Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **2-B** (1999), n.1, p. 121–126.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1999_8_2B_1_121_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1999.

Bollettino U. M. I.
(8) 2-B (1999), 121-126

Solvable Lie Algebras and the Embedding of *CR* Manifolds.

C. Denson Hill - Mauro Nacinovich

Sunto. – *In questo lavoro si dà un criterio sufficiente per l'immersione di una varietà CR astratta di codimensione arbitraria in una di codimensione CR più bassa. La condizione trovata è necessaria per l'immersione in una varietà complessa (codimensione CR uguale a zero). Essa è formulata in termini dell'esistenza di una sottoalgebra di Lie di campi di vettori complessi trasversale alla distribuzione di Cauchy-Riemann.*

1. – Introduction.

In this paper we give (Theorem 1) a sufficient condition for the local *CR* embedding of a smooth abstract *CR* manifold of type (n, k) into another smooth abstract *CR* manifold $\widetilde{M}$ of type $(n+l, k-l)$. We get an actual local embedding into $\mathbb{C}^{n+k}$ if $l = k$; and in this case the sufficient condition of our theorem is also necessary for the local embedding. The condition in the theorem concerns the existence of an l-dimensional solvable Lie subalgebra, transversal to the *CR* structure, of $\mathcal{N}/\mathcal{T}^{0,1}M$, where $\mathcal{N}$ is the normalizer of the Lie algebra $\mathcal{T}^{0,1}M$ of complex vector fields of bidegree $(0, 1)$.

It has been known for a long time (see [1]) that real analytic *CR* structures of any type (n, k) are locally embeddable in $\mathbb{C}^{n+k}$, and that smooth *CR* structures may not be (see [7], [6]); moreover, in the last decades much work has been done on the case where M is of hypersurface type $(n, 1)$, see also [5].

We give two examples in § 4. In the first example we get a complete embedding of a structure which is not real analytic. In the second example we get a *CR* embedding of a *CR* manifold M into another *CR* manifold $\widetilde{M}$, of higher *CR* dimension, in a situation where $\widetilde{M}$ is not completely embeddable.

2. – Preliminaries.

Since the notion of an abstract smooth *CR* manifold of type (n, k) is standard (n is the $CR-\dim_{\mathbb{C}}$ and k is the $CR-\operatorname{codim}_{\mathbb{R}}$), we simply refer the reader to [2], [3], [4] for a detailed discussion of all definitions and any notion that is not clear from the context. An abstract *CR* structure of type (n, k) is defined, on a manifold M of real dimension $2n+k$, by prescribing a rank n distribution $\mathcal{T}^{0,1}M$ of smooth vector fields satisfying

$$\mathcal{T}^{1,0}_p \cap \mathcal{T}^{0,1}_p = \{0\}, \qquad \text{for every } p \in M,$$

where $\mathcal{T}^{1,0}M = \overline{\mathcal{T}^{0,1}M}$,and

$$[\mathcal{T}^{0,1}M, \mathcal{T}^{0,1}M] \subset \mathcal{T}^{0,1}M .$$

If $\widetilde{M}$ is another abstract smooth *CR* manifold of type $(\tilde{n}, \tilde{k})$, we say that a map $\psi: M \to \widetilde{M}$ is a *CR*-embedding if it is a smooth embedding and moreover $\psi_* \mathcal{T}^{0,1}_p \subset \mathcal{T}^{0,1}_{\psi(p)}\widetilde{M}$, for $p \in M$. If $n + k = \tilde{n} + \tilde{k}$, the embedding is called *generic* (in general we have $n + k \leq \tilde{n} + \tilde{k}$).

3. – An embedding theorem.

Consider a smooth (abstract) *CR* manifold of type (n, k); we fix some point $p \in M$. As we shall be interested in the local analysis at p, we will work with *germs* of smooth objects at p; but to simplify notation, we will often avoid to indicate p explicitly.

Define

$$\mathcal{N} = \mathcal{N}_{(p)} = \{X \in \mathcal{T}_{(p)}M \,|\, [X, \mathcal{T}^{0,1}_{(p)}M] \subset \mathcal{T}^{0,1}_{(p)}M\},$$

where $\mathcal{T}_{(p)}M$ denotes the Lie algebra (over $\mathbb{C}$) of germs of smooth complex vector fields on M at p, and $\mathcal{T}^{0,1}_{(p)}$ the Lie subalgebra of those of type $(0, 1)$. $\mathcal{N}$ is the *normalizer* of $\mathcal{T}^{0,1}_{(p)}M$ in $\mathcal{T}_{(p)}M$; in particular, $\mathcal{T}^{0,1}_{(p)}M$ is an ideal in $\mathcal{N}$. Set

$$\widehat{\mathcal{N}} = \mathcal{N}/\mathcal{T}^{0,1}_{(p)}M ,$$

with $\pi: \mathcal{N} \to \widehat{\mathcal{N}}$ denoting projection into the quotient. Then $\widehat{\mathcal{N}}$ is a Lie algebra over $\mathbb{C}$. We set $\mathcal{N}_p = \{X_p \,|\, X \in \mathcal{N}_{(p)}\}$ and $\widehat{\mathcal{N}}_p = \mathcal{N}_p/\mathcal{T}^{0,1}_p M$. We have a natural evaluation map:

$$v_p: \widehat{\mathcal{N}} \ni \widehat{X} \to \widehat{X}_p \in \widehat{\mathcal{N}}_p .$$

We come now to the theorem (which is local at $p \in M$).

THEOREM 1. – *Assume that there exists a solvable Lie subalgebra* $\mathfrak{a}$ *of* $\widehat{\mathcal{N}}$, *which is transversal to the CR structure of M, in the sense that*

$$v_p(\mathfrak{a}) \cap v_p(\pi(\mathcal{T}^{1,0}_{(p)} \cap \mathcal{N})) = \{0\}. \tag{3.1}$$

Let $l = \dim_{\mathbb{C}} \mathfrak{a} = \dim_{\mathbb{C}} v_p(\mathfrak{a})$. *Then there exists a smooth (abstract) CR manifold* $\widetilde{M}$ *of type* $(n + l, k - l)$, *and a (generic) CR embedding of a neighborhood of* $p \in M$ *into* $\widetilde{M}$.

If $l = k$ *then there is an embedding into* $\mathbb{C}^{n+k}$.

PROOF. – We first show how to construct a smooth *CR* manifold $\widetilde{M}_1$ of type

$(n+1, k-1)$, and a local embedding of M into it. Since $\mathfrak{a}$ is a solvable Lie algebra over $\mathbb{C}$, by Lie's theorem we can find a basis $\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_l$ for $\mathfrak{a}$ such that, for each j, the $\mathbb{C}$-linear span of $\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_j$ is an ideal in $\mathfrak{a}$. In this basis the structure constants $c^h_{i,j} \in \mathbb{C}$ for the Lie algebra $\mathfrak{a}$ are determined by:

$$[X_i, X_j] = c^h_{i,j} X_h \,, \tag{3.2}$$

with $c^h_{i,j} = 0$ if $h > \min\{i, j\}$. Let us choose representatives $X_1, X_2, \ldots, X_l$ in $\mathcal{N}$. After possible multiplication of X_l by a nonzero complex number, we may assume that

$$\Re\,(X_l)_p \notin \mathcal{C}^{0,1}_p M \oplus \mathcal{C}^{1,0}_p M \,.$$

Since we only consider a local situation, we can substitute, for simplicity, M in the following, by a suitably small open neighborhood of p in M, in such a way that X_l satisfies all the above conditions at all points of this neighborhood. In $M \times \mathbb{R}_s$ we introduce a new complex field

$$\overline{Z}_{n+1} = X_l + \sqrt{-1}\,\frac{\partial}{\partial s} \,,$$

and look for complex vector fields on $M \times \mathbb{R}_s$ of the form

$$\widetilde{X}_j = X_j + \lambda^i_j(s)\, X_i \,, \tag{3.3}$$

with $\lambda^i_j(0) = 0$, such that

$$[\overline{Z}_{n+1}, \widetilde{X}_j] \equiv 0 \,, \quad (\operatorname{mod} \mathcal{C}^{0,1} M) \quad \text{for } j = 1, 2, \ldots, l-1 \,. \tag{3.4}$$

Actually we shall use a lower triangular matrix $[\lambda^i_j]_{1 \leq i, j \leq l-1}$. By using (3.2) we see that (3.4) is equivalent to the system of ordinary differential equations:

$$\sqrt{-1}\,\dot{\lambda}^h_j + c^h_{l,i}\lambda^i_j + c^h_{l,j} = 0 \quad \text{for } 1 \leq j \leq l-1 \text{ and } 1 \leq h \leq j \,. \tag{3.5}$$

These equations (3.5) have a unique explicit solution $\lambda^h_j(s)$ having zero initial conditions. On the manifold $\widetilde{M}_1 = M \times \mathbb{R}_s$ the new CR structure is defined by the complex vector distribution $\mathcal{C}^{0,1}\widetilde{M}_1$ which is generated by the vectors of $\mathcal{C}^{0,1}M$ (extended to be constant in the s-direction), and $\overline{Z}_{n+1}$. It is clear that $\mathcal{C}^{0,1}\widetilde{M}_1$ is formally integrable; hence we do have a smooth abstract CR structure of type $(n+1, k-1)$ defined on $\widetilde{M}_1$.

Next we verify that this process can be continued by induction: to this aim we consider $\mathcal{N}_1$, $\widehat{\mathcal{N}}_1$ constructed as above from $\mathcal{C}^{0,1}\widetilde{M}_1$. First we observe that the $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_{l-1}$ already constructed belong to $\mathcal{N}_1$. Next we show that their images in $\widehat{\mathcal{N}}_1$ generate a solvable Lie algebra $\mathfrak{a}_1$ over $\mathbb{C}$ satisfying on $\widetilde{M}_1$ the condition analogous to (3.1); allowing the induction to proceed.

For each fixed value of s, the $\widehat{X}_1, \ldots, \widehat{X}_{l-1}$ and $\pi(\widetilde{X}_1(s)), \ldots, \pi(\widetilde{X}_{l-1}(s))$ are just different bases for the same Lie subalgebra of $\widehat{\mathcal{N}}$. With respect to our new basis the structure constants $k_{i,j}^h(s)$ are determined by

$$[\pi(\tilde{X}_i(s)),\, \pi(\tilde{X}_j(s))] = k_{i,j}^h(s)\, \pi(\tilde{X}_h(s))\,. \tag{3.6}$$

By the Jacobi identity we deduce that

$$[\overline{Z}_{n+1}, [\tilde{X}_i,\, \tilde{X}_j]] \equiv 0 \quad (\mathrm{mod}\, \mathcal{T}^{0,1}M)\,.$$

This gives

$$\sqrt{-1}\, \dot{k}_{i,j}^h(s)\, \tilde{X}_h \equiv 0 \quad (\mathrm{mod}\, \mathcal{T}^{0,1}M)\,, \tag{3.7}$$

and it follows that $k_{i,j}^h(s) = k_{i,j}^h(0) = c_{i,j}^h$. Moreover, one easily verifies that

$$v_{p\times 0}(\mathfrak{a}_1) \cap v_{p\times 0}(\pi_1(\mathcal{T}^{1,0}_{(p\times 0)}\tilde{M}_1 \cap \mathcal{N}_1)) = \{0\}\,.$$

This procedure terminates after l steps, and leads to the local CR embedding $M \hookrightarrow M \times \{0\} \subset \tilde{M} = M \times \mathbb{R}^l$, where $\tilde{M}$ is a smooth abstract CR manifold of type $(n+l,\, k-l)$. Il $l = k$ then we arrive in the end with an integrable almost complex structure [a CR manifold $\tilde{M}$ of type $(n+k,\, 0)$]; hence by the Newlander-Nirenberg theorem, we have a local embedding of M into $\mathbb{C}^{n+k}$. This completes the proof.

REMARK. – In the case where $l = k$ one obtains a local embedding into $\mathbb{C}^{n+k}$. Conversely, if M is assumed to be locally embeddable in $\mathbb{C}^{n+k}$, then there exists an abelian (and hence solvable) Lie subalgebra $\mathfrak{a}$ of $\widehat{\mathcal{N}}$, of dimension k, which is transversal in the sense of (3.1). Indeed in this case we can take holomorphic coordinates $z_1, z_2, \ldots, z_{n+k}$ in $\mathbb{C}^{n+k}$ in such a way that M is represented near the point p by k real equations

$$\Im z_{n+j} = h_j(z)\,, \qquad 1 \leqslant j \leqslant k\,, \tag{3.8}$$

where the h_j are smooth real valued functions vanishing to the second order at the origin, corresponding to p. It follows that

$$\begin{cases} \mathcal{T}_p^{1,0}M = \mathrm{span}_{\mathbb{C}}\left\{\left(\dfrac{\partial}{\partial z_1}\right)_p, \ldots, \left(\dfrac{\partial}{\partial z_n}\right)_p\right\}, \\ \mathcal{T}_p^{0,1}M = \mathrm{span}_{\mathbb{C}}\left\{\left(\dfrac{\partial}{\partial \bar{z}_1}\right)_p, \ldots, \left(\dfrac{\partial}{\partial \bar{z}_n}\right)_p\right\}. \end{cases} \tag{3.9}$$

The pullbacks $dz_1|_M, dz_2|_M, \ldots, dz_{n+k}|_M, d\bar{z}_1|_M, d\bar{z}_2|_M, \ldots d\bar{z}_n|_M$ to M locally generate the complexified cotangent bundle to M in a neighborhood of p. Note that $dz_{n+i}|_M(p) = d\Re z_{n+i}|_M(p)$ is a real covector for $1 \leqslant i \leqslant k$. Define $X_1, X_2, \ldots, X_k$ by

$$\begin{cases} dz_j|_M(X_h) = d\bar{z}_j|_M(X_h) = 0 & \text{for } 1 \leqslant j \leqslant n \quad \text{and } 1 \leqslant h \leqslant k\,, \\ dz_{n+i}|_M(X_h) = \delta_{i,h} & \text{for } 1 \leqslant i,\, h \leqslant k\,. \end{cases} \tag{3.10}$$

If $\overline{Z} \in \mathcal{C}^{0,1}M$, then

$$dz_j\,|_M([X_h, \overline{Z}]) = X_h(dz_j\,|_M(\overline{Z})) - \overline{Z}(dz_j\,|_M(X_h)) = 0\,,$$

for $1 \leq j \leq n+k$ and $1 \leq h \leq k$, showing that $X_1, \ldots, X_k \in \mathcal{N}$. Moreover,

$$dz_j\,|_M([X_i, X_h]) = X_i(dz_j\,|_M(X_h)) - X_h(dz_j\,|_M(X_i)) = 0\,,$$

for $1 \leq j \leq n+k$ and $1 \leq i, h \leq k$, shows that the images of $X_1, \ldots, X_k$ in $\widehat{\mathcal{N}}$ generate an abelian Lie algebra $\mathfrak{a}$ of dimension k. Note that $\mathfrak{a}$ satisfies the transversality condition (3.1) because

$$X_h(p) = \left(\frac{\partial}{\partial \Re z_{n+h}}\right)_p \notin \mathcal{C}_p^{0,1}M \oplus \mathcal{C}_p^{1,0}M$$

by (3.9).

4. – Examples.

(I) For the first example, consider $M = \mathbb{R}^5$ with real coordinates (x, y, t_1, t_2, t_3) and let $z = x + \sqrt{-1}\,y$. We endow M^5 with a *CR* structure of type $(1, 3)$ by assigning the single generator for $\mathcal{C}^{0,1}M$:

$$\overline{Z} = \frac{\partial}{\partial \overline{z}} - \sqrt{-1}\,z\,\frac{\partial}{\partial t_1} + \phi(z)\,e^{t_3}\,\frac{\partial}{\partial t_2}\,. \tag{4.1}$$

Here $\phi(z)$ is any smooth complex valued function of the real variables x, y. Choose any point $p \in M$ and consider the local situation near p: then the

$$\mathrm{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3} + t_2\,\frac{\partial}{\partial t_2}\right\}$$

is a solvable Lie algebra contained in $\mathcal{N}$. It's image in $\widehat{N}$ is a 3-dimensional solvable Lie algebra which is clearly transversal to the *CR* structure of M in the sense of (3.1). So we find from Theorem 1 that our *CR* structure is locally embeddable into $\mathbb{C}^4$ with real codimension 3. If $\phi(z)$ is real analytic, then (4.1) has real analytic coefficients, so the fact that M is locally embeddable is well-known (see for instance [1]). However we can choose a function ϕ that is smooth but not real analytic anywhere, and the structure is still locally embeddable in $\mathbb{C}^4$ according to our theorem.

(II) For the second example, we also consider $M^5 = \mathbb{R}^5 = \mathbb{R}^3_x \times \mathbb{R}^2_y$ and use real coordinates $(x, y) = (x_1, x_2, x_3, y_1, y_2)$. Recently Rosay [8] constructed a smooth complex vector field L, defined in $\mathbb{R}^3_x$, which has the property that the *CR* structure of type $(1, 1)$ it defines there is strictly pseudoconvex, but is such that there exists a smooth *CR* function u, $u \not\equiv 0$, with $u \equiv 0$ on an open set, and having the point 0 on the boundary of the support of u. We use Rosay's operator L to define on M^5 a smooth *CR* structure of type $(1, 3)$, whose $\mathcal{C}^{0,1}M$ is generated by

$$\overline{Z} = L + x_3 e^{y_2} \frac{\partial}{\partial y_1} + \sqrt{-1}\, x_3^2 \frac{\partial}{\partial y_2} . \tag{4.2}$$

With p chosen as the origin, we consider the

$$\mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial y_1} , \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_1} \right\}.$$

It is a 2-dimensional solvable Lie algebra contained in $\mathcal{N}$. It's image $\mathfrak{a}$ in $\widehat{\mathcal{N}}$ is clearly transversal to the *CR* structure on M^5, and is a solvable 2-dimensional Lie algebra. In this case we find from Theorem 1 that there exists a smooth (abstract) *CR* manifold $\widetilde{M}^7$, of type $(3, 1)$, and a *CR* embedding of a neighborhood of 0 in M^5 into $\widetilde{M}^7$, having codimension 2. But this $\widetilde{M}^7$ is not locally embeddable, near the origin, as a real hypersurface in $\mathbb{C}^4$: if it could be so embedded, it would have the weak unique continuation property for its *CR* functions; however the point of Rosay's construction is that this unique continuation property is violated.

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Pervenuta in Redazione
il 2 ottobre 1997