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On Presentations of Semigroup Rings.

MARIO PETRICH - PEDRO V. SILVA

Sunto. – *Siano I un ideale di un anello R e σ una congruenza su un semigruppo S. Consideriamo l'anello semigruppo $(R/I)(S/\sigma)$ come un'immagine omomorfa dell'anello semigruppo $R(S)$. Questo è fatto in tre passi: prima studiando l'anello semigruppo $R(S/\sigma)$, poi $(R/I)(S)$ e infine combinando i due casi speciali. In ciascun caso, determiniamo l'ideale che è il nucleo dell'omomorfismo in questione. I risultati corrispondenti per le C-algebre, dove C è un anello commutativo, possono essere facilmente dedotti. Alcuni raffinamenti, casi speciali e presentazioni di anelli e semigruppi sono anche considerati.*

1. – Introduction and summary.

Given an arbitrary ring R and a semigroup S, we may form the semigroup ring $R(S)$ in the usual way; its definition is formally the same as in the case of a group ring. The first question one may ask is: if I is an ideal of R and σ is a congruence on S, is there a natural way of relating the semigroup ring $(R/I)(S/\sigma)$ with the original semigroup ring $R(S)$ modulo some ideal? As a special case, we could take R to be a free ring and S to be a free semigroup so that R/I amounts to a presentation as does S/σ. For $S = X^+$, the free semigroup on X, and $R = Z$, the ring of integers, we get that $Z(X^+)$ is a free ring on X which further enhances the consideration of the problem outlined above. For we may start with a presentation of R and a presentation of S and ask about a presentation of $R(S)$. Related to this is the structure of C-algebras where C is a commutative ring.

The theme of this paper belongs to the circle of problems concerning presentations of rings, as applied to semigroup rings. It may not be surprising, at the second look, that in the case of a semigroup ring $R(S)$, its presentation is expressible in terms of presentations of R and S.

This problem arose in the treatment of a very special semigroup ring which, in a separate publication [3], plays an important role in the study of directly finite rings.

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In Section 2, we gather the minimum of needed notation and terminology as well as prove a theorem concerning free rings and C-algebras for a commutative ring C. Section 3 contains basic results concerning the semigroup ring $R(S/\sigma)$ where R is a ring and σ is a congruence on a semigroup S in relation to the «original» semigroup ring $R(S)$. In Section 4 we consider a dual situation, namely the semigroup ring $(R/I)(S)$ where I is an ideal of R and S is a semigroup. The two cases are combined in Section 5 to yield the general situation mentioned above. In all cases, we derive corollaries concerning presentations of such semigroup rings.

2. – Background.

We assume that the reader is familiar with the basic concepts of ring and semigroup theories; for their symbolism and concepts, we follow [4] and [1], respectively. All our rings are associative, but they do not necessarily have an identity.

Let $\mathcal{V}$ denote a variety of algebras of a certain type. Given a homomorphism $\varphi\colon A \to B$ of algebras in $\mathcal{V}$, we denote by $\overline{\varphi}$ the congruence on A induced by φ. Given a $\mathcal{V}$-algebra A and a relation ϱ on A (that is, a subset of $A \times A$), we denote by $\varrho^{\sharp}$ the congruence on A generated by ϱ. The congruence $\varrho^{\sharp}$ can be described by *layers*, an important concept for inductive arguments, according to the following rules.

- Let P_1 be the reflexive and symmetric closure of ϱ.
- For every $j \geq 1$, let T_j be the transitive closure of P_j.
- For every $j \geq 1$, let

$$P_{j+1} = T_j \cup \{((a_1, \ldots, a_n)f, (b_1, \ldots, b_n)f);$$
$$f \text{ is an } n\text{-ary operation of } A \text{ and } (a_1, b_1), \ldots, (a_n, b_n) \in T_j\}.$$

It is easy to see that $\varrho^{\sharp} = \bigcup_{j \geq 1} T_j$.

Let X be a nonempty set. A *free object* of $\mathcal{V}$ on X is an ordered pair $(F\mathcal{V}(X), \iota_{\mathcal{V}})$ such that:

- $F\mathcal{V}(X) \in \mathcal{V}$;
- $\iota_{\mathcal{V}}\colon X \to F\mathcal{V}(X)$ is a map;
- for every map $\varphi\colon X \to A$, with $A \in \mathcal{V}$, there exists a unique homomorphism of $\mathcal{V}$-algebras $\Phi\colon F\mathcal{V}(X) \to A$ such that $\varphi = \iota_{\mathcal{V}}\Phi$.

A free object of $\mathcal{V}$ on X is defined up to isomorphism and $F\mathcal{V}(X)$ is often referred to as a free object on X itself. A $\mathcal{V}$-*presentation* is a formal expression of the form $\mathcal{V}\langle X;\varrho\rangle$, where X is a nonempty set and ϱ is a relation on $F\mathcal{V}(X)$. The algebra of $\mathcal{V}$ defined by the presentation $\mathcal{V}\langle X;\varrho\rangle$ is the quotient $F\mathcal{V}(X)/\varrho^\sharp$. It follows easily from the definition that every algebra $A\in\mathcal{V}$ can be defined, up to isomorphism, by a presentation of the form $\mathcal{V}\langle X;\varrho\rangle$. In such a case, we write

$$A\leftarrow\mathcal{V}\langle X;\varrho\rangle.$$

If $A\leftarrow\mathcal{V}\langle X;\varrho\rangle$ and $B\leftarrow\mathcal{V}\langle Y;\lambda\rangle$ with $X\cap Y=\emptyset$, the *free product* of A and B in $\mathcal{V}$ is defined, up to isomorphism, to be the $\mathcal{V}$-algebra defined by the presentation

$$\mathcal{V}\langle X\cup Y;\varrho\cup\lambda\rangle.$$

We denote the free product of A and B by $A*B$.

Our interest will be focused on the following varieties:

$\mathcal{S}$: the variety of semigroups - type (2);

$\mathcal{M}$: the variety of monoids - type (2, 0);

$\mathcal{R}$: the variety of rings - type (2, 2);

$\mathcal{U}$: the variety of unitary rings - type (2, 2, 0).

For every commutative ring C, we also consider

$\mathcal{A}_C$: the variety of C-algebras - type $(2,\,2,(1)_{|C|})$;

and if C is unitary, then also

$\mathcal{U}\mathcal{A}_C$: the variety of unitary C-algebras - type $(2,\,2,(1)_{|C|},\,0)$.

We shall produce concrete descriptions of the free objects in all these varieties.

Let X denote a nonempty set. A *word* over X is a finite sequence of elements of X, usually written in the form $x_1x_2\dots x_n$, with $n\geqslant 0$ and $x_1,\dots,x_n\in X$. The empty word is denoted by 1. We denote by X^+ the set of all nonempty words on X and write $X^*=X^+\cup\{1\}$, with concatenation of words as binary operation on X^*. With this multiplication, X^* is a monoid with the empty word as an identity, and X^+ is a subsemigroup of X^*. Let $\zeta\colon X\to X^+$ be the map which associates to every $x\in X$ the word x. It is well known that

- $(X^+,\,\zeta)$ is a free semigroup on X,
- $(X^*,\,\zeta)$ is a free monoid on X.

Let S be a semigroup and R a ring. Given a map $f\colon S \to R$, the *support* of f is

$$\operatorname{supp}(f) = \{s \in S \mid sf \neq 0\}.$$

We say that f has *finite support* if $\operatorname{supp}(f)$ is finite. Let $R(S)$ denote the set of all maps $f\colon S \to R$ with finite support. Given $f, g \in R(S)$, we define $f+g$ and $f \cdot g$ by

$$s(f+g) = sf + sg, \qquad s(f \cdot g) = \sum_{xy = s} (xf)(yg) \qquad (s \in S),$$

respectively. It is well known that with these operations $R(S)$ constitutes a ring. For every $r \in R$, we denote by

$$r\iota\colon S \to R(S), \qquad s \mapsto r\iota_s$$

the map defined by

$$s'(r\iota_s) = \begin{cases} r & \text{if } s' = s, \\ 0 & \text{otherwise}, \end{cases} \qquad (s' \in S).$$

If R is unitary, we write ι for 1ι. If M is a monoid and U a unitary ring, $U(M)$ is a unitary ring.

We consider also the scalar multiplication: for $r \in R$ and $f \in R(S)$, rf is defined by

$$s(rf) = r(sf) \qquad (s \in S).$$

If C is a commutative ring, then $C(S)$ can be viewed as a C-algebra by means of this scalar multiplication.

It is often simpler to describe the elements of $R(S)$ as formal sums according to the correspondence

$$f \leftrightarrow \sum_{s \in S} (sf)\, s.$$

We will often use this notation, but we will also need the more precise map notation to overcome some technical difficulties.

The next result is known ([4], Chapter 1.3) but we prove one of the claims (all the proofs are similar) for the sake of completeness. Since every ring can be viewed naturally as a Z-algebra, we can in fact derive (i)-(ii) from (iii)-(iv), respectively.

THEOREM 2.1. – *Let X be a nonempty set and C a commutative unitary ring.*

(i) *$(Z(X^+), \iota|_X)$ is a free ring on X.*

(ii) *$(Z(X^*), \iota|_X)$ is a free unitary ring on X.*

(iii) $(C(X^+), \iota|_X)$ *is a free C-algebra on X.*

(iv) $(C(X^*), \iota|_X)$ *is a free unitary C-algebra on X.*

PROOF. – (iii) We have already observed that $C(X^+)$ is a C-algebra and $\iota|_X$: $X \to C(X^+)$ is obviously a well-defined map. Let A be a C-algebra and φ: $X \to A$ a map. Define a mapping

$$\phi\colon X^+ \to A, \qquad x_1 \ldots x_n \mapsto (x_1 \varphi) \ldots (x_n \varphi) \qquad (x_1, \ldots, x_n \in X).$$

Clearly, ϕ is a semigroup homomorphism. Define a mapping

$$\Phi\colon C(X^+) \to A, \qquad \sum_{u \in X^+} c_u u \mapsto \sum_{u \in X^+} c_u (u\phi).$$

Since $c_u = 0$ for all but finitely many $u \in X^+$, Φ is well defined. Let

$$f = \sum_{u \in X^+} c_u u, \qquad g = \sum_{u \in X^+} d_u u$$

be elements of $C(X^+)$ and $b \in C$. Then

$$(f+g)\,\Phi = \Big(\sum_{u \in X^+} (c_u + d_u)\, u\Big)\,\Phi = \sum_{u \in X^+} (c_u + d_u)(u\phi)$$

$$= \Big(\sum_{u \in X^+} c_u (u\phi)\Big) + \Big(\sum_{u \in X^+} d_u (u\phi)\Big) = (f\Phi) + (g\Phi),$$

$$(f \cdot g)\,\Phi = \Big(\sum_{u,\, v \in X^+} (c_u d_v)\, uv\Big)\,\Phi = \sum_{u,\, v \in X^+} (c_u d_v)((uv)\,\phi) =$$

$$= \sum_{u,\, v \in X^+} (c_u d_v)(u\phi)(v\phi) = \Big(\sum_{u \in X^+} c_u (u\phi)\Big)\Big(\sum_{v \in X^+} d_v (v\phi)\Big) = (f\Phi)(g\Phi),$$

$$(bf)\,\Phi = \Big(\sum_{u \in X^+} (bc_u)\, u\Big)\,\Phi = \sum_{u \in X^+} (bc_u)(u\phi) = b\Big(\sum_{u \in X^+} c_u (u\phi)\Big) = b(f\Phi),$$

and so Φ is a homomorphism of C-algebras. The uniqueness of Φ follows from $X\iota$ generating $C(X^+)$ as a C-algebra. It is immediate that $x\iota\Phi = x\varphi$ for every $x \in X$. Therefore $(C(X^+), \iota|_X)$ is a free C-algebra on X. ■

3. – The semigroup ring $R(S/\sigma)$.

For a ring R and any congruence σ on a semigroup S, we shall find here a suitable isomorphic copy of $R(S/\sigma)$. Given a ring R and a relation ϱ on a semigroup S, let

$$R(\varrho\iota) = \{(r\iota_a, r\iota_b) \,|\, (a, b) \in \varrho,\ r \in R\}.$$

We are now ready for the first principal result of the paper, essentially similar to ([2], Corollary I.4.2).

THEOREM 3.1. – *Let R be a ring, S a semigroup and ϱ a relation on S. Let $\sigma = \varrho^\sharp$ and $\tau = (R(\varrho\iota))^\sharp$. For every $g \in R(S)$, define $\widehat{g} \in R(S/\sigma)$ by*

$$(a\sigma)\widehat{g} = \sum_{b \in a\sigma} bg \qquad (a \in S)$$

and a mapping

$$\alpha\colon\ R(S) \to R(S/\sigma), \qquad g \mapsto \widehat{g}\,.$$

Then α is a homomorphism of $R(S)$ onto $R(S/\sigma)$ which induces τ. In particular,

$$R(S)/\tau \cong R(S/\sigma)\,.$$

PROOF. – Since $\operatorname{supp}(\widehat{g}) \subseteq (\operatorname{supp}(g))\,\sigma$ is finite, $\widehat{g}$ is well defined. We show that α is a ring homomorphism. Let $g, h \in R(S)$. Then for every $a \in S$, we get

$$(a\sigma)(\widehat{g+h}) = \sum_{b \in a\sigma} b(g+h) = \Big(\sum_{b \in a\sigma} bg\Big) + \Big(\sum_{b \in a\sigma} bh\Big) = (a\sigma)\ \widehat{g} + (a\sigma)\ \widehat{h} = (a\sigma)(\widehat{g} + \widehat{h})$$

and thus $(g+h)\,\alpha = g\alpha + h\alpha$. Let $C, D \in S/\sigma$. For any $a \in S$ we have

$$(a\sigma)\ \widehat{g \cdot h} = \sum_{b \in a\sigma} b(g \cdot h) = \sum_{b \in a\sigma}\ \sum_{xy = b} (xg)(yh)\,,$$

$$(a\sigma)(\widehat{g} \cdot \widehat{h}) = \sum_{CD = a\sigma} (C\widehat{g})(D\widehat{h}) = \sum_{CD = a\sigma} \Big(\sum_{x \in C} xg\Big)\Big(\sum_{y \in D} yh\Big) = \sum_{CD = a\sigma}\ \sum_{x \in C}\ \sum_{y \in D} (xg)(yh)\,.$$

Hence we only need to show that the pairs (x, y) involved in both sums are the same. Let $x, y \in S$ be such that $xy = b$ for some $b \in a\sigma$. Taking $C = x\sigma$ and $D = y\sigma$, we have $x \in C$, $y \in D$ and $CD = (x\sigma)(y\sigma) = b\sigma = a\sigma$. Conversely, let $a\sigma = CD$ for some $C, D \in S/\sigma$, $x \in C$ and $y \in D$. For $b = xy$, we get

$$b\sigma = (x\sigma)(y\sigma) = CD = a\sigma$$

and so $b \in a\sigma$. We have proved that $(a\sigma)\ \widehat{g \cdot h} = (a\sigma)(\widehat{g} \cdot \widehat{h})$ for every $a \in S$ and thus $(g \cdot h)\,\alpha = (g\alpha) \cdot (h\alpha)$. Therefore α is a ring homomorphism.

Let $(p, q) \in \varrho$ and $r \in R$. For every $a \in S$, we have

$$(a\sigma)(\widehat{r\iota_p}) = \sum_{x \in a\sigma} r(x\iota_p) = \begin{cases} r & \text{if } a\,\sigma\,p\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$(a\sigma)(\widehat{r\iota_q}) = \sum_{x \in a\sigma} r(x\iota_q) = \begin{cases} r & \text{if } a\,\sigma\,q\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Since $(p, q) \in \varrho \subseteq \sigma$, it follows that $\widehat{r\iota_p} = \widehat{r\iota_q}$. This can be expressed by $R(\varrho\iota) \subseteq \overline{\alpha}$. Since α is a ring homomorphism, $\overline{\alpha}$ is a ring congruence and so $\tau = (R(\varrho\iota))^\sharp \subseteq \overline{\alpha}$. Hence α induces a ring homomorphism

$$\beta\colon\ R(S)/\tau \to R(S/\sigma)\,, \qquad g\tau \mapsto \widehat{g}\,.$$

Let S^0 be a subset of S such that S^0 contains exactly one element from each σ-class of S. For every $a \in S$, let a^0 denote the (unique) element of S^0 in $a\sigma$. For every $f \in R(S/\sigma)$, define $f' \in R(S)$ by

$$af' = \begin{cases} (a\sigma)f & \text{if } a \in S^0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For every $f \in R(S/\sigma)$, the set

$$\operatorname{supp}(f') = \{a^0 \mid a\sigma \in \operatorname{supp}(f)\}$$

is finite and so f' is well defined. We define

$$\gamma\colon\ R(S/\sigma) \to R(S)/\tau\,, \qquad f \mapsto f'\tau\,.$$

All we need now is to show that β and γ are mutually inverse mappings.

Let $f \in R(S/\sigma)$. For every $a \in S$, we have

$$(a\sigma)\,\widehat{f'} = \sum_{b \in a\sigma} bf' = (a^0\sigma)f = (a\sigma)f\,,$$

hence $\widehat{f'} = f$ and $\gamma\beta = 1$. Conversely, let $a, b \in S$ and $r \in R$. Then

$$b(\widehat{r\iota_a}') = \begin{cases} (b\sigma)\,\widehat{r\iota_a} & \text{if } b \in S^0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

If $b \in S^0$, then

$$(b\sigma)\,\widehat{r\iota_a} = \sum_{c \in b\sigma} c(r\iota_a) = \begin{cases} r & \text{if } a\,\sigma\,b\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Hence

$$b(\widehat{r\iota_a}') = \begin{cases} r & \text{if } b = a^0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and so $\widehat{r\iota_a}' = r\iota_{a^0}$. As $a\,\sigma\,a^0$, we have $(r\iota_a)\,\tau(r\iota_{a^0})$ and thus

$$((r\iota_a)\,\tau)\beta\gamma = \widehat{r\iota_a}'\,\tau = (r\iota_{a^0})\,\tau = (r\iota_a)\,\tau\,.$$

Since $\{r\iota_a \mid a \in S,\, r \in R\}$ generates $R(S)$ as a ring, it follows that $\beta\gamma = 1$ hence β is an isomorphism. ■

If R and S satisfy adequate conditions, the expression of the congruence τ in Theorem 3.1 may be simplified. Let U be a unitary ring and M a monoid. Then

$$\varrho\iota = \{(\iota_p,\, \iota_q) \mid (p,\, q) \in \varrho\}$$

is a relation on $U(M)$, and M being a monoid yields $(\varrho\iota)^\sharp = (U(\varrho\iota))^\sharp = \tau$. This equality also holds if U is generated by its identity. Since Z is generated by its identity, the next result follows immediately.

COROLLARY 3.2.

(i) *If* $S \leftarrow \mathcal{S}\langle X;\, \varrho\rangle$, *then* $Z(S) \leftarrow \mathcal{R}\langle X;\, \varrho\iota\rangle$.

(ii) *If* $M \leftarrow \mathcal{M}\langle X;\, \varrho\rangle$, *then* $Z(M) \leftarrow \mathcal{U}\langle X;\, \varrho\iota\rangle$.

Given a unitary ring U, when can we replace $U(\varrho\iota)$ by $\varrho\iota$ in Theorem 3.1 for every semigroup S? The next result describes precisely all such rings.

PROPOSITION 3.3. – *Let U be a unitary ring. Then U is generated by its identity if and only if for every semigroup S and every relation ϱ on S, we have*

$$U(S/\varrho^\sharp) \cong U(S)/(\varrho\iota)^\sharp.$$

PROOF. – Necessity was observed above. Suppose that U is not generated by its identity. Let X be an infinite uncountable set such that $|X| > |U|$, $S = X^+$ and ϱ be the universal relation on X. Obviously, $S/\varrho^\sharp$ is countable and so $U(S/\varrho^\sharp)$ is either countable or equipotent with U.

On the other hand, since U is not generated by its identity, there exists $u \in U$ such that $u \neq n \cdot 1$ for every $n \in Z$. Let I denote the $(\varrho\iota)^\sharp$-class of 0 in $U(S)$. Then I is the ideal of $U(S)$ generated by the set

$$I_0 = \{\iota_a - \iota_b \mid (a,\, b) \in \varrho\} = \{\iota_x - \iota_y \mid x,\, y \in X\}\,.$$

Let $x, y \in X$ be such that $u\iota_x - u\iota_y \in I$. Since I is generated by I_0, we must have an equality of the form

$$u\iota_x - u\iota_y = \sum_{j=1}^{n} f_j \cdot (\iota_{x_j} - \iota_{y_j}) \cdot g_j$$

where $n \geq 0$, $x_j, y_j \in X$ and the f_j, g_j are either elements of $U(S)$ or are absent. Since $S = X^+$, it follows easily from length considerations that we may assume that all f_j and g_j are absent. Suppose that $x \neq y$. Comparing the coefficient of ι_x on both sides of the above equality, we conclude that $u\iota_x = k\iota_x$ for some integer k and so $u = k \cdot 1$, a contradiction. Therefore $x = y$ and so $(u\iota_x)(\varrho\iota)^\sharp$, $x \in X$, are all distinct elements of $U(S)/(\varrho\iota)^\sharp$. Hence

$$|U(S)/(\varrho\iota)^\sharp| \geq |X| > |U(S/\varrho^\sharp)|$$

and so $U(S/\varrho^\sharp) \not\cong U(S)/(\varrho\iota)^\sharp$. ■

Given a semigroup S and a unitary ring U, we may identify S with the subsemigroup $S\iota$ of the multiplicative semigroup of $U(S)$. Keeping that in mind, the next proposition expresses the following fact: if we consider a congruence σ on S, the restriction to S of the ring congruence generated by σ in $U(S)$ is equal to σ.

PROPOSITION 3.4. – Let σ be a congruence on a semigroup S and U be a unitary ring. Then $(\sigma\iota)^\sharp|_{S\iota} = \sigma\iota$. Identifying $S\iota$ with S, $\sigma^\sharp$ is the least extension of σ to a congruence on $U(S)$; the greatest extension need not exist.

PROOF. – Trivially, we have $\sigma\iota \subseteq (\sigma\iota)^\sharp|_{S\iota}$. Conversely, let $a, b \in S$ be such that $(\iota_a, \iota_b) \in (\sigma\iota)^\sharp$. Then $(\iota_a, \iota_b) \in (U(\sigma\iota))^\sharp$ and so, applying the isomorphism

$$\beta\colon\ U(S)/(U(\sigma\iota))^\sharp \to U(S/\sigma)$$

from Theorem 3.1, we obtain

$$\widehat{\iota_a} = (\iota_a(U(\sigma\iota))^\sharp)\beta = (\iota_b(U(\sigma\iota))^\sharp)\beta = \widehat{\iota_b}.$$

Now

$$\sum_{x \in a\sigma} x\iota_b = (a\sigma)\,\widehat{\iota_b} = (a\sigma)\,\widehat{\iota_a} = \sum_{x \in a\sigma} x\iota_a = 1$$

yields $a\sigma b$ and so $(\iota_a, \iota_b) \in \sigma\iota$.

Now we identify $S\iota$ with S. We have just proved that $\sigma^\sharp$ is an extension of σ to a congruence on $U(S)$ and it is clearly the least such. We produce a counterexample where there exists no greatest extension.

Let S be a nontrivial semigroup and σ be the identity congruence on S. For all $a \in S$ and $n > 1$, let $\varrho_{a,n}$ denote the (ring) congruence on $Z(S)$ generated by $(n\iota_a, 0)$. Clearly, the $\varrho_{a,n}$-class of 0 is the ideal of $Z(S)$ generated by $n\iota_a$ and so if $f\varrho_{a,n}\, 0$, then n divides sf for every $s \in S$. It follows that $(\iota_b - \iota_c)\,\varrho_{a,n} \neq 0\varrho_{a,n}$ for all distinct $b, c \in S$ and so $\varrho_{a,n}$ is an extension of σ for all $a \in S$ and $n > 1$. However, the join of all these extensions is clearly the universal relation, which is not an extension of σ. Therefore there is no greatest extension in this case. ∎

Given a commutative ring C, a semigroup S and a relation ϱ on $C(S)$, we denote by

$\varrho^{\sharp r}$: the ring congruence on $C(S)$,

$\varrho^{\sharp a}$: the C-algebra congruence on $C(S)$,

generated by ϱ. In the next proposition, we compare the two congruences.

PROPOSITION 3.5. – *Let C be a commutative ring and S a semigroup. Let ϱ be a relation on $C(S)$ and write*

$$C(\varrho) = \{(cp, cq) \mid c \in C, (p, q) \in \varrho\}.$$

Then $(C(\varrho))^{\sharp r} = \varrho^{\sharp a}$.

PROOF. – Since $\varrho \subseteq \varrho^{\sharp a}$ and $\varrho^{\sharp a}$ is a C-algebra congruence, it follows that $C(\varrho) \subseteq \varrho^{\sharp a}$. Since all C-algebra congruences are in particular ring congruences, it follows that

$$(C(\varrho))^{\sharp r} \subseteq (C(\varrho))^{\sharp a} \subseteq \varrho^{\sharp a}.$$

We have $\varrho \subseteq (C(\varrho))^{\sharp r}$ and so, to prove the opposite inclusion, we only need to show that $(C(\varrho))^{\sharp r}$ is a C-algebra congruence. Since $(C(\varrho))^{\sharp r}$ is a ring congruence by definition, it remains to establish that

$$(p, q) \in (C(\varrho))^{\sharp r} \Rightarrow (cp, cq) \in (C(\varrho))^{\sharp r}$$

holds for every $c \in C$. To show this, we are going to use the layer description from Section 2, namely $(C(\varrho))^{\sharp r} = \bigcup_{j \geq 1} T_j$ and apply induction on j.

Fix $c \in C$ and suppose that $(p, q) \in T_1$, the equivalence relation generated by ϱ. Since $(a, b) \in C(\varrho) \Rightarrow (ca, cb) \in C(\varrho)$ holds for every (a, b), it follows easily that $(cp, cq) \in (C(\varrho))^{\sharp r}$. Suppose now that our implication holds whenever $(p, q) \in T_n$ $(n \geqslant 1)$, and let $(p, q) \in T_{n+1}$. Then there exist $w_0, \dots, w_k \in C(S)$ such that

$$p = w_0;\ q = w_k;\ \text{for every } j \in \{1, \dots, k\},\ (w_{j-1}, w_j) \in P_{n+1}\,.$$

Since $(C(\varrho))^{\sharp r}$ is transitive, we only need to prove that $(cw_{j-1}, cw_j) \in (C(\varrho))^{\sharp r}$ for every $j \in \{1, \dots, k\}$. For each such j, one of the following situations must occur:

(i) $(w_{j-1}, w_j) \in T_n$;

(ii) there exist $(x, y), (x', y') \in T_n$ such that $w_{j-1} = x + x'$ and $w_j = y + y'$;

(iii) there exist $(x, y), (x', y') \in T_n$ such that $w_{j-1} = xx'$ and $w_j = yy'$.

If (i) occurs, then $(cw_{j-1}, cw_j) \in (C(\varrho))^{\sharp r}$ at once by the induction hypothesis. If (ii) occurs, then $(cx, cy), (cx', cy') \in (C(\varrho))^{\sharp r}$ by the induction hypothesis and so

$$(cw_{j-1}, cw_j) = (cx + cx', cy + cy') \in (C(\varrho))^{\sharp r}\,.$$

Finally, if (iii) occurs, then $(cx, cy) \in (C(\varrho))^{\sharp r}$ by the induction hypothesis and so $(cw_{j-1}, cw_j) = (cxx', cyy') \in (C(\varrho))^{\sharp r}$. In any case, we obtain $(cw_{j-1}, cw_j) \in (C(\varrho))^{\sharp r}$ for every $j \in \{1, \dots, k\}$ and the equality $(C(\varrho))^{\sharp r} = \varrho^{\sharp a}$ follows by induction. ■

We now derive some consequences of Proposition 3.

COROLLARY 3.6. – *Let C be a commutative ring, S a semigroup, and ϱ a relation on $C(S)$. Then the mapping α in Theorem* 3.1 *is a C-algebra homomorphism. In particular,*

$$C(S/\varrho^{\sharp}) \cong C(S)/(\varrho\iota)^{\sharp a}$$

as C-algebras.

PROOF. – We only need to check that $(cg)\alpha = c(g\alpha)$ for all $c \in C$ and $g \in C(S)$. Let $a \in S$. Then

$$(a\sigma)\widehat{cg} = \sum_{b \in a\sigma} b(cg) = \sum_{b \in a\sigma} c(bg) = c\sum_{b \in a\sigma} bg = ((a\sigma)\widehat{g}) = (a\sigma)(c\widehat{g})$$

and thus $\widehat{cg} = c\widehat{g}$ and $(cg)\alpha = c(g\alpha)$ for all $c \in C$ and $g \in C(S)$. Therefore α is a

C-algebra homomorphism and so is the quotient isomorphism

$$\beta\colon\ C(S)/(C(\varrho))^{\sharp^r} \to C(S/\varrho^{\sharp}).$$

The result now follows from Proposition 3.5. ■

COROLLARY 3.7. – *Let C be a commutative unitary ring.*

(i) *If $S \leftarrow \mathcal{S}\langle X;\ \varrho\rangle$, then $C(S) \leftarrow \mathcal{A}_C\langle X;\ \varrho\iota\rangle$.*

(ii) *If $M \leftarrow \mathcal{M}\langle X;\ \varrho\rangle$, then $C(M) \leftarrow \mathcal{U}\mathcal{A}_C\langle X;\ \varrho\iota\rangle$.*

COROLLARY 3.8.

(i) *If S and T are semigroups, then $Z(S * T) \cong Z(S) * Z(T)$ in $\mathcal{R}$.*

(ii) *If M and N are monoids, then $Z(M * N) \cong Z(M) * Z(N)$ in $\mathcal{U}$.*

(iii) *If C is a commutative unitary ring and S and T are semigroups, then $C(S * T) \cong C(S) * C(T)$ in $\mathcal{A}_C$.*

(iv) *If C is a commutative unitary ring and M and N are monoids, then $C(M * N) \cong C(M) * C(N)$ in $\mathcal{U}\mathcal{A}_C$.*

PROOF. – (i) Let $S \leftarrow \mathcal{S}\langle X;\ \varrho\rangle$ and $T \leftarrow \mathcal{S}\langle Y;\ \lambda\rangle$ with $X \cap Y = \emptyset$. By Corollary 3.2, we have

$$Z(S) \leftarrow \mathcal{R}\langle X;\ \varrho\iota\rangle, \qquad Z(T) \leftarrow \mathcal{R}\langle Y;\ \lambda\iota\rangle,$$

$$Z((X \cup Y)^+/(\varrho \cup \lambda)^{\sharp}) \leftarrow \mathcal{R}\langle X \cup Y;\ \varrho\iota \cup \lambda\iota\rangle$$

and so $Z(S * T) \cong Z(S) * Z(T)$ as rings.

The remaining statements are proved similarly, using also Corollary 3.7. ■

4. – The semigroup ring $(R/I)(S)$.

Given an ideal I of a ring R and a semigroup S, we can view the semigroup ring $I(S)$ as an ideal of $R(S)$. Our main result here is a kind of dual of Theorem 3.1 which we formulate as follows.

THEOREM 4.1. – *Let I be an ideal of a ring R and S a semigroup. For every $g \in R(S)$, define $\tilde{g} \in (R/I)(S)$ by $a\tilde{g} = ag + I$ and a function*

$$\phi\colon R(S) \to (R/I)(S), \qquad g \mapsto \tilde{g}.$$

Then ϕ is a homomorphism of $R(S)$ onto $(R/I)(S)$ with kernel $I(S)$. In particular,

$$R(S)/I(S) \cong (R/I)(S).$$

PROOF. – Since $\operatorname{supp}(\tilde{g}) \subseteq \operatorname{supp}(g)$ is finite, $\tilde{g}$ is well defined. Let $g, h \in R(S)$. Then for every $a \in S$, we obtain

$$a(\widetilde{g+h}) = a(g+h) + I = (ag+I) + (ah+I) = a\tilde{g} + a\tilde{h} = a(\tilde{g}+\tilde{h})$$

and so $(g+h)\phi = g\phi + h\phi$; also

$$a(\widetilde{g\cdot h}) = a(g\cdot h) + I = \left(\sum_{xy=a}(xg)(yh)\right) + I = \sum_{xy=a}(xg+I)(yh+I)$$

$$= \sum_{xy=a}(x\tilde{g})(y\tilde{h}) = x(\tilde{g}\cdot\tilde{h})$$

and thus $(g\cdot h)\phi = (g\phi)\cdot(h\phi)$. Therefore ϕ is a ring homomorphism. It is easy to see that $(I(S))\phi = 0$, hence ϕ induces a ring homomorphism

$$\psi\colon R(S)/I(S) \to (R/I)(S), \qquad g + I(S) \mapsto \tilde{g}.$$

Let $R^0 \subseteq R$ be such that $|R^0 \cap (r+I)| = 1$ for every $r \in R$. For each $r \in R$, let r^0 denote the (unique) element of R^0 in $r + I$. We assume that $0^0 = 0$. For every $f \in (R/I)(S)$, define $f^* \in R(S)$ by $af^* = af \cap R^0$. Since $0^0 = 0$, we have $\operatorname{supp}(f^*) = \operatorname{supp}(f)$ for every $f \in R(S)$ and so f^* is well defined. We define a mapping

$$\theta\colon (R/I)(S) \to R(S)/I(S), \qquad f \mapsto f^* + I(S).$$

It suffices to show that θ and ψ are mutually inverse mappings.

Let $f \in (R/I)(S)$. For every $a \in S$, we have $a\widetilde{f^*} = af^* + I = af$, hence $\widetilde{f^*} = f$ and $\theta\psi = 1$.

Conversely, let $g \in R(S)$. Then for every $a \in S$, we get

$$a\tilde{g}^* = a\tilde{g} \cap R^0 = (ag + I) \cap R^0 = (ag)^0,$$

and so $a(g - \tilde{g}^*) = ag - (ag)^0 \in I$. It follows that $g - \tilde{g}^* \in I(S)$ hence

$$(g + I(S))\,\psi\theta = \tilde{g}^* + I(S) = g + I(S).$$

Thus $\psi\theta = 1$ and ψ is an isomorphism. ■

Given a ring R and a monoid M with identity 1, we may identify R with the subring $R\cdot 1$ of $R(M)$. The next result is an analogue of Proposition 3.4.

PROPOSITION 4.2. – *Let I be an ideal of a ring R, M a monoid and*

$$J = I \cdot 1 \cup (RI)(M) \cup (IR)(M).$$

Then $J \cap (R \cdot 1) = I$. If we identify $R \cdot 1$ with R, then J is the least extension of I to an ideal of $R(M)$; the greatest extension need not exist.

PROOF. – We show by a counterexample the nonexistence of a greatest extension. The rest is immediate.

Let R be a nontrivial ring and X be a nonempty set. Take $I = (0)$ and $M = X^*$ and set

$$A = \{ f \in R(M) \mid \sum_{v \in M} vf = 0 \}.$$

Then A is an ideal of $R(M)$ known as the *augmentation ideal* of $R(M)$ and $A \cap R = I$. We may consider $B = R(X^+)$ as an ideal of $R(M)$ and we also have $B \cap R = I$. Let $r \in R$ and $x \in X$. Then

$$r = (r - r\iota_x) + r\iota_x \in A + B$$

and thus $(A + B) \cap R = R$. Therefore there is no greatest extension of I in this case. ∎

We also have an analogue of Theorem 4.1 for C-algebras.

COROLLARY 4.3. – *Let I be an ideal of a commutative ring C and S a semigroup. Then the mapping ϕ in Theorem 4.1 is a C-algebra homomorphism. In particular,*

$$(C/I)(S) \cong C(S)/I(S)$$

as C-algebras.

PROOF. – By Theorem 4.1, we know that $(C/I)(S) \cong C(S)/I(S)$ as rings. In order to show that ϕ is in fact a C-algebra isomorphism, it remains to show that $(cg)\phi = c(g\phi)$ for all $c \in C$ and $g \in C(S)$. Let $a \in S$. Then

$$a(\widetilde{cg}) = a(cg) + I = c(ag) + I = c(ag + I) = c(a\tilde{g}) = a(c\tilde{g}),$$

thus $\widetilde{cg} = c\tilde{g}$ and $(cg)\phi = c(g\phi)$ for all $c \in C$ and $g \in C(S)$. Therefore ϕ is a C-algebra homomorphism and so is the quotient isomorphism $\psi: C(S)/I(S) \to (C/I)(S)$. ∎

5. – The semigroup ring $(R/I)/(S/\sigma)$.

In this section we combine results obtained in Sections 3 and 4. We formulate the main result as follows.

THEOREM 5.1. – *Let I be an ideal of a ring R and ϱ a relation on a semigroup S. Let $\sigma = \varrho^\sharp$ and*

$$\mu = (R(\varrho\iota) \cup \{(i\iota_s, 0) \mid i \in I, s \in S\})^\sharp .$$

Then

$$(R/I)/(S/\sigma) \cong R(S)/\mu .$$

PROOF. – From the proofs of Theorems 3.1 and 4.1, we have the isomorphisms:

$$\beta\colon R(S)/\tau \to R(S/\sigma), \qquad g\tau \mapsto \hat{g},$$

$$\psi\colon R(S/\sigma)/I(S/\sigma) \to (R/I)(S/\sigma), \qquad g + I(S/\sigma) \mapsto \tilde{g},$$

where $\tau = (R(\varrho\iota))^\sharp$, respectively. The congruence μ/τ on $R(S)/\tau$ is generated by the relation

$$\{(i\iota_s\tau, 0\tau) \mid i \in I, s \in S\}.$$

It follows easily from the definition of β that the image of this relation under it is the relation

$$\{(i\iota_{s\sigma}\tau, 0\tau) \mid i \in I, s \in S\}.$$

Thus β induces an isomorphism

$$\delta\colon (R(S)/\tau)/(\mu/\tau) \to R(S/\sigma)/I(S/\sigma).$$

Composing with ψ, we obtain

$$(R(S)/\tau)/(\mu/\tau) \cong (R/I)(S/\sigma).$$

Since $(R(S)/\tau)/(\mu/\tau) \cong R(S)/\mu$ by the classical isomorphism theorems, the result follows. ∎

COROLLARY 5.2. – *Let I be an ideal of a commutative ring C and ϱ a relation on a semigroup S. Let $\sigma = \varrho^\sharp$ and*

$$\mu = (\varrho\iota \cup \{(i\iota_s, 0) \mid i \in I, \; s \in S\})^\sharp .$$

Then $(C/I)/(S/\sigma) \cong C(S)/\mu$ as C-algebras.

PROOF. – By Corollaries 3.6 and 4.3, all the isomorphisms considered in the proof of Theorem 5.1 are in this case C-algebra isomorphisms. ∎

COROLLARY 5.3. – *Let I be an ideal of a commutative unitary ring C.*

(i) *If $S \leftarrow \mathcal{S}\langle X; R\rangle$, then*

$$(C/I)(S) \leftarrow \mathcal{A}_C\langle X; R\iota \cup \{i\iota_x = 0 \mid i \in I,\, x \in X\}\rangle.$$

(ii) *If $M \leftarrow \mathcal{M}\langle X; R\rangle$, then*

$$(C/I)(M) \leftarrow \mathcal{U}\mathcal{A}_C\langle X; R\iota \cup \{i\iota_1 = 0 \mid i \in I\}\rangle. \quad \blacksquare$$

We have discussed in Section 3 semigroup rings of the form $R(S/\sigma)$, in Section 4 those of the form $(R/I)/(S)$ and in Section 5 those of the form $(R/I)/(S/\sigma)$ as homomorphic images of the semigroup ring $R(S)$. Even if I runs over all ideals of R and σ over all congruences on S, obviously the stated homomorphic images of $R(S)$ are very far from exhausting all homomorphic images of $R(S)$, even up to isomorphism. Ideals of $R(S)$ may be inexpressible, at least in a straightforward way, in terms of ideals of R and congruences on S. Nevertheless, it would be of some interest to find classes of ideals of $R(S)$ which may, in some way, be extracted or, even better, expressed by means of ideals of R and congruences on S. Even in simple cases, such constructions may not guarantee description of all ideals of the semigroup ring $R(S)$. In order to see the extent of ideals which are kernels of homomorphisms of $R(S)$ onto $(R/I)/(S/\sigma)$, one would have to characterize such ideals of $R(S)$ in an abstract way. We offer these ideas as food for thought to the interested reader. This by far does not exhaust the themes related to homomorphic images of semigroup rings, for it is not even available knowledge how to characterize semigroup rings (let alone their homomorphic images) within the class of all rings.

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