclusion follows now from (2.7).

REMARK 3.10. – In Corollary 3.8 β is chosen so large that the proof works for *any* degree d' and genus g' in the range of the corollary. However, by the proof of (3.4), one knows

$$\binom{s-1}{2} \leqslant d' \leqslant \frac{5s^2 - 8s - 3}{6}.$$

So having a fixed d' in this range, we can usually choose β smaller. For instance if d' is close to the upper bound, $\beta \geqslant 4 + 2(s-3)/5$ suffices.

We think it should be possible to improve (3.8) by proving the vanishing of $H^1(\mathfrak{I}_D(t))$ (and the smoothness of D) under weaker conditions. Even if we succeed, there will, however, be lots of cases for which Conjecture 0.2 can not be proved by this method. To treat these, we think one should have to introduce other smooth surfaces. For instance if one systematically can prove the existence of maximal rank curves of negative arithmetic genus sitting on smooth surfaces as in (2.9) such that the linked curves of (2.9) are smooth, we can easily prove (0.1) and (0.2) in a larger range. Or one may use smoothing techniques as in [31], § 8, to construct such families, but it seems to us that we have to improve upon the smoothing lemmas to see if we can get the smoothed curves to sit on smoothed surfaces.

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