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The Fibre of the Prym Map in Genus Four.

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Sunto. – *In questa nota si dà una descrizione della fibra della mappa di Prym in genere 4. Se JX è la Jacobiana di una curva di genere 3, allora la fibra della mappa di Prym in JX si ottiene dalla varietà di Kummer KX mediante due scoppiamenti:* $\sigma_1 \colon KX(0) \to KX$ *che è lo scoppiamento di KX nell'origine e* $\sigma_2 \colon \widetilde{KX(0)} \to KX(0)$ *che è lo scoppiamento lungo una curva isomorfa a X.*

Introduction.

Let C be a non-singular curve of genus g, let $\pi \colon \widetilde{C} \to C$ be a connected unramified double cover. The Prym variety $\mathfrak{p}(\widetilde{C}, C, \pi)$ is defined to be the connected component of the origin in $\ker Nm$, where $Nm \colon J(\widetilde{C}) \to J(C)$ is the norm map. By Mumford [M] $\mathfrak{p}(\widetilde{C}, C, \pi)$ is a principally polarized abelian variety of dimension $g-1$.

The above construction yields a morphism, the Prym map

$$\mathfrak{p}_{g-1} \colon \mathfrak{R}_g \to \mathfrak{A}_{g-1}.$$

where $\mathfrak{R}_g$ is the coarse moduli space of curves C of genus g, together with an unramified double cover $\widetilde{C}$ and $\mathfrak{A}_g$ is the coarse moduli space of principally polarized abelian varieties of dimension g.

The natural projection $p \colon \mathfrak{R}_g \to \mathfrak{M}_g$ is finite, of degree $2^{2g}-1$, where $\mathfrak{M}_g$ denotes the coarse moduli space of smooth, projective irreducible curves of genus g defined over $\mathbb{C}$. Given a C, its double covers correspond to the nonzero $(\mathbb{Z}/2\mathbb{Z})$-homology 1-classes.

The problem of extending $\mathfrak{p}_{g-1}$ to possibly singular and ramified covers was attempted by Fay [F] and Mumford [M]. Masiewicki in [Ma] had the correct notion of allowable double cover, but applied it only in the special case of plane quintics. Beauville in [Be.1], gave the precise definiton of allowable double cover and in § 6 he obtained a proper map $\mathfrak{P}_{g-1}$ which on a dense open

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set factors through the previously described $\mathfrak{p}_{g-1}$. The map $\mathfrak{P}_{g-1}\colon \overline{\mathfrak{R}}_g \to \mathfrak{A}_{g-1}$ is dominant for $g \leqslant 6$, [DS], generically finite for $g = 6$ [DS] and generically injective for $g \geqslant 7$ [FS]. Donagi and Smith show that the fibre of $\mathfrak{P}_5$ has the structure of the 27 lines on a cubic surface [DS]. Donagi [Do.] studied the fibre of $\mathfrak{P}_4$ at a general (A, Θ), and shows that it is isomorphic to a double cover of the Fano surface of the lines of a cubic threefold, and for $g = 3$ Verra [V] proves that the fibre of $\mathfrak{P}_2$ is biregular to the Siegel modular quartic threefold V.

In this paper we study the extended Prym map:

$$\mathfrak{P}_3\colon \overline{\mathfrak{R}}_4 \to \mathfrak{A}_3 .$$

We will work over the complex field. Let X be a smooth, reduced, irreducible, projective non-hyperelliptic curve of genus 3, and assume that all its Weierstrass points are normal. Let $JX = \mathrm{Pic}^0 X$ be the Jacobian variety of X with $\Theta \subset JX$ a symmetric theta divisor, $\mathrm{Pic}^n X$ the divisors on X of degree n modulo linear equivalence, $W_n \subset \mathrm{Pic}^n X$ the image of the n^{th}–Abel-Jacobi map, $X - X \subset JX$ the image of the map $X^2 \to JX$, $(p, q) \mapsto \mathcal{O}(p-q)$ and let $JX(0)$ denote the blow-up of JX at the origin.

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The fibre of the Prym map.

In this section our aim is to describe the fibre of the Prym map over a general principally polarized abelian variety of dimension 3.

In [Re.1] it is proved that the fiber of the Prym map $\mathfrak{p}\colon \mathfrak{R}_4 \to \mathfrak{A}_3$ over a general Jacobian JX is birational to $KX = JX/\pm 1$, following Beauville, this map extends to a dominant, proper map $\mathfrak{P}_3\colon \overline{\mathfrak{R}}_4 \to \mathfrak{A}_3$, where $\overline{\mathfrak{R}}_4$ denotes the moduli space of allowable double covers of genus 4.

One should recall [Be.1, Prop. 5.2 pg. 174] that if $\widetilde{C}$ is a stable curve and ι an involution on it then the pair $(\widetilde{C}, \iota)$ is allowable iff, the components of $\widetilde{C}$ can be grouped as $\widetilde{C} = A \cup A' \cup \widetilde{B}$ where ι interchanges A, A' and fixes $\widetilde{B}$, each component of A is «tree-like» and either

(i) $\widetilde{B} = \emptyset$, A connected, and $\# A \cap A' = 2$, or

(ii) $A \cap A' = \emptyset$, $\#(\widetilde{B} \cap A_i) = 1$ for each component A_i of A, the fixed points of ι in $\widetilde{B}$ are precisely the nodes, and the two branches at the nodes are never exchanged.

As a consequence of the above, one has two further types of double covers of stable curves which occur in the completed fibre of $\mathfrak{P}_3$ over a generic Jacobian.

The first type of allowable double cover from Beauville's list is the Wirtinger type cover [Be.1, pg. 175], let $p, q \in X$ be two distinct points, and let $C = X/_{p \sim q}$ be the singular curve of genus 4 obtained from X by identifying p, q. The curve C has an ordinary double point $p = q$. Let $\widetilde{C} = X_1 \cup X_2/_{p_1 \sim q_2,\, p_2 \sim q_1}$ where X_i are copies of X. $\widetilde{C}$ has two ordinay double points, and like C is a stable curve. The natural map $\pi\colon \widetilde{C} \to C$ is an allowable double cover in the sense of Beauville and $\mathfrak{P}_3(\widetilde{C}, C) = JX$.

The second type of singular double cover from Beauville's list is the elliptic tail [Be.1, pg. 74]. Consider a point $p \in X$, an elliptic curve $\widetilde{E}$ and a non zero point $\tilde{\eta} \in \widetilde{E}$ of order two. Then pick $\tilde{e} \in \widetilde{E}$ and consider the curve $\widetilde{C} = X_1 \cup \widetilde{E} \cup X_2/_{p_1 \sim \tilde{e},\, p_2 \sim \tilde{e} + \tilde{\eta}}$, (where X_i are copies of X) with the involution the one which exchanges the components of X and on $\widetilde{E}$ is defined as $p \mapsto p + \tilde{\eta}$, $2\tilde{\eta} = 0$, (this construction does not depend on $\tilde{e}$). Let $\pi\colon \widetilde{C} \to C$ the quotient map by the involution, then this cover is allowable and $\mathfrak{P}(\widetilde{C}, C) = JX$.

Observe that for elliptic tails the image of the Abel-Prym map $\widetilde{C} \to \mathfrak{P}(\widetilde{C}, C)$ is two copies of X intersecting at p.

Let $\mathfrak{R}_E$ denote the hypersurface in $\overline{\mathfrak{R}}_4$ parametrizing the elliptic tails, then the fibre of the restricted Prym map $\mathfrak{R}_E \to \mathfrak{A}_3$ over a generic Jacobian JX is isomorphic to $X \times \mathbb{P}^1$ [DS, Lemma 1.3.1, pg. 62].

Summing up, set-theoretically the fibre of the Prym map $\mathfrak{P}_3(JX)$ has three distinguished subsets, the allowable double covers of irreducible curves (possibly singular), the covers of Wirtinger type and the covers of elliptic tails.

In this section we prove the following:

THEOREM 1. – *Given a non-hyperelliptic curve X, let (JX, Θ) denote its Jacobian variety, and assume* $\mathrm{Aut}(JX) = \mathbb{Z}/2\mathbb{Z}$, *then the inverse image of (JX, Θ) by the Prym map $\mathfrak{P}_3$ is obtained from the Kummer variety KX by a sequence of two blow-up's, σ_1 and σ_2, where $\sigma_1\colon KX(0) \to KX$ blows up KX at the origin point and $\sigma_2\colon \widetilde{KX(0)} \to KX(0)$ is centered along a curve $\mathfrak{X}$ which is birational to the curve X.*

The proof of this theorem is divided in three parts. In the first part we extend the trigonal construction as given in [Rec.2] to obtain a family $S \to JX(0)$, and an open subset S' of $JX(0)$ in which the geometric points parametrize allowable double covers whose Prym's are JX.

In the second part we prove that there exist a birational morphism $\mathfrak{P}_3^{-1}(JX) \to KX(0)$, and at the end we apply the general properties of the blow-up to prove that the fibre of the Prym map is isomorphic to $\widetilde{KX(0)}$.

Proposition 2. – *There exist a natural family* $p\colon S \to JX(0)$, *of* 1-*cycles of* JX *numerically equivalent to* $\Theta . \Theta$ *that is, for* $c \in JX(0)$, $p^{-1}(c) \underset{\text{num.}}{\equiv} \Theta \bullet \Theta$.

Proof. – We consider the stable curves $\Theta \cap \Theta_c$, $c \in JX$, $c \neq 0$. These curves have an involution $i_c\colon \Theta \cap \Theta_c \to \Theta \cap \Theta_c$, $i_c(x) = c - x$ which is fixed point free for c generic.

To describe those curves as a family, let θ be a nonzero section of $\mathcal{O}_{JX}(\Theta)$ and consider the divisor

$$\mathcal{D} = \{\theta(z - c) = 0 \mid (c, z) \in JX \times \Theta\} \subset JX \times \Theta$$

and the first projection $\mathcal{D} \to JX$. Observe that using the second projection one can show that $\mathcal{D}$ is smooth, irreducible with fiber biregular to Θ.

To extend our family to the origin, we need to consider the intersection of Θ with an infinitesimal translation of itself. To do this, let us consider the morphism:

$$\pi\colon JX(0) \times \Theta \to JX \times \Theta$$

which in the first factor is the blow up at the origin and the identity on the second.

Let $D = \pi^* \mathcal{D}$, then $D = S + E \times \Theta$, where S is the strict transform of $\mathcal{D}$ and $E \subset JX(0)$ is the exceptional divisor, and using again the first projection, we obtain a family of 1-cycles of JX:

$$S \to JX(0).$$

To describe the family over $E \subset JX(0)$ let us write locally $S = \{f(c, z) = 0 \mid (c, z) \in JX(0) \times \Theta\}$ and $E = \{g(c) = 0 \mid c \in JX(0)\}$, so $D = \{fg = 0\}$.

Consider now $v \in E$ and let $\tilde{v}$ be a normal vector to E at v (in what follows, we will denote by $v \neq 0$ either an element of $T_0 JX$ or the direction it defines: $v \in \mathbb{P}((T_0 JX))$.

Then $\tilde{v}(fg)_{|v} = \tilde{v}(f)_{|v} \cdot g(v) + f(v) \cdot \tilde{v}(g)_{|v}$.

We observe that $g(v) = 0$ since $g = 0$ is the equation of E and $\tilde{v}(g)_{|v} \neq 0$ since $E \times \Theta$ is reduced.

So we have that

$$\tilde{v}(fg)_{|v} = 0 \quad \text{if and only if } f(v) = 0 .$$

Since $D = \pi^*(\Theta)$, by the chain rule we have:

$$\tilde{v}(fg)_{|v} = 0 \quad \text{if and only if } \left.\frac{\partial \theta(z - c)}{\partial v}\right|_{c = 0} = 0$$

hence

$$f(v) = 0 \quad \text{if and only if} \quad \frac{\partial\theta(z-c)}{\partial v}\Big|_{c=0} = 0\,.$$

So the fibers of the extended family over E are described by the equations

$$D_v = \left\{\theta = 0,\ \frac{\partial\theta}{\partial v}\Big|_{c=0} = 0\right\}.$$

It is well known that these curves are the pullback of lines under the Gauss map. ■

Following [Rec.2], let $m\colon \mathrm{Pic}^2 X \to \mathrm{Pic}^4 X$ be the morphism defined as $L \mapsto L^{\otimes 2}$ and $\psi\colon \mathrm{Pic}^4 X \to \mathrm{Pic}^0 X$ the canonical isomorphism $L \mapsto K \otimes L^{-1}$. Then define $R = \psi(mW_2)$, which is of dimension 2, and given $U \subset JX$, let $\widehat{U} \subset JX(0)$ denote the proper transform of U, then:

PROPOSITION 3. – *For $c \in JX(0)$, let $Z_c = \Theta \bullet \Theta_c$ and $i_c(x) = c - x$ be as before. Then*

(1) *If $c \in \widehat{(X-X)} - \widehat{R}$ then Z_c has two components, each isomorphic to X. The components are exchanged under the involution i_c, they intersect transversally in two points, which in turn are exchanged by i_c. Wirtinger type.*

(2) *If $c \in \widehat{(X-X)} \cap \widehat{R}$, then Z_c has two components, each isomorphic to X, those components are exchanged by i_c, they intersect non transversally in a unique point, which is a fixed point of i_c. Non allowable type.*

(3) *If $c \in \widehat{R} - \widehat{(X-X)}$, then Z_c is irreducible but not smooth, has ordinary double points and at most 3 of them, such points are fixed under the involution, which does not interchange the branches at the point. Lower genus type.*

(4) *If $c \notin \widehat{R} \cup \widehat{(X-X)}$, then Z_c is irreducible and non singular. Generic type.*

(5) *The above are the only possible fixed points of the involutions i_c.*

PROOF. – For $c \in JX - \{0\}$ the proof is given at [Rec. 2, Proposition 2.14] and the proof is similar in the case $c \in \mathbb{P}(T_0 JX)$. ■

As a consequence of the above we have that

COROLLARY 4. – *The pair (Z_c, ι_c) is allowable if and only if $c \notin \widehat{(X-X)} \cap \widehat{R}$.*

Observe that we can describe the sets appearing in Proposition 3 in terms of the g_4^1's on X.

(1) The point c is in $\widehat{(X-X)} - \widehat{R}$ iff the corresponding g_4^1 has a base point and without divisors of type $2a+2b$.

(2) The point c is in $\widehat{(X-X)} \cap \widehat{R}$ iff the corresponding g_4^1 has a base point and a divisor of type $2a+2b$. This is equivalent to $p+q+2r \in |K_X|$ for some $r \in X$, and $g_4^1 = |K_X - p + q|$.

(3) The point c is in $\widehat{R} - \widehat{(X-X)}$ iff the g_4^1 has no base point and has divisors of type $2a+2b$, at most 3 of them.

(4) The set $JX(0) - (\widehat{R} \cup \widehat{(X-X)})$ parametrizes the g_4^1 without base points and divisors of type $2a+2b$.

Fix an embedding

$$JX(0) \times \Theta \to \mathbb{P}^M .$$

Let $\varphi\colon S \to \mathbb{P}^M$ be the restriction of such embedding, then the diagram

$$\begin{array}{ccc} S & \xrightarrow{(p,\varphi)} & JX(0) \times \mathbb{P}^M \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & JX(0) & \end{array}$$

embeds our family S.

Observe that our family S is flat and the fibre is allowable if and only if $c \in R' := JX(0) - (\widehat{(X-X)} \cap \widehat{R})$.

Denote by

$$\mathcal{Z} \to R'$$

the restriction of the family to R'.

PROPOSITION 5. – *There is a rational dominant map*

$$KX(0) \to \mathfrak{P}^{-1}(JX).$$

PROOF. – Since $\mathcal{Z} \to R'$ is a flat family of allowable double covers of curves of genus 4, then by the universal property of $\overline{\mathfrak{R}}_4$, there exists a morphism $g\colon R' \to \overline{\mathfrak{R}}_4$ such that for each closed point $c \in R'$, $\mathcal{Z}_c$ is in the isomorphism class of allowable double covers of curves determined by the point $g(c) \in \overline{\mathfrak{R}}_4$. Since the Prym of $(\mathcal{Z}_c, \iota_c)$ is JX, then the image of this morphism is contained in $\mathfrak{P}^{-1}(JX)$.

Observe that, $\mathcal{Z}_c$ is isomorphic to $\mathcal{Z}_{-c}$ by the involution $c \mapsto -c$ on JX, and that the Gauss map $\mathcal{G}$ commutes with this isomorphism, so by the above, we obtain a morphism $\phi: R'' \to \mathfrak{P}^{-1}(JX)$, where R'' is the image of R' into $KX(0)$ and $KX(0) \to KX = JX/\pm 1$ is the blow up of the Kummer variety at the origin point. ■

PROPOSITION 6 [Be.2]. – *Let X be a non-hyperelliptic curve of genus 3, (JX, Θ) its Jacobian variety. Let Z be an effective algebraic 1-cycle in JX with fundamental class $[Z] = \bigwedge^2 [\Theta]$ in $H^4(JX, \mathbb{Z})$, then Z is one of the following*:

(i) $Z = \Theta \cap \Theta_a$ *for some* $a \in JX$, *or*

(ii) $Z = \mathcal{G}^{-1}(l)$ *for some line* $l \subset \mathbb{P}^2$, *where* $\mathcal{G}: \Theta \to \mathbb{P}^2$ *denotes the Gauss map.*

PROOF. – Since JX is an abelian variety, as a consequence of the strong Lefschetz theorem [K, pg 367] $\alpha: H^4(JX, \mathbb{Z}) \to H^2(JX, \mathbb{Z})$ is an isomorphism.

On the other hand, since Θ is a smooth divisor in JX, $\beta: H^2(JX, \mathbb{Z}) \to H^2(\Theta, \mathbb{Z})$ is an injective morpism [K, pg. 368].

As a consequence of the above, we can identify $[Z]$ with its image in $H^2(\Theta, \mathbb{Z})$.

From the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

on JX and Θ we get a diagram of exact sequences

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H^1(JX, \mathcal{O}^*_{JX}) & \xrightarrow{c_1} & H^2(JX, \mathbb{Z}) & \longrightarrow & H^2(JX, \mathcal{O}_{JX}) & \longrightarrow \dots \\
& \downarrow & & \downarrow \beta & & \downarrow & \\
\dots \longrightarrow & H^1(\Theta, \mathcal{O}^*_{\Theta}) & \xrightarrow{c_1} & H^2(\Theta, \mathbb{Z}) & \xrightarrow{j_*} & H^2(\Theta, \mathcal{O}_{\Theta}) & \longrightarrow \dots .
\end{array}$$

Then either $\beta([Z]) \in c_1(H^1(\Theta, \mathcal{O}^*_{\Theta}))$ or $0 \neq j_*(\beta([Z]) \in H^2(\Theta, \mathcal{O}_{\Theta})$.

In the first case, since β is injective we have that $[Z] \in c_1(H^1(JX, \mathcal{O}^*_{JX}))$, so $[Z_{|\Theta}] \in c_1(H^1(\Theta, \mathcal{O}^*_{\Theta}))$.

Since $\Theta \subset JX$ induces an isomorphism $\mathrm{Pic}^0(JX) \to \mathrm{Pic}^0(\Theta)$ (see [MP 2, Thm. 1.2]), and since $JX \to \mathrm{Pic}^0(JX)$, $a \mapsto [\Theta_a - \Theta]$, is an isomorphism, there exist a unique $a \in JX$ such that $Z = \Theta \cap \Theta_a$.

In the second case, by Kodaira-Serre theorem $H^2(\Theta, \mathcal{O}_{\Theta}) \simeq H^0(\Theta, \mathcal{O}_{\Theta}(\Theta))^*$, hence $j_*(\beta([Z]))$ gives an element v of $H^0(\Theta, \mathcal{O}_{\Theta}(\Theta))^*$, i.e. an hyperplane l_v in $\mathbb{P}^2$ the canonical space of the curve X, and as in the proof of the proposition 2, it follows that $Z = \mathcal{G}^{-1}(l_v)$ ■

In other words every effective algebraic 1-cycle Z in JX with $[Z] = \bigwedge^2[\Theta]$ belongs to the family $S \to JX(0)$ of the Proposition 2.

As a consequence of the Propositions 3, 6 and [DC-R], we have the following

COROLLARY 7. – *Let* $(Z, \iota) \in \mathfrak{P}^{-1}(JX)$ *and let* $\alpha: Z \to JX$ *be the Abel-Prym map, then*:

(1) $(Z, \iota) \in \mathfrak{P}^{-1}(JX)$ *is an elliptic tail if and only if* $\alpha(Z) = X_1 \cup X_2/_{p_1 \sim p_2}$ *if and only if* $\alpha(Z) = \Theta \cap \Theta_a$ *with* $a \in \widehat{(X-X)} \cap \widehat{R}$.

(2) $(Z, \iota) \in \mathfrak{P}^{-1}(JX)$ *is an elliptic-hyperelliptic curve if and only if* $\alpha(Z) = \Theta \cap \Theta_a$ *with* $a \in J_2X - \{0\}$.

(3) $(Z, \iota) \in \mathfrak{P}^{-1}(JX)$ *is a curve such that* Z_ι *has a vanishing theta null, i.e. only one* g_3^1 *if and only if* $\alpha(Z) = \Theta \cap \Theta_a$ *with* $a \in E$, *where* E *denotes the exceptional divisor on* $JX(0)$.

Let $\overline{\mathfrak{M}}_g^{(n)}$ denote the moduli space of stable curves of genus g with a level n-structure. H. Popp proved in [P] than $\overline{\mathfrak{M}}_g^{(n)}$ is a fine moduli space for $n \geq 3$, moreover, it is a separate, complete irreducible scheme [DM]. In particular there exist a universal curve $q: \overline{\Gamma}_g^{(n)} \to \overline{\mathfrak{M}}_g^{(n)}$, i.e. a morphism of complete irreducible schemes which has a universal property.

As a consequence of the representability of the functor of $\overline{\mathfrak{M}}_{2g-1}^{(n)}$-involutions of $\overline{\Gamma}_{2g-1}^{(n)}$, there is a $\overline{\mathfrak{M}}_{2g-1}^{(n)}$-scheme I which in the geometric points parametrizes curves $\tilde{C}$ of genus $2g-1$ with a level n-structure and with an involution. Following Beauville, [DS] constructs a $\overline{\mathfrak{M}}_{2g-1}^{(n)}$-scheme $\overline{\mathfrak{R}}_g^{(n)} \subset I$, the moduli space of allowable double covers of curves of genus g with a level n-structure, this space contains $\mathfrak{R}_g^{(n)}$ (the moduli space of smooth double covers of curves of genus g with a level n-structure) as a dense open subset and is contained in the closure in I of $\mathfrak{R}_g^{(n)}$. In particular [DS] proved that $\overline{\mathfrak{R}}_g^{(n)}$ admits a universal curve i.e. there exist a family of stable curves $q: \tilde{\mathcal{C}} \to \overline{\mathfrak{R}}_g^{(n)}$ and a $\overline{\mathfrak{R}}_g^{(n)}$-involution $\iota: \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ such that:

(a) For each $s \in \overline{\mathfrak{R}}_g^{(n)}$, the induced involution $\iota_s: \tilde{\mathcal{C}}_s \to \tilde{\mathcal{C}}_s$ is different from the identity on each component of $\tilde{\mathcal{C}}_s$.

(b) $\tilde{\mathcal{C}}_s$ has arithmetic genus $2g-1$, and the quotient curve $\tilde{\mathcal{C}}_s/\iota_s$ has arithmetic genus g.

(c) Any pair $(\tilde{C}, \iota)$ where $\tilde{C} \in \mathfrak{M}_g$, ι a fixed-point free involution is isomorphic to $(\tilde{\mathcal{C}}_s, \iota_s)$ for some $s \in \mathfrak{R}_g^{(n)} \subset \overline{\mathfrak{R}}_g^{(n)}$.

On the other hand, the symplectic group $Sp(4g-2, \mathbb{Z}/n\mathbb{Z})$ acts on $\overline{\mathfrak{M}}_{2g-1}^{(n)}$ with quotient $\overline{\mathfrak{M}}_{2g-1}$. This action lifts to I, with quotient $\mathfrak{R}' = I/\mathrm{Sp}(4g-2, \mathbb{Z}/n\mathbb{Z})$. $\mathfrak{R}'$ is finite over $\overline{\mathfrak{M}}_{2g-1}$, so $\mathfrak{R}'$ is a scheme. In particular $\overline{\mathfrak{R}}_g^{(n)}$

is a finite base extension of $\mathfrak{R}_g^{(n)}$, the coarse moduli space of allowable double covers of curves of genus g.

By [Be.1], there is a proper Prym morphism $\overline{\mathfrak{P}}: \overline{\mathfrak{R}}_g^{(n)} \to \mathfrak{A}_{g-1}$. Since by [DS] $\overline{\mathfrak{R}}_g^{(n)}$ is stable under $\mathrm{Sp}(4g-2, \mathbb{Z}/n\mathbb{Z})$ with quotient $\overline{\mathfrak{R}}_g$, and $\overline{\mathfrak{P}}$ commutes with this action, we obtain the induced map $\mathfrak{P}_{g-1}: \mathfrak{R}_g \to \mathfrak{A}_{g-1}$.

Let $\mathcal{P}$ denote the subscheme of $\overline{\mathfrak{R}}_4^{(n)}$ parametrizing the inverse image of JX by the morphism $\overline{\mathfrak{P}}$, and let $\overline{\Gamma} \to \mathcal{P}$ denote the restriction of the universal curve $\tilde{\mathfrak{C}}$ to $\mathcal{P}$

$$\begin{array}{ccc} \overline{\Gamma} & \longrightarrow & \tilde{\mathfrak{C}} \\ \downarrow & & \downarrow q \\ \mathcal{P} & \longrightarrow & \overline{\mathfrak{R}}_3^{(n)} \end{array}$$

We denote by ι the $\mathcal{P}$-involution given by the restriction of the $\overline{\mathfrak{R}}_4^{(n)}$-involution ι to $\mathcal{P}$.

Consider the Jacobian map

$$\begin{array}{ccc} \varphi: \overline{\Gamma} & \longrightarrow & J(\overline{\Gamma}/\mathcal{P}) =: \mathfrak{J} \\ & \searrow \quad \swarrow & \\ & \mathcal{P} & \end{array}$$

which is injective over closed points [GIT, pg. 142]. The $\overline{\mathfrak{R}}_4^{(n)}$-involution $\iota: \tilde{\mathfrak{C}} \to \tilde{\mathfrak{C}}$ induces (by descent theory) a $\mathcal{P}$-involution $\iota^*: \mathfrak{J} \to \mathfrak{J}$. Denote by α the $\mathcal{P}$-morphism $(1_{\mathfrak{J}} - \iota^*) \cdot \varphi: \overline{\Gamma} \to \mathfrak{J}$ and let $P := \mathrm{Im}(1_{\mathfrak{J}} - \iota^*)$.

On the diagram

$$\begin{array}{ccc} \tilde{\Gamma} & \xrightarrow{\alpha} & P \\ & \searrow \quad \swarrow & \\ & \mathcal{P} & \end{array}$$

over closed points of $\mathcal{P}$, we get on P the Prym varieties and α is the corresponding Abel Prym map.

Let $\mathfrak{X} \subset KX(0)$ denote the image of $\widehat{(X-X)} \cap \widehat{R} \subset JX(0)$ by the induced morphism $JX(0) \to KX(0)$. The map $X \to KX$, $x \mapsto \pm[p-q]$ where $2x + p + q \in |K_X|$ induces an isomorphism between $\mathfrak{X}$ (with the reduced structure) and the curve X.

Moreover, for a closed point $b \in \mathcal{P}$, $\mathcal{P}_b$ is the Prym of $(\overline{\Gamma}_b, \iota_b)$, so $\alpha_*(\overline{\Gamma}_b) \subset \mathcal{P}_b$ has class $[\Theta \centerdot \Theta]$ and by the definition of $\mathcal{P}$ we know that $\mathcal{P}_b \simeq JX$ (as p.p.a.v). Actually $\alpha_*(\overline{\Gamma}_b) = \Theta \centerdot \Theta_{a(b)}$ for some $\pm a(b) \in KX(0)$. By Proposition 6 $a(b)$ is unique (here we assume X is general, so $\mathrm{Aut}(JX) = \pm 1$). This defines a morphism $\mathcal{P} \to KX(0)$ which descends to a morphism $\psi: \mathfrak{P}^{-1}(JX) \to KX(0)$.

We are now ready to prove the theorem. Recall that we have a morphism

$\phi\colon R'' \to \mathfrak{P}^{-1}(JX)$, $R'' = KX(0) - \mathfrak{X}$. Moreover we have $\psi \cdot \phi = id_{R''}$ and $\phi \cdot \psi_{|\phi(R'')} = id_{\phi(R'')}$.

The morphism $\psi_{|\psi^{-1}(R'')}\colon \psi^{-1}(KX(0) - \mathfrak{X}) \to KX(0) - \mathfrak{X}$ is an isomorphism, and $\psi^{-1}(R'')$ is the maximal open set $U \subset \mathfrak{P}^{-1}(JX)$ such that the restriction $\psi_{|U}$ is an isomorphism. Since $\mathrm{Sing}\, KX(0) \cap \mathfrak{X} = \emptyset$ there exists a sheaf of ideals $\mathfrak{J}$ on $KX(0)$ with support $\mathfrak{X}$ such that $\mathfrak{P}^{-1}(JX)$ is isomorphic to the blow up $p\colon \widetilde{KX(0)} \to KX(0)$ with respect to $\mathfrak{J}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{P}^{-1}(JX) & \xrightarrow{\ \simeq\ } & \widetilde{KX(0)} \\ & \psi \searrow \quad \swarrow p & \\ & KX(0) & \end{array}$$

Moreover since the fibre of the restricted Prym map $\mathfrak{R}_E \to \mathfrak{A}_3$ over a generic Jacobian JX is isomorphic to $X \times \mathbb{P}^1$ [DS, Lemma 1.3.1, pg. 62], then $\mathfrak{J}$ must be the ideal sheaf of $\mathfrak{X}$ (recall $\mathfrak{X} := \widetilde{(X - X)} \cap \widehat{R}$ with the reduced structure) [Har. pg. 166, 171].

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